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Connection between the Hadamard and matrix products with an application to matrix-variate Birnbaum–Saunders distributions

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In this paper, we establish a connection between the Hadamard product and the usual matrix multiplication. In addition, we study some new properties of the Hadamard product and explore the inverse problem associated with the established connection, which facilitates diverse applications. Furthermore, we propose a matrix-variate generalized Birnbaum–Saunders (GBS) distribution. Three representations of the matrix-variate GBS density are provided, one of them by using the mentioned connection. The main motivation of this article is based on the fact that the representation of the matrix-variate GBS density based on element-by-element specification does not allow matrix transformations. Consequently, some statistical procedures based on this representation, such as multivariate data analysis and statistical shape theory, cannot be performed. For this reason, the primary goal of this work is to obtain a matrix representation of the matrix-variate GBS density that is useful for some statistical applications. When the GBS density is expressed by means of a matrix representation based on the Hadamard product, such a density is defined in terms of the original matrices, as is common for many matrix-variate distributions, allowing matrix transformations to be handled in a natural way and then suitable statistical procedures to be developed.

1. Introduction

The Hadamard product is a type of matrix multiplication that is commutative and simpler than the usual product; see [23]. The first ingredient of the present work is the Hadamard product. Halmos [20] was the pioneer to give a name to this product due to the early work of the French mathematician Jaques Hadamard (1865–1963); see [19]. The Hadamard multiplication is also known as the entry-wise or Schur product due to an earlier work of the German mathematician Issai Schur (1875–1941). However, the first known work dedicated to this topic was due to Sylvester who in 1867 proposed a recurrent method to construct certain type of Hadamard matrices. For an elaborate historical review about this product, interested readers may refer to Styan [33], Agaian [1, pp. 1–4], Beder [5], and the references therein. The importance and applicability of the Hadamard multiplication are well known. In mathematics, for example, this multiplication is used (i) for constructing discrete equipments by means of integer orthogonal matrices that allow fast transformations, and (ii) for finding the maximum of a determinant. This product has also been used in combinatorial analysis, finite geometry, group theory, number theory, and regular graphs. Applications of the Hadamard product can also be found in other fields, for

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example, in (i) correcting codes in satellite transmissions and cryptography, (ii) communication and information theory, (iii) signal processing and pattern recognition, (iv) neural behavior, and (v) lossy compression algorithms as images in JPEG format. In statistics, some applications of the Hadamard product pertain to (i) interrelations between Hadamard matrices and different combinatorial configurations such as block-designs, Latin square, and orthogonal F-square, (ii) linear models, (iii) maximum likelihood estimation of the variances in a multivariate normal population, (iv) multivariate statistical analysis, and (v) multivariate Tchebycheff equalities. For more details and applications of the Hadamard product, interested readers may refer to Vijayan [35], Hedayat and Wallis [21], Styan [33], Agaian [1], Seberry and Yamada [32], and the references therein. For an excellent review of Hadamard matrices and their statistical applications until 1978, see [21]. More recent references are [23,8,2,4].

The second ingredient of this work corresponds to the matrix-variate symmetric distributions. The normal (Gaussian) model has dominated the landscape of distribution theory and statistical applications for over 100 years, with the univariate case being the most treated, followed by the multivariate case and, to a lesser degree, the matrix-variate case. The remarkable properties of this model have made it the most widely used one in theoretical as well as applied statistics; one may refer to Johnson et al. [24, pp. 80–206], Kotz et al. [26, pp. 105–333], Tulino and Verdú [34], and Anderson et al. [3, pp. 19–20], for more details about the normal distribution. However, the random matrix theory based on elliptic distributions has emerged as an alternative to the Gaussian theory providing families of symmetric distributions with different shapes yielding a greater level of flexibility on the kurtosis, i.e., heavier-and-lighter tails than those of the normal distribution, thus enabling to describe different types of multivariate data. For more details about elliptic distributions, see [14,15,18,12,9].

The third and final ingredient of the present work corresponds to the asymmetric (skewed) distributions. Recently, many skewed distributions (as opposed to symmetric models such as the normal model) have been proposed and discussed in the literature. One such asymmetric distribution, defined on the positive real line, with two parameters (shape and scale) and positive skewness, is the Birnbaum–Saunders (BS) model. This model has received considerable attention in the past four decades due mainly to its attractive properties and its simple relationship with the normal distribution; see [7,25, pp. 651–663]. A univariate generalization of the BS distribution based on elliptic distributions, proposed by Díaz-García and Leiva [13], is known as the generalized Birnbaum–Saunders (GBS) distribution, which includes the BS distribution as a particular case; see also Sanhueza et al. [31]. An extension of the BS distribution to the bivariate case has been developed by Kundu et al. [27], while Díaz-García and Domínguez-Molina [10,11] proposed multivariate versions of the GBS distribution.

Our main motivation here comes from the classical matrix-variate distribution theory; see, e.g., Anderson et al. [3]. In general, a univariate distribution (as the normal model) or a family of univariate distributions (as the elliptic models) can be easily extended to the multivariate and matrix-variate cases by representing the Jacobian and kernel (or generator of the density) of the distribution in terms of the original matrices and of the usual matrix product. However, this is not the case with the univariate BS distribution. Indeed, until now, the multivariate GBS density only has been represented by means of the elements of the random vector (element-by-element representation); see [10,27]. In the matrix-variate GBS case, the presence of the elements of the random matrix and of their reciprocals in the corresponding density does not allow simple matrix operations, such as the representation of the respective kernel of this density in terms of a suitable quadratic form. Nevertheless, this GBS density can be easily represented in terms of the original matrices by using the Hadamard product. This is because the inverse of the GBS random matrix with respect to this product contains precisely those reciprocals. Thus, the Hadamard product arises naturally in the matrix-variate GBS density instead of forcing the use of the usual matrix product. Indeed, by using the Hadamard representation, there are many advantages in computations and algebraic manipulations such as (i) the products are entry-wise, (ii) the multiplication is commutative, (iii) the inverse is very easy to obtain, and (iv) the computation of power matrices is straightforward. This is not the first time that a matrix-variate density is expressed in terms of unusual products instead of the usual matrix product; see, e.g., Bentler and Lee [6]. The matrix representation of the matrix-variate GBS density based on the Hadamard product can be useful in several important statistical procedures, such as multivariate data analytic methods and statistical shape theory, where matrix transformations are required to facilitate affine, orthogonal matrix triangularization (QR), polar coordinate, and singular value (SV) decompositions. These are some of the reasons as to why the primary goal of the present paper is to obtain a matrix representation of the matrix-variate GBS density. Such a representation will specifically allow us to achieve the following. First, to obtain the associated generalized Wishart distribution, which would enable us to develop statistical procedures based on it, perform a QR decomposition and integrate over the Stiefel manifold, whose integration is feasible due to the fact that zonal polynomials arise naturally in this density; see [30]. Second, to obtain the affine configuration of the shape theory by transformations, since by the use of zonal polynomial theory, the integration over positive definite spaces and Stiefel manifold yields the required affine shape; see [9]. Third, to obtain Euclidean transformations useful for shape theory by QR shape coordinates constructed in several steps, requiring integration over the Stiefel manifold by using once again zonal polynomial integrals; see [17]. Fourth and finally, to obtain logarithmic matrix transformations that would be useful in fixed and mixed effect log-linear models; see [28], for the univariate case. Clearly, all these applications cannot be achieved if we leave the matrix-variate GBS density in terms of the elements of the corresponding random matrix.

The above mentioned motivation and goals pose several challenges since some problems relating to Hadamard matrices are still unanswered. For example, to the best of our knowledge, there does not exist an explicit formula that connects the Hadamard and matrix products, excepting a relationship with the Kronecker product that requires the introduction of some permutation matrices; see [22]. Through this connection, some new properties and applications of the Hadamard product can also be explored. Therefore, the aims of this study are as follows: (i) to provide a connection between the

Hadamard and matrix products, (ii) to study new properties and applications of the Hadamard product that could be useful for different problems concerning matrix-variate distributions, (iii) to propose a matrix-variate GBS distribution, and (iv) to use the established connection between the Hadamard and usual products to obtain a matrix representation of the matrix-variate GBS density based on the Hadamard product that would be useful for many multivariate statistical applications.

The rest of this paper is organized as follows. In Section 2, we provide the required background for the Hadamard product and the univariate and multivariate GBS distributions. In Section 3, we establish a connection between the Hadamard and matrix products. In Section 4, we study some new properties of the Hadamard product and explore the inverse problem associated with the established connection, which leads to some further applications in multivariate analysis. In Section 5, we introduce a matrix-variate GBS distribution and propose three representations for its density, one of them based on the established connection between the Hadamard and matrix products. Finally, in Section 6, we make some concluding remarks, point out some open problems, and also suggest some possible future work.

2. Background

In this section, we describe the basic notion of Hadamard products and then present univariate and multivariate versions of GBS distributions.

2.1. Hadamard product

Let $\boldsymbol{A} = (a_{ij})$ and $\boldsymbol{B} = (b_{ij})$ be $m \times n$ and $n \times p$ matrices (not necessarily square), respectively. Then, the usual matrix product between these two matrices, denoted by $\boldsymbol{A} \cdot \boldsymbol{B}$, is an $m \times p$ matrix given by

$$\boldsymbol{A} \cdot \boldsymbol{B} = \left(\sum_{k=1}^{m} a_{ik}\, b_{kj} \right),$$

where $a_{ik}\, b_{kj}$ denotes the usual scalar product between the elements a_{ij} and b_{ij} of the corresponding matrices.

