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Metric characterization of apartments in dual polar spaces

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ABSTRACT

Let Π be a polar space of rank n and let $\mathcal{G}_k(\Pi)$, $k \in \{0, \dots, n-1\}$ be the polar Grassmannian formed by k-dimensional singular subspaces of Π. The corresponding Grassmann graph will be denoted by $\Gamma_k(\Pi)$. We consider the polar Grassmannian $\mathcal{G}_{n-1}(\Pi)$ formed by maximal singular subspaces of Π and show that the image of every isometric embedding of the n-dimensional hypercube graph H_n in $\Gamma_{n-1}(\Pi)$ is an apartment of $\mathcal{G}_{n-1}(\Pi)$. This follows from a more general result concerning isometric embeddings of H_m, $m \leqslant n$ in $\Gamma_{n-1}(\Pi)$. As an application, we classify all isometric embeddings of $\Gamma_{n-1}(\Pi)$ in $\Gamma_{n'-1}(\Pi')$, where Π' is a polar space of rank $n' \geqslant n$.

1. Introduction

The problem discussed in this note was first considered in [12] and motivated by the well-known metric characterization of apartments in Tits buildings (Theorem 1). By [14], a *building* is a simplicial complex Δ containing a family of subcomplexes called *apartments* and satisfying certain axioms. One of the axioms says that all apartments are isomorphic to a certain Coxeter complex – the simplicial complex associated with a Coxeter system. The diagram of this Coxeter system defines the type of the building Δ.

Maximal simplices of Δ, they are called *chambers*, have the same cardinal number n (the rank of Δ). Two chambers are said to be *adjacent* if their intersection consists of $n-1$ vertices. We write $\mathrm{Ch}(\Delta)$ for the set of all chambers and denote by $\Gamma_{\mathrm{ch}}(\Delta)$ the graph whose vertex set is $\mathrm{Ch}(\Delta)$ and whose edges are pairs of adjacent chambers. Let $\mathcal{A}$ be the intersection of $\mathrm{Ch}(\Delta)$ with an apartment of Δ and let $\Gamma(\mathcal{A})$ be the restriction of the graph $\Gamma_{\mathrm{ch}}(\Delta)$ to $\mathcal{A}$.

Theorem 1. *(See [2, p. 90].) A subset of* $\mathrm{Ch}(\Delta)$ *is the intersection of* $\mathrm{Ch}(\Delta)$ *with an apartment of* Δ *if and only if it is the image of an isometric embedding of* $\Gamma(\mathcal{A})$ *in* $\Gamma_{\mathrm{ch}}(\Delta)$.

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The vertex set of Δ can be naturally decomposed in n disjoint subsets called *Grassmannians*: the vertex set is labeled by the nodes of the associated diagram (such a labeling is unique up to a permutation on the set of nodes) and all vertices corresponding to the same node form a Grassmannian. More general Grassmannians defined by parts of the diagram were investigated in [13].

Let $\mathcal{G}$ be a Grassmannian of Δ. We say that $a, b \in \mathcal{G}$ are *adjacent* if there exists a simplex $P \in \Delta$ such that $P \cup \{a\}$ and $P \cup \{b\}$ both are chambers; in this case, the set of all $c \in \mathcal{G}$ such that $P \cup \{c\}$ is a chamber will be called the *line* joining a and b. The Grassmannian $\mathcal{G}$ together with the set of all such lines is a partial linear space; it is called the *Grassmann space* corresponding to $\mathcal{G}$. The associated *Grassmann graph* is the graph $\Gamma_{\mathcal{G}}$ whose vertex set is $\mathcal{G}$ and whose edges are pairs of adjacent vertices; in other words, $\Gamma_{\mathcal{G}}$ is the collinearity graph of the Grassmann space. It is well known that this graph is connected. The intersections of $\mathcal{G}$ with apartments of Δ are called *apartments* of the Grassmannian $\mathcal{G}$.

In [4,5,9,10] apartments of some Grassmannians were characterized in terms of the associated Grassmann spaces. We are interested in a metric characterization of apartments in Grassmannians similar to Theorem 1.