Let $\boldsymbol{A} = (a_{ij})$ and $\boldsymbol{B} = (b_{ij})$ be now two $m \times n$ matrices, i.e., of the same dimension but not necessarily square. Then, the Hadamard product between these two matrices, denoted by $\boldsymbol{A} \odot \boldsymbol{B}$, is an $m \times n$ matrix given by

$$\boldsymbol{A} \odot \boldsymbol{B} = (a_{ij}\, b_{ij}). \tag{1}$$

Now, let $\boldsymbol{A} = (a_{ij})$ and $\boldsymbol{B} = (b_{ij})$ be $m \times n$ and $p \times q$ matrices (not necessarily square), respectively. Then, the Kronecker product between these two matrices, denoted by $\boldsymbol{A} \otimes \boldsymbol{B}$, is an $m \times n$-by-$p \times q$ block matrix given by

$$\boldsymbol{A} \otimes \boldsymbol{B} = \begin{pmatrix} a_{11}\boldsymbol{B} & \cdots & a_{1n}\boldsymbol{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\boldsymbol{B} & \cdots & a_{mn}\boldsymbol{B} \end{pmatrix}.$$

Notice that the Hadamard product is a submatrix of the Kronecker product, but the latter is not commutative.

Let us introduce the following notation that is used through out this paper. Let $\boldsymbol{X} = (X_{ij})$ be an $n \times k$ matrix. As mentioned earlier, the real powers of a matrix with respect to the Hadamard product are simpler than the powers with respect to the usual matrix product. These powers are denoted here by

$$\boldsymbol{X}^{aH} = (X_{ij}^{a}), \quad a \in \mathbb{R}. \tag{2}$$

Thus, we have the following particular cases of interest:

(i) $\boldsymbol{X}^{\frac{1}{2}H} = (X_{ij}^{1/2})$ denotes the positive square root of $\boldsymbol{X}$ with respect to the Hadamard product, such that $\boldsymbol{X}^{\frac{1}{2}H} \odot \boldsymbol{X}^{\frac{1}{2}H} = \boldsymbol{X}$; and
(ii) $\boldsymbol{X}^{-H} = (1/X_{ij})$ denotes the inverse matrix of $\boldsymbol{X}$ with respect to the Hadamard product (that we call Hadamard inverse), such that $\boldsymbol{X} \odot \boldsymbol{X}^{-H} = \boldsymbol{J}$, where $\boldsymbol{J}$ is an $n \times k$ matrix consisting of ones.

2.2. A univariate GBS distribution

A random variable T with univariate BS distribution has shape and scale parameters to be $\alpha > 0$ and $\beta > 0$, respectively. In addition, β is the median of the distribution. We use the notation $T \sim \mathrm{BS}(\alpha, \beta)$ in this case. Random variables T and Z with BS and standard normal distributions, respectively, satisfy the relationships

$$T = \beta \left[\frac{\alpha Z}{2} + \sqrt{\left\{ \frac{\alpha Z}{2} \right\}^2 + 1} \right]^2 \sim \mathrm{BS}(\alpha, \beta) \tag{3}$$

and

$$Z = \frac{1}{\alpha}\left[\sqrt{\frac{T}{\beta}} - \sqrt{\frac{\beta}{T}}\right] = a(T) \sim \mathrm{N}(0, 1). \tag{4}$$

Thus, a BS random variable T is simply a transformation of the standard normal random variable Z. If the normality assumption in (4) is relaxed by allowing Z to follow any standard symmetric distribution in $\mathbb{R}$, then we obtain the class of univariate GBS distributions. In this case, we use the notation $T \sim \mathrm{GBS}(\alpha, \beta; g)$. The density of T is $f_T(t) = c\, g\big(a(t)^2\big)\, \mathrm{d}a(t)/\mathrm{d}t$, for $t > 0$, where g is the kernel of Z given by $f_Z(z) = c\, g(z^2)$, with $z \in \mathbb{R}$, and c is the normalizing constant such that $\int_{-\infty}^{+\infty} g(z^2)\, \mathrm{d}z = 1/c$. In addition, the derivative of $a(t)$, given in (4), with respect to t is $\mathrm{d}a(t)/\mathrm{d}t = t^{-3/2}\,[t + \beta]/[2\,\alpha\,\beta^{1/2}]$.

2.3. A multivariate GBS distribution

The class of univariate GBS distributions can be extended to the multivariate (vector) case by using the family of elliptic distributions.

Let $\boldsymbol{x}$ be an $n \times 1$ random vector with elliptic distribution characterized by a location vector $\boldsymbol{\mu} \in \mathbb{R}^n$, a scale matrix $\boldsymbol{\Sigma} \in \mathbb{R}^{n\times n}$, with $\mathrm{rank}(\boldsymbol{\Sigma}) = n$, and the corresponding kernel g. By denoting $\boldsymbol{x} \sim \mathrm{EC}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}; g)$, we have the density of $\boldsymbol{x}$ as

$$f_{\boldsymbol{x}}(\boldsymbol{x}) = c\,|\boldsymbol{\Sigma}|^{-\frac{1}{2}}\, g\left([\boldsymbol{x} - \boldsymbol{\mu}]^\top \cdot \boldsymbol{\Sigma}^{-1} \cdot [\boldsymbol{x} - \boldsymbol{\mu}]\right), \quad \boldsymbol{x} \in \mathbb{R}^n, \tag{5}$$

where once again c is the normalizing constant. Let $\boldsymbol{z} = (Z_1, \ldots, Z_n)^\top \sim \mathrm{EC}_n(\boldsymbol{0}, \boldsymbol{I}_n; g)$ and $\boldsymbol{t} = (T_1, \ldots, T_n)^\top$, where

$$T_i = \beta_i \left[\frac{\alpha_i Z_i}{2} + \sqrt{\left\{\frac{\alpha_i Z_i}{2}\right\}^2 + 1}\right]^2, \quad \alpha_i > 0, \beta_i > 0, i = 1, \ldots, n.$$

Then, the random vector $\boldsymbol{t}$ generates the multivariate GBS distribution, denoted by $\boldsymbol{t} \sim \mathrm{GBS}_n(\alpha, \beta; g)$, where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)^\top$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)^\top$.

3. Connection between Hadamard and matrix products

In this section, we point out a direct connection between Hadamard and matrix multiplications, without the introduction of permutation matrices or the over-dimension due to the Kronecker product.

Let $\boldsymbol{A} = (a_{ij})$ be a $k \times k$ matrix, and consider the set $P_{12\cdots k}$ of k cyclic permutations of $12\cdots k$ given by

$$P_{12\cdots k} = \{123\cdots(k-1)k, 23\cdots(k-1)k1, \ldots, k123\cdots(k-2)(k-1)\}. \tag{6}$$

If $\boldsymbol{a}_i$ is the ith column of the $k \times k$ matrix $\boldsymbol{A}$, then for a particular element $p = p_1p_2\cdots p_k$ of $P_{12\cdots k}$ in (6), let us define

$$\boldsymbol{A}_{(p)} = (\boldsymbol{a}_{p_1}|\boldsymbol{a}_{p_2}|\cdots|\boldsymbol{a}_{p_k}), \tag{7}$$

i.e., $\boldsymbol{A}_{(p)}$ is the matrix $\boldsymbol{A}$ with permuted columns according to the permutation $p = p_1p_2\cdots p_k$. For example, for the permutation $34\cdots k12$, we have

$$\boldsymbol{A}_{(34\cdots k12)} = \begin{pmatrix} a_{13} & a_{14} & \cdots & a_{1k} & a_{11} & a_{12} \\ a_{23} & a_{24} & \cdots & a_{2k} & a_{21} & a_{22} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{(k-1)3} & a_{(k-1)4} & \cdots & a_{(k-1)k} & a_{(k-1)1} & a_{(k-1)2} \\ a_{k3} & a_{k4} & \cdots & a_{kk} & a_{k1} & a_{k2} \end{pmatrix}.$$

A particular case of interest is obtained when $\boldsymbol{A} = \boldsymbol{I}_k$, in which case $\boldsymbol{I}_{(p)}$ is the column-permutation of the identity matrix under the permutation $p \in P_{12\cdots k}$. Then, it is easy to see that $\boldsymbol{A}_{(p)} = \boldsymbol{A} \cdot \boldsymbol{I}_{(p)}$, for all p. Also, for a $k \times k$ matrix $\boldsymbol{B}$, let us define

$$\boldsymbol{B}_{[p]} = \begin{pmatrix} b_{p_1 1} & b_{p_2 2} & \cdots & b_{p_k k} \\ b_{p_1 1} & b_{p_2 2} & \cdots & b_{p_k k} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p_1 1} & b_{p_2 2} & \cdots & b_{p_k k} \end{pmatrix}. \tag{8}$$

With these notation, we have the following theorem.

Theorem 1. *Let $\boldsymbol{A} = (a_{ij})$ and $\boldsymbol{B} = (b_{ij})$ be $k \times k$ matrices. Then,*

$$\boldsymbol{A} \cdot \boldsymbol{B} = \sum_p \boldsymbol{A}_{(p)} \odot \boldsymbol{B}_{[p]} = \sum_p (\boldsymbol{A} \cdot \boldsymbol{I}_{(p)}) \odot \boldsymbol{B}_{[p]},$$

where $\boldsymbol{A}_{(p)}$ and $\boldsymbol{B}_{[p]}$ are as defined in (7) and (8), respectively, for a particular permutation $p \in P_{12\cdots k}$.

Proof. When $\boldsymbol{A} = (a_{ij})$ and $\boldsymbol{B} = (b_{ij})$ are $k \times k$ matrices, we find

$$\boldsymbol{A} \cdot \boldsymbol{B} = \begin{pmatrix} \sum_{j=1}^{k} a_{1j}\, b_{j1} & \cdots & \sum_{j=1}^{k} a_{1j}\, b_{jk} \\ \sum_{j=1}^{k} a_{2j}\, b_{j1} & \cdots & \sum_{j=1}^{k} a_{2j}\, b_{jk} \\ \vdots & \ddots & \vdots \\ \sum_{j=1}^{k} a_{kj}\, b_{j1} & \cdots & \sum_{j=1}^{k} a_{kj}\, b_{jk} \end{pmatrix} = \left(\sum_{j=1}^{k} a_{ij}\, b_{j1} \left| \sum_{j=1}^{k} a_{ij}\, b_{j2} \right| \cdots \left| \sum_{j=1}^{k} a_{ij}\, b_{jk} \right. \right).$$

Clearly, the usual product of matrices $\boldsymbol{A} \cdot \boldsymbol{B}$ decomposes uniquely as the sum of k matrices $\boldsymbol{C}_p$, where, as before, $p = p_1 p_2 \cdots p_k \in P_{12\cdots k}$. Such a decomposition can be constructed as follows. The first matrix $\boldsymbol{C}_{12\cdots k}$ is obtained by taking the first summand of the first column, the second summand of the second column, and so on, i.e.,

$$\boldsymbol{C}_{12\cdots k} = (a_{i1}\, b_{11} | a_{i2}\, b_{22} | \cdots | a_{ik}\, b_{kk}) = \boldsymbol{A}_{(12\cdots k)} \odot \boldsymbol{B}_{[12\cdots k]}.$$

In other words, $\boldsymbol{C}_{12\cdots k}$ is constructed by extracting the summands of each column according to the permutation $12 \cdots k$. The second matrix is selected according to the permutation $23 \cdots k1$, i.e., by selecting the second summand of the first column, the third summand of the second column, and so on until the first summand of the last column. The resulting matrix is given by

$$\boldsymbol{C}_{23\cdots k1} = \left(a_{i2}\, b_{21} | a_{i3}\, b_{32} | \cdots | a_{ik}\, b_{k(k-1)} | a_{i1}\, b_{1k} \right) = \boldsymbol{A}_{(23\cdots k1)} \odot \boldsymbol{B}_{[23\cdots k1]}.$$

Following this procedure and taking the matrices according to the complete set of k cyclic permutations of $12 \cdots k$ in (6), we obtain the $(k-1)$th matrix as

$$\boldsymbol{C}_{(k-1)k1\cdots(k-3)(k-2)} = \boldsymbol{A}_{((k-1)k1\cdots(k-3)(k-2))} \odot \boldsymbol{B}_{[(k-1)k1\cdots(k-3)(k-2)]}.$$

Finally, the matrix corresponding to the ultimate permutation $k1 \cdots (k-2)(k-1)$ is formed by the remaining summands as

$$\begin{aligned} \boldsymbol{C}_{k1\cdots(k-2)(k-1)} &= \left(a_{ik} b_{k1} | a_{i1} b_{12} | \cdots | a_{i(k-2)} b_{(k-2)(k-1)} | a_{i(k-1)} b_{(k-1)k} \right) \\ &= \boldsymbol{A}_{(k1\cdots(k-2)(k-1))} \odot \boldsymbol{B}_{[k1\cdots(k-2)(k-1)]}. \end{aligned}$$

Thus, we have

$$\begin{aligned} \boldsymbol{A} \cdot \boldsymbol{B} &= \boldsymbol{A}_{(12\cdots k)} \odot \boldsymbol{B}_{[12\cdots k]} + \boldsymbol{A}_{(23\cdots k1)} \odot \boldsymbol{B}_{[23\cdots k1]} + \boldsymbol{A}_{(34\cdots k12)} \odot \boldsymbol{B}_{[34\cdots k12]} + \cdots \\ &\quad + \boldsymbol{A}_{((k-1)k1\cdots(k-3)(k-2))} \odot \boldsymbol{B}_{[(k-1)k1\cdots(k-3)(k-2)]} + \boldsymbol{A}_{(k1\cdots(k-2)(k-1))} \odot \boldsymbol{B}_{[k1\cdots(k-2)(k-1)]}, \end{aligned}$$

which is the required result. □

Let us now consider some examples to illustrate the result established in Theorem 1. When $k = 1$, the equivalence is trivial. Now, when $k = 2$, we have

$$\begin{aligned} \boldsymbol{A} \cdot \boldsymbol{B} &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}\, b_{11} + a_{12}\, b_{21} & a_{11}\, b_{12} + a_{12}\, b_{22} \\ a_{21}\, b_{11} + a_{22}\, b_{21} & a_{21}\, b_{12} + a_{22}\, b_{22} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \odot \begin{pmatrix} b_{11} & b_{22} \\ b_{11} & b_{22} \end{pmatrix} + \begin{pmatrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{pmatrix} \odot \begin{pmatrix} b_{21} & b_{12} \\ b_{21} & b_{12} \end{pmatrix} \\ &= \boldsymbol{A}_{(12)} \odot \boldsymbol{B}_{[12]} + \boldsymbol{A}_{(21)} \odot \boldsymbol{B}_{[21]}. \end{aligned}$$

Then, when $k = 3$, we have

$$\begin{aligned} \boldsymbol{A} \cdot \boldsymbol{B} &= \begin{pmatrix} a_{11}\, b_{11} + a_{12}\, b_{21} + a_{13}\, b_{31} & a_{11}\, b_{12} + a_{12}\, b_{22} + a_{13}\, b_{32} & a_{11}\, b_{13} + a_{12}\, b_{23} + a_{13}\, b_{33} \\ a_{21}\, b_{11} + a_{22}\, b_{21} + a_{23}\, b_{31} & a_{21}\, b_{12} + a_{22}\, b_{22} + a_{23}\, b_{32} & a_{21}\, b_{13} + a_{22}\, b_{23} + a_{23}\, b_{33} \\ a_{31}\, b_{11} + a_{32}\, b_{21} + a_{33}\, b_{31} & a_{31}\, b_{12} + a_{32}\, b_{22} + a_{33}\, b_{32} & a_{31}\, b_{13} + a_{32}\, b_{23} + a_{33}\, b_{33} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \odot \begin{pmatrix} b_{11} & b_{22} & b_{33} \\ b_{11} & b_{22} & b_{33} \\ b_{11} & b_{22} & b_{33} \end{pmatrix} + \begin{pmatrix} a_{12} & a_{13} & a_{11} \\ a_{22} & a_{23} & a_{21} \\ a_{32} & a_{33} & a_{31} \end{pmatrix} \odot \begin{pmatrix} b_{21} & b_{32} & b_{13} \\ b_{21} & b_{32} & b_{13} \\ b_{21} & b_{32} & b_{13} \end{pmatrix} \end{aligned}$$

$$+\begin{pmatrix} a_{13} & a_{11} & a_{12} \\ a_{23} & a_{21} & a_{22} \\ a_{33} & a_{31} & a_{32} \end{pmatrix} \odot \begin{pmatrix} b_{31} & b_{12} & b_{23} \\ b_{31} & b_{12} & b_{23} \\ b_{31} & b_{12} & b_{23} \end{pmatrix}$$
$$= \boldsymbol{A}_{(123)} \odot \boldsymbol{B}_{[123]} + \boldsymbol{A}_{(231)} \odot \boldsymbol{B}_{[231]} + \boldsymbol{A}_{(312)} \odot \boldsymbol{B}_{[312]}.$$

A similar expansion of $\boldsymbol{A} \cdot \boldsymbol{B}$ when the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are not square is also of interest. In the case of non-singular matrix-variate distributions (e.g., for Wishart distributions), we just need to study the case when $n \le k$, for an $n \times k$ matrix $\boldsymbol{A}$ and a $k \times n$ matrix $\boldsymbol{B}$. This result follows from Theorem 1. Specifically, let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $n \times k$ and $k \times n$ matrices, respectively, and define the set of cyclic permutations of $12 \cdots n \cdots k$ according to (6), with $n \le k$. Given that the number of rows of $\boldsymbol{A}$ is less than (or equal to) its number of columns, then the permutations $p \in P_{12\cdots n \cdots k}$ involved in the proof of Theorem 1 just have the first n indices, i.e., $\boldsymbol{A} \cdot \boldsymbol{B} = \sum_p \boldsymbol{A}_{(p)} \odot \boldsymbol{B}_{[p]}$, where the summation runs over the k permutations of $P_{12\cdots n \cdots k}$ for a particular p, each one consisting of the first n indices. In other words, the $n \times n$ matrix $\boldsymbol{A}_{(p)}$ consists of the first n columns of $\boldsymbol{A}$ according to the permutation p (with only the first n indices). The $n \times n$ matrix $\boldsymbol{B}_{[p]}$ is as defined in (8), but with the restriction that $n \le k$, i.e.,

$$\boldsymbol{B}_{[p]} = \begin{pmatrix} b_{p_1 1} & b_{p_2 2} & \cdots & b_{p_n n} \\ b_{p_1 1} & b_{p_2 2} & \cdots & b_{p_n n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p_1 1} & b_{p_2 2} & \cdots & b_{p_n n} \end{pmatrix}. \tag{9}$$

Finally, notice again that $\boldsymbol{A}_{(p)}$ can be simplified and expressed in terms of the original matrix $\boldsymbol{A}$ and of $\boldsymbol{I}_{(p)}$. In this case, it is easy to see that $\boldsymbol{I}_{(p)}$ is the $k \times n$ matrix constituted by the first n columns of the $k \times k$ identity matrix ($n \le k$) permuted according to the corresponding permutation $p = p_1 p_2 \cdots p_n \in P_{12\cdots n}$. Thus, once again $\boldsymbol{A}_{(p)} = \boldsymbol{A} \cdot \boldsymbol{I}_{(p)}$. This immediately results in the following corollary.

Corollary 1. *Let $\boldsymbol{A} = (a_{ij})$ and $\boldsymbol{B} = (b_{ij})$ be $n \times k$ and $k \times n$ matrices, respectively, with $n \le k$. Then,*

$$\boldsymbol{A} \cdot \boldsymbol{B} = \sum_p \boldsymbol{A}_{(p)} \odot \boldsymbol{B}_{[p]} = \sum_p (\boldsymbol{A} \cdot \boldsymbol{I}_{(p)}) \odot \boldsymbol{B}_{[p]},$$

where the summation runs over all cyclic permutations $p = p_1 p_2 \cdots p_n$ (consisting of the first n indices) of $P_{12\cdots n \cdots k}$. For a particular $p = p_1 p_2 \cdots p_n \in P_{12\cdots n \cdots k}$, $\boldsymbol{A}_{(p)}$ is as given in (7) with k replaced by n and $\boldsymbol{B}_{[p]}$ is as in (8).