Every building of type A_{n-1}, $n \geqslant 4$ is the flag complex of an n-dimensional vector space V (over a division ring). The Grassmannians of this building are the usual Grassmannians $\mathcal{G}_k(V)$, $k \in \{1, \dots, n-1\}$ formed by k-dimensional subspaces of V. Two elements of $\mathcal{G}_k(V)$ are adjacent if their intersection is $(k-1)$-dimensional. The associated Grassmann graph is denoted by $\Gamma_k(V)$. If $k = 1, n-1$ then any two distinct vertices of $\Gamma_k(V)$ are adjacent and the corresponding Grassmann space is the projective space Π_V associated with V or the dual projective space Π_V^*, respectively. Every apartment of $\mathcal{G}_k(V)$ is defined by a certain base $B \subset V$: it consists of all k-dimensional subspaces spanned by subsets of B. For every $S, U \in \mathcal{G}_k(V)$ the distance $d(S, U)$ in $\Gamma_k(V)$ is equal to

$$k - \dim(S \cap U)$$

and all apartments of $\mathcal{G}_k(V)$ are the images of isometric embeddings of the Johnson graph $J(n, k)$ in $\Gamma_k(V)$. However, the image of every isometric embedding of $J(n, k)$ in $\Gamma_k(V)$ is an apartment of $\mathcal{G}_k(V)$ if and only if $n = 2k$. This follows from the classification of isometric embeddings of Johnson graphs $J(l, m)$, $1 < m < l - 1$ in the Grassmann graph $\Gamma_k(V)$, $1 < k < n - 1$ given in [12].

Every building of type C_n is the flag complex of a rank n polar space Π, i.e. it consists of all flags formed by singular subspaces of Π. Apartments of this building are defined by frames of Π and the associated Grassmannians are the polar Grassmannians $\mathcal{G}_k(\Pi)$, $k \in \{0, \dots, n-1\}$ consisting of k-dimensional singular subspaces of Π. We restrict ourself to the Grassmannian $\mathcal{G}_{n-1}(\Pi)$ formed by maximal singular subspaces of Π. Two elements of $\mathcal{G}_{n-1}(\Pi)$ are adjacent if their intersection is $(n-2)$-dimensional. The corresponding Grassmann graph is denoted by $\Gamma_{n-1}(\Pi)$. The associated Grassmann space is known as the *dual polar space* of Π. We show that apartments of $\mathcal{G}_{n-1}(\Pi)$ can be characterized as the images of isometric embeddings of the n-dimensional hypercube graph H_n in $\Gamma_{n-1}(\Pi)$. This is a partial case of a more general result (Theorem 2) concerning isometric embeddings of H_m, $m \leqslant n$ in $\Gamma_{n-1}(\Pi)$. As an application, we describe all isometric embeddings of $\Gamma_{n-1}(\Pi)$ in $\Gamma_{n'-1}(\Pi')$, where Π' is a polar space of rank $n' \geqslant n$ (Theorem 3). This result generalizes classical Chow's theorem [3] on automorphisms of the graph $\Gamma_{n-1}(\Pi)$.

2. Basics

2.1. Graph theory

Let Γ be a connected graph. The *distance* $d(v, w)$ between two vertices $v, w \in \Gamma$ is defined as the smallest number i such that there exists a path of length i (a path consisting of i edges) between v and w; a path connecting v and w is called a *geodesic* if it consists of $d(v, w)$ edges. The number

$$\max\{d(v, w):\ v, w \in \Gamma\}$$

is called the *diameter* of Γ. Suppose that it is finite. Two vertices of Γ are said to be *opposite* if the distance between them is maximal (is equal to the diameter).

An *isometric embedding* of a graph Γ in a graph Γ' is an injection of the vertex set of Γ to the vertex set of Γ' preserving the distance between vertices. The existence of isometric embeddings of Γ in Γ' implies that the diameter of Γ is not greater than the diameter of Γ'. An isometric embedding of Γ in Γ' is an isomorphism of Γ to a subgraph of Γ'; the converse fails, an isomorphism of Γ to a proper subgraph of Γ' needs not to be an isometric embedding. We refer [6] for the general theory of isometric embeddings of graphs.

2.2. Hypercube graphs

Let

$$J := \{1, \dots, n, -1, \dots, -n\}.$$

A subset $X \subset J$ is said to be *singular* if

$$i \in X \quad \Rightarrow \quad -i \notin X$$

for all $i \in I$. Every maximal singular subset consists of n elements and for every $i \in \{1, \dots, n\}$ it contains i or $-i$. The *n-dimensional hypercube graph* H_n can be defined as the graph whose vertices are maximal singular subsets of J and two such subsets are adjacent (connected by an edge) if their intersection consists of $n-1$ elements. This graph is connected and for every maximal singular subsets $X, Y \subset J$ the distance $d(X, Y)$ in the graph H_n is equal to

$$n - |X \cap Y|.$$

The diameter of H_n is equal to n and X, Y are opposite vertices of H_n if and only if $X \cap Y = \emptyset$.

2.3. Partial linear spaces

Let P be a non-empty set and let $\mathcal{L}$ be a family of proper subsets of P. Elements of P and $\mathcal{L}$ will be called *points* and *lines*, respectively. Two or more points are said to be *collinear* if there is a line containing all of them. Suppose that the pair $\Pi = (P, \mathcal{L})$ is a *partial linear space*, i.e. the following axioms hold:

- each line contains at least two points,
- every point belongs to a line,
- for any distinct collinear points $p, q \in P$ there is precisely one line containing them, this line is denoted by pq.

We say that $S \subset P$ is a *subspace* of Π if for any distinct collinear points $p, q \in S$ the line pq is contained in S. A *singular* subspace is a subspace where any two points are collinear (the empty set and a single point are singular subspaces).

For every subset $X \subset P$ the minimal subspace containing X (the intersection of all subspaces containing X) is called *spanned* by X and denoted by $\langle X \rangle$. We say that X is *independent* if $\langle X \rangle$ is not spanned by a proper subset of X.

Let S be a subspace of Π (possible $S = P$). An independent subset $X \subset S$ is called a *base* of S if $\langle X \rangle = S$. The *dimension* of S is the smallest cardinality α such that S has a base of cardinality $\alpha + 1$. The dimension of the empty set and a single point is equal to -1 and 0 (respectively), lines are 1-dimensional subspaces.

Two partial linear spaces $\Pi = (P, \mathcal{L})$ and $\Pi' = (P', \mathcal{L}')$ are *isomorphic* if there exists a bijection $f : P \to P'$ such that $f(\mathcal{L}) = \mathcal{L}'$; this bijection is called a *collineation* of Π to Π'. We say that an injection of P to P' is an *embedding* of Π in Π' if it sends lines to subsets of lines such that distinct lines go to subsets of distinct lines.

2.4. Polar spaces

Following [1], we define a *polar space* as a partial linear space $\Pi = (P, \mathcal{L})$ satisfying the following axioms:

- each line contains at least three points,
- there is no point collinear with all points,
- if $p \in P$ and $L \in \mathcal{L}$ then p is collinear with one or all points of the line L,
- any flag formed by singular subspaces is finite.

If there exists a maximal singular subspace of Π containing more than one line then all maximal singular subspaces of Π are projective spaces of the same dimension $n \geqslant 2$; the number $n+1$ is called the *rank* of Π.

The collinearity relation on Π will be denoted by $\perp$: we write $p \perp q$ if $p, q \in P$ are collinear and $p \not\perp q$ otherwise. If $X, Y \subset P$ then $X \perp Y$ means that every point of X is collinear with all points of Y.

Lemma 1. *The following assertions are fulfilled*:

(1) *If $X \subset P$ and $X \perp X$ then the subspace $\langle X \rangle$ is singular and $p \perp X$ implies that $p \perp \langle X \rangle$.*
(2) *If S is a maximal singular subspace of Π then $p \perp S$ implies that $p \in S$.*

Proof. See, for example, Subsection 4.1.1 in [11]. □

2.5. Dual polar spaces

Let $\Pi = (P, \mathcal{L})$ be a polar space of rank $n \geqslant 3$. For every $k \in \{0, 1, \dots, n-1\}$ we denote by $\mathcal{G}_k(\Pi)$ the Grassmannian consisting of all k-dimensional singular subspaces of Π. Then $\mathcal{G}_{n-1}(\Pi)$ is formed by maximal singular subspaces of Π. Recall that two elements of $\mathcal{G}_{n-1}(\Pi)$ are adjacent if their intersection is $(n-2)$-dimensional and the associated Grassmann graph is denoted by $\Gamma_{n-1}(\Pi)$.

Let M be an m-dimensional singular subspace of Π. If $m < k$ then we write $[M\rangle_k$ for the set of all elements of $\mathcal{G}_k(\Pi)$ containing M. In the case when $m = n-2$, the subset $[M\rangle_{n-1}$ is called a *line* of $\mathcal{G}_{n-1}(\Pi)$. The Grassmannian $\mathcal{G}_{n-1}(\Pi)$ together with the set of all such lines is a partial linear space; it is called the *dual polar space* of Π. Two distinct points of the dual polar space are collinear if and only if they are adjacent elements of $\mathcal{G}_{n-1}(\Pi)$. Note that every maximal singular subspace of the dual polar space is a line.