As a simple example, let us take

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}.$$

Here, $P_{123} = \{123, 231, 312\}$ as usual, but the permutations that we consider have only the first two parts, viz., $p = 12$, 23, or 31. Then,

$$\begin{aligned} \boldsymbol{A} \cdot \boldsymbol{B} &= \begin{pmatrix} a_{11} b_{11} + a_{12} b_{21} + a_{13} b_{31} & a_{11} b_{12} + a_{12} b_{22} + a_{13} b_{32} \\ a_{21} b_{11} + a_{22} b_{21} + a_{23} b_{31} & a_{21} b_{12} + a_{22} b_{22} + a_{23} b_{32} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \odot \begin{pmatrix} b_{11} & b_{22} \\ b_{11} & b_{22} \end{pmatrix} + \begin{pmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{pmatrix} \odot \begin{pmatrix} b_{21} & b_{32} \\ b_{21} & b_{32} \end{pmatrix} + \begin{pmatrix} a_{13} & a_{11} \\ a_{23} & a_{21} \end{pmatrix} \odot \begin{pmatrix} b_{31} & b_{12} \\ b_{31} & b_{12} \end{pmatrix} \\ &= \boldsymbol{A}_{(12)} \odot \boldsymbol{B}_{[12]} + \boldsymbol{A}_{(23)} \odot \boldsymbol{B}_{[23]} + \boldsymbol{A}_{(31)} \odot \boldsymbol{B}_{[31]} \\ &= (\boldsymbol{A} \cdot \boldsymbol{I}_{(12)}) \odot \boldsymbol{B}_{[12]} + (\boldsymbol{A} \cdot \boldsymbol{I}_{(23)}) \odot \boldsymbol{B}_{[23]} + (\boldsymbol{A} \cdot \boldsymbol{I}_{(31)}) \odot \boldsymbol{B}_{[31]}, \end{aligned}$$

where

$$\boldsymbol{I}_{(12)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \boldsymbol{I}_{(23)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{I}_{(31)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix},$$

corresponding to the first two columns of $\boldsymbol{I}_{(123)}, \boldsymbol{I}_{(231)}$ and $\boldsymbol{I}_{(312)}$, respectively.

The case $n > k$ is not of interest for the main application of this paper, i.e., for the representation of the GBS density. However, this case can also be handled by exploiting the cyclic permutations detailed above. But, in such a case, we need to consider permutations with augmented indices instead of permutations with less indices.

4. Hadamard inverse and related problems

In this section, we focus our attention on the computation of the Hadamard inverse and on some related problems. For this purpose, we need an expression for the Hadamard inverse matrix in terms of the original matrix and of the usual product. Then, for example, a QR decomposition of the original matrix can be obtained. However, a general formula for this issue is not available in the literature, since, even for the 2×2 case, the required expression is quite difficult to obtain as is evident from the following theorem which provides the inverse of a 2×2 matrix with respect to the Hadamard product.

4.1. A 2×2 Hadamard inverse matrix

Theorem 2. *Let $\boldsymbol{Z} = (Z_{ij})$ be a 2×2 matrix and $\boldsymbol{N} = \boldsymbol{I}_{(21)}$. Then, the Hadamard inverse of $\boldsymbol{Z}$ is given by*

$$\boldsymbol{Z}^{-H} = 2\left(\boldsymbol{Z}^{\top} - \frac{1}{2}\mathrm{tr}(\boldsymbol{N}\cdot\boldsymbol{Z}\cdot\boldsymbol{N}\cdot\boldsymbol{Z}^{\top})\,\boldsymbol{N}\cdot\boldsymbol{Z}^{-1}\cdot\boldsymbol{N}\right)^{-1}.$$

Proof. Let $\boldsymbol{Z} = (Z_{ij})$ be a 2×2 matrix, $\boldsymbol{N} = \boldsymbol{I}_{(21)}$, and $\boldsymbol{Z}_I = \boldsymbol{Z}^{-H} = (1/Z_{ij})$ denote the Hadamard inverse of $\boldsymbol{Z}$. Consider the QR decompositions $\boldsymbol{Z} = \boldsymbol{Q}^{\top}\cdot\boldsymbol{R}$ and $\boldsymbol{Z}_I = \boldsymbol{Q}_I^{\top}\cdot\boldsymbol{R}_I$, where

$$\boldsymbol{Q} = \frac{1}{\sqrt{Z_{11}^2+Z_{21}^2}}\begin{pmatrix} Z_{11} & Z_{21} \\ Z_{21} & -Z_{11}\end{pmatrix}, \qquad \boldsymbol{R} = \frac{1}{\sqrt{Z_{11}^2+Z_{21}^2}}\begin{pmatrix} Z_{11}^2+Z_{21}^2 & Z_{11}Z_{12}+Z_{21}Z_{22} \\ 0 & Z_{12}Z_{21}-Z_{11}Z_{22}\end{pmatrix},$$

$$\boldsymbol{Q}_I = \begin{pmatrix} \dfrac{1/Z_{11}}{\sqrt{1/Z_{11}^2+1/Z_{21}^2}} & \dfrac{1/Z_{21}}{\sqrt{1/Z_{11}^2+1/Z_{21}^2}} \\ -\dfrac{Z_{11}}{\sqrt{Z_{11}^2+1/Z_{21}^2}} & \dfrac{Z_{21}}{\sqrt{Z_{11}^2+1/Z_{21}^2}}\end{pmatrix} = \boldsymbol{Q}\cdot\boldsymbol{N}, \quad \text{and}$$

$$\boldsymbol{R}_I = \begin{pmatrix} \sqrt{\dfrac{1}{Z_{11}^2}+\dfrac{1}{Z_{21}^2}} & \dfrac{Z_{11}Z_{12}+Z_{21}Z_{22}}{Z_{11}Z_{12}Z_{21}Z_{22}\sqrt{1/Z_{11}^2+1/Z_{21}^2}} \\ 0 & \dfrac{Z_{12}Z_{21}-Z_{11}Z_{22}}{Z_{12}Z_{22}\sqrt{Z_{11}^2+Z_{21}^2}}\end{pmatrix}$$

are the corresponding orthogonal and triangular matrices. Thus, in terms of the permutation 21 of 12, we have $\boldsymbol{Q}_I = \boldsymbol{Q}_{(21)}$. Now, using this fact and the QR decompositions of $\boldsymbol{Z}$ and $\boldsymbol{Z}_I$, it is easy to show that $\boldsymbol{Z}_I = \boldsymbol{N}\cdot\boldsymbol{Z}\cdot\boldsymbol{R}^{-1}\cdot\boldsymbol{R}_I$. Notice that $(\boldsymbol{R}^{-1}\cdot\boldsymbol{R}_I)^{-1} = \boldsymbol{Z}_I^{-1}\cdot\boldsymbol{N}\cdot\boldsymbol{Z}$ and, similarly, we obtain $(\boldsymbol{Z}_I^{\top})^{-1}\cdot\boldsymbol{N}\cdot\boldsymbol{Z}^{\top}$. However, we have $\boldsymbol{Z}^{\top}\cdot\boldsymbol{N}\cdot\boldsymbol{Z}-\boldsymbol{Z}\cdot\boldsymbol{N}\cdot\boldsymbol{Z}^{\top} = 2\,\boldsymbol{Z}_I^{-1}\cdot\boldsymbol{N}\cdot\boldsymbol{Z}-2\,(\boldsymbol{Z}_I^{\top})^{-1}\cdot\boldsymbol{N}\cdot\boldsymbol{Z}^{\top}$, which simplifies to the interesting relationship $(\boldsymbol{Z}^{\top}-2\,\boldsymbol{Z}_I^{-1})\cdot\boldsymbol{N}\cdot\boldsymbol{Z} = (\boldsymbol{Z}^{\top}-2\,\boldsymbol{Z}_I^{-1})^{\top}\cdot\boldsymbol{N}\cdot\boldsymbol{Z}^{\top}$. This allows us to obtain

$$\boldsymbol{R}\cdot(\boldsymbol{Z}^{\top}-2\,\boldsymbol{Z}_I^{-1})^{-1}\cdot\boldsymbol{R} = \frac{1}{\langle\boldsymbol{Z}\rangle}\,\boldsymbol{Z},$$

where $\langle\boldsymbol{Z}\rangle = Z_{11}Z_{22}+Z_{21}Z_{12}$ is the permanent of $\boldsymbol{Z}$, i.e., the non-signed determinant of $|\boldsymbol{Z}|$. Thus, given that

$$\langle\boldsymbol{Z}\rangle = \frac{1}{2}\,\mathrm{tr}(\boldsymbol{N}\cdot\boldsymbol{Z}\cdot\boldsymbol{N}\cdot\boldsymbol{Z}^{\top}), \tag{10}$$

the theorem is established. □

4.2. A $k \times k$ Hadamard inverse matrix

Based on Theorem 2, we can obtain the Hadamard inverse of any $k \times k$ matrix as presented in the following theorem.