Let M be as above. If $m < n-2$ then $[M\rangle_{n-1}$ is a non-singular subspace of the dual polar space. Subspaces of such type are called *parabolic* [5]. We will use the following fact: *the parabolic subspace $[M\rangle_{n-1}$ is isomorphic to the dual polar space of a rank $n-m-1$ polar space.*

Consider $[M\rangle_{m+1}$. A subset $\mathcal{X} \subset [M\rangle_{m+1}$ is called a *line* if there exists $N \in [M\rangle_{m+2}$ such that $\mathcal{X}$ consists of all elements of $[M\rangle_{m+1}$ contained in N. Then, by Lemma 4.4 in [11], $[M\rangle_{m+1}$ together with the set of all such lines is a polar space of rank $n-m-1$. If $\mathcal{Y} \subset [M\rangle_{m+1}$ is a maximal singular subspace of this polar space then there exists $S \in [M\rangle_{n-1}$ such that $\mathcal{Y}$ consists of all elements of $[M\rangle_{m+1}$ contained in S. So, we can identify maximal singular subspaces of $[M\rangle_{m+1}$ with elements of $[M\rangle_{n-1}$. This correspondence is a collineation between $[M\rangle_{n-1}$ and the dual polar space of $[M\rangle_{m+1}$.

For every $S, U \in \mathcal{G}_{n-1}(\Pi)$ the distance $d(S, U)$ in the graph $\Gamma_{n-1}(\Pi)$ is equal to

$$n - 1 - \dim(S \cap U).$$

The diameter of $\Gamma_{n-1}(\Pi)$ is equal to n and two vertices of $\Gamma_{n-1}(\Pi)$ are opposite if and only if they are disjoint elements of $\mathcal{G}_{n-1}(\Pi)$.

3. Apartments of dual polar spaces

3.1. Main result

Let $\Pi = (P, \mathcal{L})$ be a polar space of rank n. Apartments of $\mathcal{G}_k(\Pi)$ are defined by frames of Π. A subset $\{p_1, \ldots, p_{2n}\} \subset P$ is called a *frame* if for every $i \in \{1, \ldots, 2n\}$ there exists unique $\sigma(i) \in \{1, \ldots, 2n\}$ such that $p_i \not\perp p_{\sigma(i)}$. Frames are independent subsets of Π. This guarantees that any k mutually collinear points in a frame span a $(k-1)$-dimensional singular subspace.

Let $B = \{p_1, \ldots, p_{2n}\}$ be a frame of Π. The associated apartment $\mathcal{A} \subset \mathcal{G}_{n-1}(\Pi)$ is formed by all maximal singular subspaces spanned by subsets of B – the subspaces of type $\langle p_{i_1}, \ldots, p_{i_n}\rangle$ such that

$$\{i_1, \ldots, i_n\} \cap \big\{\sigma(i_1), \ldots, \sigma(i_n)\big\} = \emptyset.$$

Thus every element of $\mathcal{A}$ contains precisely one of the points p_i or $p_{\sigma(i)}$ for each i. By Subsection 2.2, $\mathcal{A}$ is the image of an isometric embedding of H_n in $\Gamma_{n-1}(\Pi)$.

Let M be an $(n-m-1)$-dimensional singular subspace of Π and let B be a frame of Π such that M is spanned by a subset of B. The intersection of the associated apartment of $\mathcal{G}_{n-1}(\Pi)$ with the parabolic subspace $[M\rangle_{n-1}$ is called an *apartment* of $[M\rangle_{n-1}$. This parabolic subspace can be identified with the dual polar space of the rank m polar space $[M\rangle_{n-m}$, see Subsection 2.5; and every apartment of $[M\rangle_{n-1}$ is defined by a frame of this polar space, see [11, p. 180]. All apartments of $[M\rangle_{n-1}$ are the images of isometric embeddings of H_m in $\Gamma_{n-1}(\Pi)$.

Theorem 2. *The image of every isometric embedding of H_m, $m \leqslant n$ in $\Gamma_{n-1}(\Pi)$ is an apartment in a parabolic subspace $[M\rangle_{n-1}$, where M is an $(n-m-1)$-dimensional singular subspace of Π. In particular, the image of every isometric embedding of H_n in $\Gamma_{n-1}(\Pi)$ is an apartment of $\mathcal{G}_{n-1}(\Pi)$.*

Remark 1. There are no isometric embeddings of H_m in $\Gamma_{n-1}(\Pi)$ if $m > n$ (in this case, the diameter of H_m is greater than the diameter of $\Gamma_{n-1}(\Pi)$). However, there exists a subset $\mathcal{X} \subset \mathcal{G}_{n-1}(\Pi)$ such that the restriction of the graph $\Gamma_{n-1}(\Pi)$ to $\mathcal{X}$ is isomorphic to H_{n+1}, see Example 2 in [4].

3.2. Lemmas

To prove Theorem 2 we will use the following lemmas.

Lemma 2. *If $X_0, X_1, \ldots, X_m$ is a geodesic in $\Gamma_{n-1}(\Pi)$ then*

$$X_0 \cap X_m \subset X_i$$

for every $i \in \{1, \ldots, m-1\}$.

Proof. We prove induction by i that $M := X_0 \cap X_m$ is contained in every X_i. The statement is trivial if $i = 0$. Suppose that $i \geqslant 1$ and $M \subset X_{i-1}$.

Since $X_0, X_1, \ldots, X_m$ is a geodesic, we have

$$d(X_i, X_m) < d(X_{i-1}, X_m)$$

and, by the distance formula given in Subsection 2.5,

$$\dim(X_i \cap X_m) > \dim(X_{i-1} \cap X_m).$$

The latter implies the existence of a point

$$p \in (X_i \cap X_m) \setminus X_{i-1}.$$

Then X_i is spanned by the $(n-2)$-dimensional singular subspace $X_{i-1} \cap X_i$ and the point p. On the other hand,

$$(X_{i-1} \cap X_i) \perp M$$

(M is contained in X_{i-1} by inductive hypothesis) and $p \perp M$ (p and M both are contained in X_m). By the first part of Lemma 1, $X_i \perp M$. Since X_i is a maximal singular subspace, the second part of Lemma 1 guarantees that $M \subset X_i$. □

Lemma 3. *In the hypercube graph H_m for every vertex v there is a unique vertex opposite to v. If vertices $v, w \in H_m$ are opposite then for every vertex $u \in H_m$ there is a geodesic connecting v with w and passing through u.*

Proof. An easy verification. □

Lemma 4. *The image of every isometric embedding of H_m, $m \leqslant n$ in $\Gamma_{n-1}(\Pi)$ is contained in a parabolic subspace $[M\rangle_{n-1}$, where M is an $(n-m-1)$-dimensional singular subspace of Π.*

Proof. Let f be an isometric embedding of H_m in $\Gamma_{n-1}(\Pi)$. We take any opposite vertices $v, w \in H_m$. Then

$$d\big(f(v), f(w)\big) = m$$

and, by the distance formula,

$$M := f(v) \cap f(w)$$

is an $(n-m-1)$-dimensional singular subspace of Π. Lemmas 2 and 3 show that M is contained in $f(u)$ for every $u \in H_m$. □

3.3. Proof of Theorem 2

If M is an $(n-m-1)$-dimensional singular subspace of Π then $[M\rangle_{n-m}$ is a polar space of rank m and $[M\rangle_{n-1}$ can be identified with the associated dual polar space, see Subsection 2.5; moreover, every apartment of $[M\rangle_{n-1}$ is defined by a frame of the polar space $[M\rangle_{n-m}$. Therefore, by Lemma 4, it is sufficient to prove Theorem 2 only in the case when $m = n$.

Let $\{p_1, \ldots, p_{2n}\}$ be a frame of Π. Denote by $\mathcal{A}$ the associated apartment of $\mathcal{G}_{n-1}(\Pi)$. The restriction of $\Gamma_{n-1}(\Pi)$ to $\mathcal{A}$ is isomorphic to H_n. Suppose that $f : \mathcal{A} \to \mathcal{G}_{n-1}(\Pi)$ is an injection which induces an isometric embedding of H_n in $\Gamma_{n-1}(\Pi)$. Let $\mathcal{X}$ be the image of f.