Theorem 3. *Let $\boldsymbol{Z} = (Z_{ij})$ be a $k \times k$ matrix such that $Z_{ij} \neq 0$, for all $i, j = 1, \ldots, k$. Hence,*

(i) *If $k = 2n$ (even), $\boldsymbol{N}$ is as given in Theorem 2, and $\boldsymbol{Z} = (Z_{ij})$ is partitioned in n^2 2×2 blocks $\boldsymbol{Z}_{rs}$, for $r, s = 1, \ldots, n$, where*

$$\boldsymbol{Z}_{rs} = \begin{pmatrix} Z_{(2r-1)(2s-1)} & Z_{(2r-1)2s} \\ Z_{2r(2s-1)} & Z_{2r\,2s}\end{pmatrix},$$

then the Hadamard inverse of $\boldsymbol{Z}$, in terms of the matrix product, is given by $\boldsymbol{Z}^{-H} = (\boldsymbol{Z}_{rs}^{-H})$, with $\boldsymbol{Z}_{rs}^{-H} = 2\,(\boldsymbol{Z}_{rs}^{\top} - \frac{1}{2}\,\mathrm{tr}(\boldsymbol{N}\cdot\boldsymbol{Z}_{rs}\cdot\boldsymbol{N}\cdot\boldsymbol{Z}_{rs}^{\top})\,\boldsymbol{N}\cdot\boldsymbol{Z}_{rs}^{-1}\cdot\boldsymbol{N})^{-1}$, for $r, s = 1, \ldots, n$; and

(ii) *If $k = 2n+1$ (odd) and $\boldsymbol{Z} = (Z_{ij})$ is partitioned as*

$$\boldsymbol{Z} = \begin{pmatrix} \boldsymbol{Z}_{(k-1)(k-1)} & \boldsymbol{Z}_{(k-1)\,1} \\ \boldsymbol{Z}_{1\,(k-1)} & Z_{kk}\end{pmatrix},$$

then the Hadamard inverse of $\boldsymbol{Z}$ is given by

$$\boldsymbol{Z}^{-H} = \begin{pmatrix} \boldsymbol{Z}_{(k-1)\,(k-1)}^{-H} & \boldsymbol{Z}_{(k-1)\,1}^{-H} \\ \boldsymbol{Z}_{1\,(k-1)}^{-H} & \dfrac{1}{Z_{kk}}\end{pmatrix},$$

where, for $r, s = 1, \ldots, (k-1)/2$,

$$\boldsymbol{Z}^{-H}_{(k-1)(k-1)} = \boldsymbol{Z}^{-H}_{rs} = 2\left(\boldsymbol{Z}^{\top}_{rs} - \frac{1}{2}\operatorname{tr}(\boldsymbol{N}\cdot\boldsymbol{Z}_{rs}\cdot\boldsymbol{N}\cdot\boldsymbol{Z}^{\top}_{rs})\,\boldsymbol{N}\cdot\boldsymbol{Z}^{-1}_{rs}\cdot\boldsymbol{N}\right)^{-1},$$

$$\boldsymbol{Z}^{-H}_{(k-1)1} = \left(\frac{1}{Z_{1k}}\cdots\frac{1}{Z_{(k-1)k}}\right)^{\top} \quad \textit{and} \quad \boldsymbol{Z}^{-H}_{1(k-1)} = \left(\frac{1}{Z_{k1}}\cdots\frac{1}{Z_{k(k-1)}}\right).$$

Proof. (i) Let k be an even number. Then, $\boldsymbol{Z}\odot\boldsymbol{Z}^{-H} = \boldsymbol{J}$, where $\boldsymbol{J}$ is a $k\times k$ matrix consisting of ones. Partitioning $\boldsymbol{Z}$ into 2×2 blocks and then using Theorem 2, we obtain the required result. (ii) When k is an odd number, the required proof follows by applying the result for the even case replacing k by $k-1$. □

When the matrix $\boldsymbol{Z}$ in Theorem 3 is not a square matrix, we can make use of the same partial (or complete) partition into 2×2 blocks and Theorem 2 for obtaining the Hadamard inverse.

4.3. Inverse problem in the connection of Hadamard and matrix products

We have already established in Theorem 1 an expression for the matrix product in terms of the Hadamard product. Now, we consider the inverse problem, which is useful in the study of matrix-variate distributions; for example, to find an expansion of the Hadamard product in terms of the matrix product. This expansion is also of interest for the proposed matrix-variate GBS distribution, since it can simplify the form of its density in terms of the original matrices. This could facilitate the computation of Euclidean matrix transformations and of marginal densities after integration over, for example, the Steifel manifold. In general, the above mentioned expansion involves a function f that must relate the Hadamard product ($\odot$) only with the original square matrices, say $\boldsymbol{A}$ and $\boldsymbol{B}$, and with the usual matrix product ($\cdot$), i.e.,

$$\boldsymbol{A}\odot\boldsymbol{B} = f(\boldsymbol{A}, \boldsymbol{B}, \cdot).$$

This is an elementary problem to state, but, apparently and even for the simplest case of dimension 2, it does not seem to have been resolved in the literature. We have found even this simplest case to be quite complicated and feel strongly that a general expression for any order would be very hard to obtain, if not impossible. We now discuss this problem by employing the class of cyclic permutations used in Section 3.

First, instead of considering the entire matrix $\boldsymbol{A}\odot\boldsymbol{B}$, in the context of Theorem 1, let us take $\boldsymbol{C}$ to be a $k\times k$ matrix. Then, it is easy to check that $\boldsymbol{C}$ decomposes uniquely as $\boldsymbol{C} = \sum_p \boldsymbol{C}_{\{p\}}$, where $\boldsymbol{C}_{\{p\}} = \boldsymbol{C}\odot\boldsymbol{I}_p$, for $p \in P_{12\cdots k}$. Thus, we have

$$\boldsymbol{A}\odot\boldsymbol{B} = \sum_{p\in P_{12\cdots k}} \boldsymbol{A}\odot\boldsymbol{B}\odot\boldsymbol{I}_p. \tag{11}$$

Note that the columns of $\boldsymbol{A}\odot\boldsymbol{B}\odot\boldsymbol{I}_p$ can be permuted by a unique $\boldsymbol{I}_{(p')}$, with $p' \in P_{12\cdots k}$, in such a way that we obtain a diagonal matrix $(\boldsymbol{A}\odot\boldsymbol{B}\odot\boldsymbol{I}_{(p)})\cdot\boldsymbol{I}_{(p')}$. We call p and p' conjugate cyclic permutations due to an analogy with partition theory relating to two partitions that hold some property involving diagonal symmetry. Thus, the considered problem reduces to the task of finding a representation of a diagonal matrix as a function of $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{I}_{(p)}$, $\boldsymbol{I}_{(p')}$ and of the usual matrix product.

As an example, let us take $k = 3$. In this case, we have $P_{123} = \{123, 231, 312\}$,

$$\boldsymbol{I}_{(123)} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \quad \boldsymbol{I}_{(231)} = \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}, \quad \boldsymbol{I}_{(312)} = \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}, \quad \boldsymbol{A} = \left(a_{ij}\right) \quad \text{and} \quad \boldsymbol{B} = \left(b_{ij}\right).$$

Then,

$$\begin{aligned}\boldsymbol{A}\odot\boldsymbol{B} &= \begin{pmatrix}a_{11}b_{11}&0&0\\0&a_{22}b_{22}&0\\0&0&a_{33}b_{23}\end{pmatrix} + \begin{pmatrix}0&0&a_{13}b_{13}\\a_{21}b_{21}&0&0\\0&a_{32}b_{32}&0\end{pmatrix} + \begin{pmatrix}0&a_{12}b_{12}&0\\0&0&a_{23}b_{23}\\a_{31}b_{31}&0&0\end{pmatrix}\\ &= \boldsymbol{A}\odot\boldsymbol{B}\odot\boldsymbol{I}_{(123)} + \boldsymbol{A}\odot\boldsymbol{B}\odot\boldsymbol{I}_{(231)} + \boldsymbol{A}\odot\boldsymbol{B}\odot\boldsymbol{I}_{(312)}.\end{aligned} \tag{12}$$

The matrices in each one of the summands of (12) can be trivially turned into diagonal matrices as $\boldsymbol{A}\odot\boldsymbol{B}\odot\boldsymbol{I}_{(123)}$ by $(\boldsymbol{A}\odot\boldsymbol{B}\odot\boldsymbol{I}_{(123)})\cdot\boldsymbol{I}_{(123)}$, $\boldsymbol{A}\odot\boldsymbol{B}\odot\boldsymbol{I}_{(231)}$ by $(\boldsymbol{A}\odot\boldsymbol{B}\odot\boldsymbol{I}_{(231)})\cdot\boldsymbol{I}_{(312)}$, and $\boldsymbol{A}\odot\boldsymbol{B}\odot\boldsymbol{I}_{(312)}$ by $(\boldsymbol{A}\odot\boldsymbol{B}\odot\boldsymbol{I}_{(312)})\cdot\boldsymbol{I}_{(231)}$. Hence, it is sufficient to study the representation of one of the above diagonal matrices in terms of the involved matrices.

Now, we focus on the inverse problem for 2×2 matrices. First, we need a preliminary result presented in the following lemma.

Lemma 2. *Let* $\boldsymbol{A} = (a_{ij})$ *and* $\boldsymbol{B} = (b_{ij})$ *be* 2×2 *non-singular matrices with non-null entries, and* $\boldsymbol{N} = \boldsymbol{I}_{(21)}$. *Then,*

$$\boldsymbol{A}\odot\boldsymbol{B} = \frac{1}{|\boldsymbol{A}|}\frac{1}{|\boldsymbol{B}|}\boldsymbol{A}\cdot\left[\left(|\boldsymbol{A}\odot\boldsymbol{I}| + \frac{1}{2}\langle\boldsymbol{A}\rangle\right)\boldsymbol{I} - \frac{1}{2}\boldsymbol{N}\cdot\boldsymbol{A}^{\top}\cdot\boldsymbol{N}\cdot\boldsymbol{A}\right]\cdot\boldsymbol{B}\cdot\left[\left(|\boldsymbol{B}\odot\boldsymbol{I}| + \frac{1}{2}\langle\boldsymbol{B}\rangle\right)\boldsymbol{I} - \frac{1}{2}\boldsymbol{N}\cdot\boldsymbol{B}^{\top}\cdot\boldsymbol{N}\cdot\boldsymbol{B}\right]$$

$$+\frac{1}{|\boldsymbol{A}|}\frac{1}{|\boldsymbol{B}|}\boldsymbol{A}\cdot\boldsymbol{N}\cdot\left[\left(|\boldsymbol{A}\cdot\boldsymbol{N}\odot\boldsymbol{I}|+\frac{1}{2}\langle\boldsymbol{A}\rangle\right)\boldsymbol{I}-\frac{1}{2}\boldsymbol{A}^{\top}\cdot\boldsymbol{N}\cdot\boldsymbol{A}\cdot\boldsymbol{N}\right]$$
$$\cdot\boldsymbol{B}\cdot\boldsymbol{N}\cdot\left[\left(|\boldsymbol{B}\cdot\boldsymbol{N}\odot\boldsymbol{I}|+\frac{1}{2}\langle\boldsymbol{B}\rangle\right)\boldsymbol{I}-\frac{1}{2}\boldsymbol{B}^{\top}\cdot\boldsymbol{N}\cdot\boldsymbol{B}\cdot\boldsymbol{N}\right]\cdot\boldsymbol{N},$$

where $\langle\boldsymbol{A}\rangle$ *and* $\langle\boldsymbol{B}\rangle$ *are as given in* (10).