Each $\mathcal{A} \cap [p_i\rangle_{n-1}$ is an apartment in the parabolic subspace $[p_i\rangle_{n-1}$. Since $[p_i\rangle_{n-1}$ is the dual polar space of the rank $n-1$ polar space $[p_i\rangle_1$, see Subsection 2.5, the restriction of $\Gamma_{n-1}(\Pi)$ to $\mathcal{A} \cap [p_i\rangle_{n-1}$ is isomorphic to H_{n-1}. Lemma 4 implies the existence of points $q_1, \ldots, q_{2n}$ such that

$$f\big(\mathcal{A} \cap [p_i\rangle_{n-1}\big) \subset \mathcal{X} \cap [q_i\rangle_{n-1}$$

for every i.

For every $X \in \mathcal{A} \setminus [p_i\rangle_{n-1}$ there exists $Y \in \mathcal{A} \cap [p_i\rangle_{n-1}$ disjoint from X. We have

$$d(X, Y) = d\big(f(X), f(Y)\big) = n.$$

Then $f(X)$ and $f(Y)$ are disjoint and $q_i \in f(Y)$. This means that $q_i \notin f(X)$ and $f(X)$ does not belong to $\mathcal{X} \cap [q_i\rangle_{n-1}$. Since $f(\mathcal{A}) = \mathcal{X}$, we get the equality

$$f\big(\mathcal{A} \cap [p_i\rangle_{n-1}\big) = \mathcal{X} \cap [q_i\rangle_{n-1}.$$

If $i \neq j$ then $\mathcal{A} \cap [p_i\rangle_{n-1}$ and $\mathcal{A} \cap [p_j\rangle_{n-1}$ are distinct subsets of $\mathcal{A}$ and their images $\mathcal{X} \cap [q_i\rangle_{n-1}$ and $\mathcal{X} \cap [q_j\rangle_{n-1}$ are distinct.

Therefore, $q_i \neq q_j$ if $i \neq j$. For every $X \in \mathcal{A}$ we have

$$p_i \in X \quad \Leftrightarrow \quad q_i \in f(X).$$

Also, $q_i \perp q_j$ if $j \neq \sigma(i)$ (we take any $X \in \mathcal{A}$ which contains p_i and p_j, then q_i and q_j both belong to $f(X)$).

Lemma 5. *For any* $\{i_1, \ldots, i_n\} \subset \{1, \ldots, 2n\}$ *satisfying*

$$\{i_1, \ldots, i_n\} \cap \{\sigma(i_1), \ldots, \sigma(i_n)\} = \emptyset,$$

$\langle q_{i_1}, \ldots, q_{i_n} \rangle$ *is a maximal singular subspace of* Π *and*

$$f(\langle p_{i_1}, \ldots, p_{i_n} \rangle) = \langle q_{i_1}, \ldots, q_{i_n} \rangle.$$

Proof. Suppose that q_{i_n} belongs to the singular subspace $\langle q_{i_1}, \ldots, q_{i_{n-1}} \rangle$. Let X and Y be the elements of $\mathcal{A}$ spanned by

$$p_{i_1}, \ldots, p_{i_{n-1}}, p_{\sigma(i_n)} \quad \text{and} \quad p_{\sigma(i_1)}, \ldots, p_{\sigma(i_{n-1})}, p_{i_n},$$

respectively. These subspaces are disjoint and the same holds for $f(X)$ and $f(Y)$. We have

$$\langle q_{i_1}, \ldots, q_{i_{n-1}} \rangle \subset f(X) \quad \text{and} \quad q_{i_n} \in f(Y)$$

which contradicts $q_{i_n} \in \langle q_{i_1}, \ldots, q_{i_{n-1}} \rangle$.

Therefore, $q_{i_1}, \ldots, q_{i_n}$ form an independent subset of Π and $\langle q_{i_1}, \ldots, q_{i_n} \rangle$ is an element of $\mathcal{G}_{n-1}(\Pi)$. Since $f(\langle p_{i_1}, \ldots, p_{i_n} \rangle)$ contains $q_{i_1}, \ldots, q_{i_n}$, this subspace coincides with $\langle q_{i_1}, \ldots, q_{i_n} \rangle$. $\square$

By Lemma 5, every element of $\mathcal{X}$ is spanned by some $q_{i_1}, \ldots, q_{i_n}$. We need to show that the points $q_1, \ldots, q_{2n}$ form a frame of Π. Since $q_i \perp q_j$ if $j \neq \sigma(i)$, it is sufficient to establish that $q_i \not\perp q_{\sigma(i)}$ for all i.

Suppose that $q_i \perp q_{\sigma(i)}$ for a certain i. Then $q_i \perp q_j$ for every $j \in \{1, \ldots, 2n\}$ and $q_i \perp X$ for all $X \in \mathcal{X}$. Since $\mathcal{X}$ is formed by maximal singular subspaces of Π, the second part of Lemma 1 implies that q_i belongs to every element of $\mathcal{X}$. Then the distance between any two elements of $\mathcal{X}$ is not greater than $n-1$ which is impossible.

4. Application of Theorem 2

4.1. Let $\Pi = (P, \mathcal{L})$ and $\Pi' = (P', \mathcal{L}')$ be polar spaces of rank n and n', respectively. In this section Theorem 2 will be exploited to study isometric embeddings of $\Gamma_{n-1}(\Pi)$ in $\Gamma_{n'-1}(\Pi')$. The existence of such embeddings implies that n (the diameter of $\Gamma_{n-1}(\Pi)$) is not greater than n' (the diameter of $\Gamma_{n-1}(\Pi')$). So, we assume that $n \leqslant n'$.

Suppose that $n = n'$. Every mapping $f : P \to P'$ sending frames of Π to frames of Π' is an embedding of Π in Π', see Subsection 4.9.6 in [11]; moreover, for every singular subspace S of Π the subset $f(S)$ spans a singular subspace whose dimension is equal to the dimension of S. The mapping of $\mathcal{G}_{n-1}(\Pi)$ to $\mathcal{G}_{n-1}(\Pi')$ transferring every S to $\langle f(S) \rangle$ is an isometric embedding of $\Gamma_{n-1}(\Pi)$ in $\Gamma_{n-1}(\Pi')$.

Now consider the general case. Let M be an $(n'-n-1)$-dimensional singular subspace of Π'. By Subsection 2.5, $[M\rangle_{n'-n}$ is a polar space of rank n and $[M\rangle_{n'-1}$ can be identified with the associated dual polar space (if $n = n'$ then $M = \emptyset$ and $[M\rangle_{n'-n}$ coincides with P'). As above, every frames preserving mapping of Π to $[M\rangle_{n'-n}$ (a mapping which sends frames to frames) induces an isometric embedding of $\Gamma_{n-1}(\Pi)$ in $\Gamma_{n'-1}(\Pi')$.

Theorem 3. *Let* $f : \mathcal{G}_{n-1}(\Pi) \to \mathcal{G}_{n'-1}(\Pi')$ *be an isometric embedding of* $\Gamma_{n-1}(\Pi)$ *in* $\Gamma_{n'-1}(\Pi')$. *There exists an* $(n'-n-1)$*-dimensional singular subspace* M *of* Π' *such that the image of* f *is contained in* $[M\rangle_{n'-1}$ *and* f *is induced by a frames preserving mapping of* Π *to* $[M\rangle_{n'-n}$.

Remark 2. Theorem 3 generalizes classical Chow's theorem [3]: if $n = n'$ then every isomorphism of $\Gamma_{n-1}(\Pi)$ to $\Gamma_{n-1}(\Pi')$ is induced by a collineation of Π to Π'. In [3] this theorem was proved for the dual polar spaces of non-degenerate reflexive forms; but Chow's method works in the general case, see Subsection 4.6.4 in [11]. Some interesting results concerning adjacency preserving transformations of symplectic dual polar spaces were established in [7,8].

4.2. Proof of Theorem 3

We will use the following.

Theorem 4. *Let M be an $(n'-n-1)$-dimensional singular subspace of Π'. Every mapping $f:\mathcal{G}_{n-1}(\Pi)\to[M\rangle_{n'-1}$ sending apartments of $\mathcal{G}_{n-1}(\Pi)$ to apartments of $[M\rangle_{n'-1}$ is induced by a frames preserving mapping of Π to $[M\rangle_{n'-n}$.*

Proof. This follows from Theorem 4.17 in [11]. □

Theorem 3 will be a consequence of Theorems 2, 4 and the following lemma.