Proof. For the case $k = 2$, the set of cyclic permutations is just $P_{12} = \{12, 21\}$. Hence, for 2×2 matrices $\boldsymbol{A} = (a_{ij})$ and $\boldsymbol{B} = (b_{ij})$ with non-null entries, we obtain from (11) that

$$\begin{aligned}\boldsymbol{A}\odot\boldsymbol{B} &= (\boldsymbol{A}\odot\boldsymbol{B})\odot\boldsymbol{I}_{(12)}+(\boldsymbol{A}\odot\boldsymbol{B})\odot\boldsymbol{I}_{(21)}\\ &= (\boldsymbol{A}\odot\boldsymbol{B})\odot\boldsymbol{I}+(\boldsymbol{A}\odot\boldsymbol{B})\odot\boldsymbol{N}\\ &= (\boldsymbol{A}\odot\boldsymbol{I})\odot(\boldsymbol{B}\odot\boldsymbol{I})+(\boldsymbol{A}\cdot\boldsymbol{N}\odot\boldsymbol{I})\odot(\boldsymbol{B}\cdot\boldsymbol{N}\odot\boldsymbol{I})\\ &= (\boldsymbol{A}\odot\boldsymbol{I})\cdot(\boldsymbol{B}\odot\boldsymbol{I})+(\boldsymbol{A}\cdot\boldsymbol{N}\odot\boldsymbol{I})\cdot(\boldsymbol{B}\cdot\boldsymbol{N}\odot\boldsymbol{I}),\end{aligned} \tag{13}$$

where once again $\boldsymbol{N} = \boldsymbol{I}_{(21)}$. The expression in (13) reduces the problem of finding the representation of the diagonal matrices

$$(\boldsymbol{A}\odot\boldsymbol{B})\odot\boldsymbol{I}=\begin{pmatrix}a_{11}\,b_{11} & 0\\ 0 & a_{22}\,b_{22}\end{pmatrix}\quad\text{and}\quad((\boldsymbol{A}\odot\boldsymbol{B})\odot\boldsymbol{N})\cdot\boldsymbol{N}=\begin{pmatrix}a_{12}\,b_{12} & 0\\ 0 & a_{21}\,b_{21}\end{pmatrix}$$

in terms of the matrix product. Now, let $\boldsymbol{Z} = (Z_{ij})$ be a 2×2 matrix. Then, by using Theorem 2, we have

$$\begin{aligned}\begin{pmatrix}Z_{11} & 0\\ 0 & Z_{22}\end{pmatrix} &= \frac{1}{|\boldsymbol{Z}|}\boldsymbol{Z}\cdot\begin{pmatrix}Z_{11}\,Z_{22} & -Z_{12}\,Z_{22}\\ -Z_{11}\,Z_{21} & Z_{11}\,Z_{22}\end{pmatrix}\\ &= \frac{1}{|\boldsymbol{Z}|}\boldsymbol{Z}\cdot[|\boldsymbol{Z}\odot\boldsymbol{I}|\,\boldsymbol{I}-\boldsymbol{N}\cdot\boldsymbol{Z}_I^{-1}\cdot\boldsymbol{N}\cdot\boldsymbol{Z}]\\ &= \frac{1}{|\boldsymbol{Z}|}\boldsymbol{Z}\cdot\left[\left(|\boldsymbol{Z}\odot\boldsymbol{I}|+\frac{1}{4}\langle\boldsymbol{Z}\rangle\right)\boldsymbol{I}-\frac{1}{2}\boldsymbol{N}\cdot\boldsymbol{Z}^{\top}\cdot\boldsymbol{N}\cdot\boldsymbol{Z}\right].\end{aligned} \tag{14}$$

Finally, the representation of

$$\begin{pmatrix}0 & Z_{12}\\ Z_{21} & 0\end{pmatrix}$$

is obtained replacing $\boldsymbol{Z}$ by $\boldsymbol{Z}\cdot\boldsymbol{N}$ in (14). Upon using these results in (13), we obtain the required result. □

The presence of the Hadamard product in the determinant of Lemma 2 suggests a further simplification. Then, how can we express $|\boldsymbol{Z}\odot\boldsymbol{I}|$ in terms of the 2×2 non-singular matrix $\boldsymbol{Z}$ avoiding the Hadamard multiplication? Once again, the answer to this question does not seem to be available in the literature, and even the simplest case of dimension 2 requires some tricky algebra, as seen in the following theorem.

Theorem 4. *Let* $\boldsymbol{A} = (a_{ij})$ *and* $\boldsymbol{B} = (b_{ij})$ *be* 2×2 *non-singular matrices with non-null entries. Then, we have*

$$\begin{aligned}\boldsymbol{A}\odot\boldsymbol{B} = {}&\frac{1}{|\boldsymbol{A}|}\frac{1}{|\boldsymbol{B}|}\boldsymbol{A}\cdot\left[\left(\boldsymbol{D}_1+\frac{1}{2}\langle\boldsymbol{A}\rangle\right)\boldsymbol{I}-\frac{1}{2}\boldsymbol{N}\cdot\boldsymbol{A}^{\top}\cdot\boldsymbol{N}\cdot\boldsymbol{A}\right]\cdot\boldsymbol{B}\cdot\left[\left(\boldsymbol{D}_2+\frac{1}{2}\langle\boldsymbol{B}\rangle\right)\boldsymbol{I}-\frac{1}{2}\boldsymbol{N}\cdot\boldsymbol{B}^{\top}\cdot\boldsymbol{N}\cdot\boldsymbol{B}\right]\\ &+\frac{1}{|\boldsymbol{A}|}\frac{1}{|\boldsymbol{B}|}\boldsymbol{A}\cdot\boldsymbol{N}\cdot\left[\left(\boldsymbol{D}_3+\frac{1}{2}\langle\boldsymbol{A}\rangle\right)\boldsymbol{I}-\frac{1}{2}\boldsymbol{A}^{\top}\cdot\boldsymbol{N}\cdot\boldsymbol{A}\cdot\boldsymbol{N}\right]\cdot\boldsymbol{B}\cdot\boldsymbol{N}\cdot\left[\left(\boldsymbol{D}_4+\frac{1}{2}\langle\boldsymbol{B}\rangle\right)\boldsymbol{I}-\frac{1}{2}\boldsymbol{B}^{\top}\cdot\boldsymbol{N}\cdot\boldsymbol{B}\cdot\boldsymbol{N}\right]\cdot\boldsymbol{N},\end{aligned}$$

where $\boldsymbol{D_1} = |\boldsymbol{A}\odot\boldsymbol{I}|$, $\boldsymbol{D_2} = |\boldsymbol{B}\odot\boldsymbol{I}|$, $\boldsymbol{D_3} = |\boldsymbol{A}\cdot\boldsymbol{N}\odot\boldsymbol{I}|$ *and* $\boldsymbol{D_4} = |\boldsymbol{B}\cdot\boldsymbol{N}\odot\boldsymbol{I}|$.

Proof. Consider the ordered partitions (2, 0) and (1, 1) of the integer number 2. Then, from [16], we have an unusual representation of a determinant of second order in terms of powers of traces given by $|\boldsymbol{A}| = \frac{1}{2}\left[(\text{tr}(\boldsymbol{A}))^2-\text{tr}(\boldsymbol{A}\cdot\boldsymbol{A})\right]$. From Lemma 2, we know that

$$\boldsymbol{Z}\odot\boldsymbol{I}=\frac{1}{|\boldsymbol{Z}|}\boldsymbol{Z}\cdot\left[\left(|\boldsymbol{Z}\odot\boldsymbol{I}|+\frac{1}{2}\langle\boldsymbol{Z}\rangle\right)\boldsymbol{I}-\frac{1}{2}\boldsymbol{N}\cdot\boldsymbol{Z}^{\top}\cdot\boldsymbol{N}\cdot\boldsymbol{Z}\right].$$

Then, since

$$|\boldsymbol{Z}\odot\boldsymbol{I}|=\frac{1}{|\boldsymbol{Z}|}\left|\left(|\boldsymbol{Z}\odot\boldsymbol{I}|+\frac{1}{2}\langle\boldsymbol{Z}\rangle\right)\boldsymbol{I}-\frac{1}{2}\boldsymbol{N}\cdot\boldsymbol{Z}^{\top}\cdot\boldsymbol{N}\cdot\boldsymbol{Z}\right|,$$

by solving the quadratic equation for $|\boldsymbol{Z} \odot \boldsymbol{I}|$, we get that one root $Z_{11}\,Z_{22}$ gives the required result, which is

$$|\boldsymbol{Z} \odot \boldsymbol{I}| = \frac{1}{2\,w}\left[1 - r\,w - 2\,w\,y - (1 - 2\,r\,w - r^2\,w^2 - 4\,w\,y + 2\,w^2\,x)^{\frac{1}{2}}\right], \tag{15}$$

where

$$r = \mathrm{tr}(\boldsymbol{M}), \quad \boldsymbol{M} = -\frac{1}{2}\boldsymbol{N}\cdot\boldsymbol{Z}^\top\cdot\boldsymbol{N}\cdot\boldsymbol{Z}, \qquad w = \frac{1}{|\boldsymbol{Z}|}, \qquad y = \frac{1}{2}\langle\boldsymbol{Z}\rangle, \quad \text{and} \quad x = \mathrm{tr}(\boldsymbol{M}\cdot\boldsymbol{M}).$$

The second root $Z_{12}\,Z_{21}$ equals $|\boldsymbol{Z}\cdot\boldsymbol{N} \odot \boldsymbol{I}|$. Recall now that $|\boldsymbol{Z}| = |\boldsymbol{Z} \odot \boldsymbol{I}| - |\boldsymbol{Z}\cdot\boldsymbol{N} \odot \boldsymbol{I}|$. We thus obtain the expansion free of Hadamard products as required. □

As noted, the representation of $\boldsymbol{A}\odot\boldsymbol{B}$ in terms of matrix products is reduced to the study of diagonal matrices of the type $\boldsymbol{Z}\odot\boldsymbol{I}$, which requires expressions of $|\boldsymbol{Z} \odot \boldsymbol{I}|$ free of Hadamard products. At least, the solution for the simplest case of dimension 2 shows that the task is feasible, but may be tedious.

5. A matrix-variate generalized Birnbaum–Saunders distribution

It is useful to reiterate that the main motivation for proposing matrix representations of the density of matrix-variate GBS distributions, instead of an element-by-element-representation of this density, comes from the use of transformations of random matrices. As mentioned earlier, the matrix representation is needed for some statistical procedures based on the GBS distribution, such as hypothesis testing, linear models, multivariate analysis of variance, principal components analysis, and statistical shape theory. In this section, we propose an extension of the vector GBS distribution to the matrix case.

5.1. A matrix-variate GBS distribution

The class of multivariate GBS distributions introduced in Section 2.3 can be extended to the matrix-variate case through elliptic random matrices. Next, we present the matrix-variate elliptic distributions. Then, we define a matrix-variate GBS distribution and propose three representations for the matrix-variate GBS density, viz., (i) element-by-element representation, (ii) a first matrix representation via diagonal matrices, and (iii) a second matrix representation involving the connection between the Hadamard and matrix products, which in fact forms the main result of this section.

Let $\boldsymbol{X} = (X_{ij})$ be an $n \times k$ random matrix with an elliptic distribution characterized by a location matrix $\boldsymbol{M} \in \mathbb{R}^{n\times k}$, scale matrices $\boldsymbol{\Omega} \in \mathbb{R}^{k\times k}$ with $\mathrm{rank}(\boldsymbol{\Omega}) = k$, and $\boldsymbol{\Sigma} \in \mathbb{R}^{n\times n}$ with $\mathrm{rank}(\boldsymbol{\Sigma}) = n$, and a kernel function g. By denoting $\boldsymbol{X} \sim \mathrm{EC}_{n\times k}(\boldsymbol{M}, \boldsymbol{\Omega}, \boldsymbol{\Sigma}; g)$, we have the density of $\boldsymbol{X}$ to be

$$f_{\boldsymbol{X}}(\boldsymbol{X}) = c\,|\boldsymbol{\Omega}|^{-\frac{n}{2}}\,|\boldsymbol{\Sigma}|^{-\frac{k}{2}}\,g\left(\mathrm{tr}\left(\boldsymbol{\Omega}^{-1}\cdot[\boldsymbol{X}-\boldsymbol{M}]^\top\cdot\boldsymbol{\Sigma}^{-1}\cdot[\boldsymbol{X}-\boldsymbol{M}]\right)\right), \quad \boldsymbol{X} \in \mathbb{R}^{n\times k},$$

where c is the normalizing constant for the kernel g.

Definition 1. Let $\boldsymbol{Z} = (Z_{ij}) \sim \mathrm{EC}_{n\times k}(\boldsymbol{0}, \boldsymbol{I}_k, \boldsymbol{I}_n; g)$ and $\boldsymbol{T} = \left(T_{ij}\right)$, where

$$T_{ij} = \beta_{ij}\left[\frac{\alpha_{ij}\,Z_{ij}}{2} + \sqrt{\left\{\frac{\alpha_{ij}\,Z_{ij}}{2}\right\}^2 + 1}\,\right]^2, \quad \alpha_{ij} > 0, \beta_{ij} > 0, i = 1, \ldots, j = 1, \ldots, k.$$

Then, the random matrix $\boldsymbol{T}$ generates the matrix-variate GBS distribution, denoted by $\boldsymbol{T} \sim \mathrm{GBS}_{n\times k}(\boldsymbol{A}, \boldsymbol{B}; g)$, where $\boldsymbol{A} = (\alpha_{ij})$ and $\boldsymbol{B} = (\beta_{ij})$.

5.1.1. An element-by-element representation of the matrix-variate GBS density

First, we propose a natural representation of the GBS density by adopting the vectorial approach, i.e., we start with a representation involving the elements of a GBS random matrix.

Lemma 3. *Let $\boldsymbol{T} = (T_{ij}) \sim \mathrm{GBS}_{n\times k}(\boldsymbol{A}, \boldsymbol{B}; g)$, with $\boldsymbol{A} = (\alpha_{ij})$ and $\boldsymbol{B} = (\beta_{ij})$. Then, the density of $\boldsymbol{T}$ is given by*

$$f_{\boldsymbol{T}}(\boldsymbol{T}) = \frac{c}{2^{n+k}}\,g\left(\sum_{i=1}^{n}\sum_{j=1}^{k}\frac{1}{\alpha_{ij}^2}\left[\frac{T_{ij}}{\beta_{ij}} + \frac{\beta_{ij}}{T_{ij}} - 2\right]\right)\prod_{i=1}^{n}\prod_{j=1}^{k}\frac{T_{ij}^{-3/2}\,[T_{ij}+\beta_{ij}]}{\alpha_{ij}\,\sqrt{\beta_{ij}}}, \quad T_{ij} > 0,$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, k$.

Proof. The result follows from a trivial extension of the proof given for the multivariate case in Díaz-García and Domínguez-Molina [10, Theorem 3.1]. □

We must stress here that the density given in Lemma 3 is not expressed either in terms of the GBS random matrix $\boldsymbol{T}$ or of its parameter matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, as commonly done in the case of matrix-variate distributions; see, e.g., Muirhead [30], Fang and Zhang [15] and Gupta and Varga [18]. Hence, the representation in Lemma 3 is not a proper matrix-variate version of the GBS distribution in this sense. In fact, the density of the multivariate GBS distribution presented in Section 2.3 has the same problem as well. Apparently, the matrix-variate GBS density so expressed seems to be easily computable (conditional to parameter matrices), but it does not allow any matrix transformation based on affine, polar coordinates, and QR and SV decompositions. For this reason, as mentioned, some statistical procedures based on matrix-variate GBS distributions cannot be performed.

5.1.2. A first matrix representation of the GBS density

In the following lemma, we construct a matrix representation of the matrix-variate GBS density. For this construction, we need an operation called diagonalization, denoted by "diag", which is very similar to the one known as vectorization, denoted by "vec". The diag operation is defined as follows. Let $\boldsymbol{X} = (X_{ij})$ be an $n \times k$ matrix and $\boldsymbol{X}_d$ be the $(n+k) \times (n+k)$ diagonal matrix defined as

$$\boldsymbol{X}_d = \text{diag}\{X_{11}, \ldots, X_{1k}, X_{21}, \ldots, X_{2k}, \ldots, X_{n1}, \ldots, X_{nk}\}. \tag{16}$$

Lemma 4. *Let $\boldsymbol{T} = (T_{ij}) \sim \text{GBS}_{n\times k}(\boldsymbol{A}, \boldsymbol{B}; g)$, with $\boldsymbol{A} = (\alpha_{ij})$ and $\boldsymbol{B} = (\beta_{ij})$. Then, the density of $\boldsymbol{T}$ is given by*

$$f_{\boldsymbol{T}}(\boldsymbol{T}) = \frac{c}{2^{n+k}} \left|\boldsymbol{T}_d^{-3} \cdot \boldsymbol{A}_d^{-2} \cdot \boldsymbol{B}_d^{-1}\right|^{\frac{1}{2}} |\boldsymbol{T}_d + \boldsymbol{B}_d| \times g\left(\text{tr}(\boldsymbol{T}_d \cdot \boldsymbol{A}_d^{-2} \cdot \boldsymbol{B}_d^{-1} + \boldsymbol{T}_d^{-1} \cdot \boldsymbol{A}_d^{-2} \cdot \boldsymbol{B}_d - 2\,\boldsymbol{A}_d)\right),$$

where $\boldsymbol{T}_d$, $\boldsymbol{A}_d$ and $\boldsymbol{B}_d$ are as defined in (16), with respect to $\boldsymbol{T}$, $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively.

Proof. This result follows in a straightforward manner by applying the diagonalization operation in the representation in Lemma 3.

Clearly, we can now perform matrix transformations over the GBS random matrix $\boldsymbol{T}$ given in Lemma 4. However, some properties of the diagonalization operation under the Euclidean and affine transformations need to be studied first, and we hope to consider this problem in our future work. □

Remark 1. Diagonal matrices of the type given in (16) correspond to matrices $\hat{\boldsymbol{X}}$ of Magnus and Neudecker [29]. As can be checked there, the algebra associated with this type of matrices is completely unexplored. Only a simple relationship with the vectorization is known in Magnus and Neudecker [29, Lemma 2, p. 487].

5.1.3. A second matrix representation of the GBS density

Instead of considering arrangements of the elements of the GBS random matrix for its density as given in Lemma 3 and parameter matrices into augmented diagonal matrices as given in Lemma 4, we can express the kernel of the matrix-variate GBS density in terms of the original matrices, as is the case with many matrix-variate distributions. Then, by using the connection between Hadamard and matrix products established in Theorem 1, the GBS density given in Lemma 4 can be expressed in terms of the original matrices, as presented in the following theorem.