Lemma 6. *Let $f:\mathcal{G}_{n-1}(\Pi)\to\mathcal{G}_{n'-1}(\Pi')$ be an isometric embedding of $\Gamma_{n-1}(\Pi)$ in $\Gamma_{n'-1}(\Pi')$. There exists an $(n'-n-1)$-dimensional singular subspace M of Π' such that the image of f is contained in $[M\rangle_{n'-1}$.*

Proof. Let X_0 and Y_0 be opposite vertices of $\Gamma_{n-1}(\Pi)$. Then

$$d\big(f(X_0),f(Y_0)\big)=n$$

and, by the distance formula given in Subsection 2.5,

$$M:=f(X_0)\cap f(Y_0)$$

is an $(n'-n-1)$-dimensional singular subspace of Π'. Let $X\in\mathcal{G}_{n-1}(\Pi)$ and let

$$X_0,X_1,\dots,X_m=X$$

be a path in $\Gamma_{n-1}(\Pi)$ connecting X_0 with X. We show that every $f(X_i)$ belongs to $[M\rangle_{n'-1}$.

It is clear that X_1 is opposite to Y_0 or $d(X_1,Y_0)=n-1$. In the second case, we take a geodesic of $\Gamma_{n-1}(\Pi)$ containing X_0,X_1,Y_0. The mapping f transfers it to a geodesic of $\Gamma_{n'-1}(\Pi')$ containing $f(X_0),f(X_1),f(Y_0)$. Lemma 2 guarantees that $f(X_1)\in[M\rangle_{n'-1}$.

In the case when X_1 is opposite to Y_0, we have

$$\dim\big(f(X_1)\cap f(Y_0)\big)=n'-n-1.$$

If the subspace $f(X_1)\cap f(Y_0)$ coincides with M then $M\subset f(X_1)$. If these subspaces are distinct then there exists a point

$$p\in\big(f(X_1)\cap f(Y_0)\big)\setminus M.$$

This point does not belong to $f(X_0)\cap f(X_1)$ (otherwise $p\in f(X_0)\cap f(Y_0)=M$ which is impossible). Thus $f(X_1)$ is spanned by the $(n'-2)$-dimensional singular subspace $f(X_0)\cap f(X_1)$ and the point p. As in the proof of Lemma 2,

$$\big[f(X_0)\cap f(X_1)\big]\perp M\quad\text{and}\quad p\perp M.$$

Hence $M\perp f(X_1)$ and $M\subset f(X_1)$.

So, for every $X\in\mathcal{G}_{n-1}(\Pi)$ adjacent to X_0 we have $f(X)\in[M\rangle_{n'-1}$. The same holds for every $X\in\mathcal{G}_{n-1}(\Pi)$ adjacent to Y_0 (the proof is similar).

Now we establish the existence of $Y_1\in\mathcal{G}_{n-1}(\Pi)$ opposite to X_1 and satisfying $f(Y_1)\in[M\rangle_{n'-1}$.

It was noted above that X_1 is opposite to Y_0 or $d(X_1,Y_0)=n-1$. If X_1 and Y_0 are opposite then $Y_1=Y_0$ is as required. In the case when $d(X_1,Y_0)=n-1$, the intersection of X_1 and Y_0 is a single point. We take any $(n-2)$-dimensional singular subspace $U \subset Y_0$ which does not contain this point. There exists a frame of Π whose subsets span X_1 and U, see Proposition 4.7 in [11]. The associated apartment of $\mathcal{G}_{n-1}(\Pi)$ contains an element Y_1 such that $U \subset Y_1$ and $X_1 \cap Y_1 = \emptyset$. It is clear that Y_0 and Y_1 are adjacent; hence $f(Y_1) \in [M\rangle_{n'-1}$.

Since

$$d\big(f(X_1), f(Y_1)\big) = n,$$

the subspace $f(X_1) \cap f(Y_1)$ is $(n'-n-1)$-dimensional. On the other hand, $f(X_1)$ and $f(Y_1)$ both belong to $[M\rangle_{n'-1}$ and we get

$$f(X_1) \cap f(Y_1) = M.$$

We apply the arguments given above to X_1, Y_1, X_2 instead of X_0, Y_0, X_1 and establish that $f(X_2) \in [M\rangle_{n'-1}$. Step by step, we show that each $f(X_i)$ belongs to $[M\rangle_{n'-1}$. $\square$

Let f be, as above, an isometric embedding of $\Gamma_{n-1}(\Pi)$ in $\Gamma_{n'-1}(\Pi')$. Lemma 6 implies the existence of an $(n'-n-1)$-dimensional singular subspace M of Π' such that the image of f is contained in $[M\rangle_{n'-1}$. By Theorem 2, f transfers apartments of $\mathcal{G}_{n-1}(\Pi)$ to apartments of $[M\rangle_{n'-1}$. Theorem 4 gives the claim.

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