Theorem 5. *Let $\boldsymbol{T} = (T_{ij}) \sim \text{GBS}_{n\times k}(\boldsymbol{A}, \boldsymbol{B}; g)$, with $\boldsymbol{A} = (\alpha_{ij})$ and $\boldsymbol{B} = (\beta_{ij})$. Then, the density of $\boldsymbol{T}$ is given by*

$$\begin{aligned} f_{\boldsymbol{T}}(\boldsymbol{T}) = {} & \frac{c}{2^{n+k}} \left|\left(\boldsymbol{A}^{-H} \odot \boldsymbol{B}^{-\frac{H}{2}} \odot \boldsymbol{T}^{-\frac{H}{2}} + \boldsymbol{A}^{-H} \odot \boldsymbol{B}^{\frac{H}{2}} \odot \boldsymbol{T}^{-\frac{3H}{2}}\right)_d\right| \\ & \times g\left(\sum_{p\in P_{1\cdots k}} \text{tr}\left(\left(\boldsymbol{A}^{-\text{H}} \odot \boldsymbol{B}^{-\frac{H}{2}} \odot \boldsymbol{T}^{\frac{H}{2}}\right)_{(p)} \odot \left((\boldsymbol{A}^{-H})^\top \odot (\boldsymbol{B}^{-\frac{H}{2}})^\top \odot (\boldsymbol{T}^{\frac{H}{2}})^\top\right)_{[p]}\right.\right. \\ & + \left(\boldsymbol{A}^{-H} \odot \boldsymbol{B}^{\frac{H}{2}} \odot \boldsymbol{T}^{-\frac{H}{2}}\right)_{(p)} \odot \left((\boldsymbol{A}^{-H})^\top \odot (\boldsymbol{B}^{\frac{H}{2}})^\top \odot (\boldsymbol{T}^{-\frac{H}{2}})^\top\right)_{[p]} \\ & \left.\left. -2\,\left(\boldsymbol{A}^{-H} \odot \boldsymbol{B}^{-\frac{H}{2}} \odot \boldsymbol{T}^{\frac{H}{2}}\right)_{(p)} \odot \left((\boldsymbol{A}^{-H})^\top \odot (\boldsymbol{B}^{\frac{H}{2}})^\top \odot (\boldsymbol{T}^{-\frac{H}{2}})^\top\right)_{[p]}\right)\right). \end{aligned}$$

Proof. Consider $\boldsymbol{T} = (T_{ij}) \sim \text{GBS}_{n\times k}(\boldsymbol{A}, \boldsymbol{B}; g)$, with $\boldsymbol{A} = (\alpha_{ij})$ and $\boldsymbol{B} = (\beta_{ij})$. Then, if

$$\boldsymbol{Z} = \left(\frac{1}{\alpha_{ij}}\left[\sqrt{\frac{T_{ij}}{\beta_{ij}}} - \sqrt{\frac{\beta_{ij}}{T_{ij}}}\right]\right),$$

we have

$$g(\text{tr}(\boldsymbol{Z} \cdot \boldsymbol{Z}^\top)) = g\left(\sum_{i=1}^{n}\sum_{j=1}^{k} \frac{1}{\alpha_{ij}^2}\left(\frac{T_{ij}}{\beta_{ij}} + \frac{\beta_{ij}}{T_{ij}} - 2\right)\right). \tag{17}$$

However, we are now seeking an expression for $g(\mathrm{tr}(\boldsymbol{Z}\cdot\boldsymbol{Z}^\top))$ in (17) in terms of the matrices $\boldsymbol{T}$, $\boldsymbol{A}$ and $\boldsymbol{B}$ rather than in terms of particular elements of the GBS random matrix. Thus, starting with

$$\boldsymbol{Z}=\left(\frac{1}{\alpha_{ij}}\left[\sqrt{\frac{t_{ij}}{\beta_{ij}}}-\sqrt{\frac{\beta_{ij}}{t_{ij}}}\right]\right)=\boldsymbol{A}^{-H}\odot(\boldsymbol{B}^{-\frac{H}{2}}\odot\boldsymbol{T}^{\frac{H}{2}}-\boldsymbol{B}^{\frac{H}{2}}\odot\boldsymbol{T}^{-\frac{H}{2}}), \tag{18}$$

we obtain

$$\begin{aligned}\boldsymbol{Z}\cdot\boldsymbol{Z}^\top &= (\boldsymbol{A}^{-H}\odot\boldsymbol{B}^{-\frac{H}{2}}\odot\boldsymbol{T}^{\frac{H}{2}}-\boldsymbol{A}^{-H}\odot\boldsymbol{B}^{\frac{H}{2}}\odot\boldsymbol{T}^{-\frac{H}{2}})\cdot(\boldsymbol{A}^{-H}\odot\boldsymbol{B}^{-\frac{H}{2}}\odot\boldsymbol{T}^{\frac{H}{2}}-\boldsymbol{A}^{-H}\odot\boldsymbol{B}^{\frac{H}{2}}\odot\boldsymbol{T}^{-\frac{H}{2}})^\top\\
&=(\boldsymbol{A}^{-H}\odot\boldsymbol{B}^{-\frac{H}{2}}\odot\boldsymbol{T}^{\frac{H}{2}})\cdot((\boldsymbol{A}^{-H})^\top\odot(\boldsymbol{B}^{-\frac{H}{2}})^\top\odot(\boldsymbol{T}^{\frac{H}{2}})^\top)\\
&\quad+(\boldsymbol{A}^{-H}\odot\boldsymbol{B}^{\frac{H}{2}}\odot\boldsymbol{T}^{-\frac{H}{2}})\cdot((\boldsymbol{A}^{-H})^\top\odot(\boldsymbol{B}^{\frac{H}{2}})^\top\odot(\boldsymbol{T}^{-\frac{H}{2}})^\top)\\
&\quad-(\boldsymbol{A}^{-H}\odot\boldsymbol{B}^{-\frac{H}{2}}\odot\boldsymbol{T}^{\frac{H}{2}})\cdot((\boldsymbol{A}^{-H})^\top\odot(\boldsymbol{B}^{\frac{H}{2}})^\top\odot(\boldsymbol{T}^{-\frac{H}{2}})^\top)\\
&\quad-(\boldsymbol{A}^{-H}\odot\boldsymbol{B}^{\frac{H}{2}}\odot\boldsymbol{T}^{-\frac{H}{2}})\cdot((\boldsymbol{A}^{-H})^\top\odot(\boldsymbol{B}^{-\frac{H}{2}})^\top\odot(\boldsymbol{T}^{\frac{H}{2}})^\top).\end{aligned}$$

Now, by using Theorem 1, we obtain for (17) the expression

$$\begin{aligned}g\left(\mathrm{tr}(\boldsymbol{Z}\cdot\boldsymbol{Z}^\top)\right) &= g\Big(\mathrm{tr}((\boldsymbol{A}^{-H}\odot\boldsymbol{B}^{-\frac{H}{2}}\odot\boldsymbol{T}^{\frac{H}{2}})\cdot((\boldsymbol{A}^{-H})^\top\odot(\boldsymbol{B}^{-\frac{H}{2}})^\top\odot(\boldsymbol{T}^{\frac{H}{2}})^\top)\\
&\quad+(\boldsymbol{A}^{-H}\odot\boldsymbol{B}^{\frac{H}{2}}\odot\boldsymbol{T}^{-\frac{H}{2}})\cdot((\boldsymbol{A}^{-H})^\top\odot(\boldsymbol{B}^{\frac{H}{2}})^\top\odot(\boldsymbol{T}^{-\frac{H}{2}})^\top)\\
&\quad-\ 2(\boldsymbol{A}^{-H}\odot\boldsymbol{B}^{-\frac{H}{2}}\odot\boldsymbol{T}^{\frac{H}{2}})\cdot((\boldsymbol{A}^{-H})^\top\odot(\boldsymbol{B}^{\frac{H}{2}})^\top\odot(\boldsymbol{T}^{-\frac{H}{2}})^\top))\Big)\\
&= g\Bigg(\sum_{p\in P_{1\cdots k}}\mathrm{tr}\Bigg(\left(\boldsymbol{A}^{-H}\odot\boldsymbol{B}^{-\frac{H}{2}}\odot\boldsymbol{T}^{\frac{H}{2}}\right)_{(p)}\odot\left((\boldsymbol{A}^{-H})^\top\odot(\boldsymbol{B}^{-\frac{H}{2}})^\top\odot(\boldsymbol{T}^{\frac{H}{2}})^\top\right)_{[p]}\\
&\quad+\left(\boldsymbol{A}^{-H}\odot\boldsymbol{B}^{\frac{H}{2}}\odot\boldsymbol{T}^{-\frac{H}{2}}\right)_{(p)}\odot\left((\boldsymbol{A}^{-H})^\top\odot(\boldsymbol{B}^{\frac{H}{2}})^\top\odot(\boldsymbol{T}^{-\frac{H}{2}})^\top\right)_{[p]}\\
&\quad-2\left(\boldsymbol{A}^{-H}\odot\boldsymbol{B}^{-\frac{H}{2}}\odot\boldsymbol{T}^{\frac{H}{2}}\right)_{(p)}\odot\left((\boldsymbol{A}^{-H})^\top\odot(\boldsymbol{B}^{\frac{H}{2}})^\top\odot(\boldsymbol{T}^{-\frac{H}{2}})^\top\right)_{[p]}\Bigg)\Bigg).\end{aligned} \tag{19}$$

The corresponding Jacobian of Lemma 3 can be written in terms of Hadamard products and diagonal matrices as

$$\prod_{i=1}^{n}\prod_{j=1}^{k}\frac{t_{ij}^{-3/2}\,[t_{ij}+\beta_{ij}]}{\alpha_{ij}\sqrt{\beta_{ij}}}=\left|\left(\boldsymbol{A}^{-H}\odot\boldsymbol{B}^{-\frac{H}{2}}\odot\boldsymbol{T}^{-\frac{H}{2}}+\boldsymbol{A}^{-H}\odot\boldsymbol{B}^{\frac{H}{2}}\odot\boldsymbol{T}^{-\frac{3H}{2}}\right)_d\right|. \tag{20}$$

The required result is then established by multiplying the constant $c/2^{n+k}$ with (19) and (20). □

Remark 2. Note that if the normalization constant and the kernel given in Theorem 5 are taken as

$$c=\frac{2^{\frac{nk}{2}}}{\pi^{\frac{nk}{2}}}\quad\text{and}\quad g(u)=\exp\left(-\frac{u}{2}\right),$$

then a matrix-variate classical BS distribution is obtained. Thus, just as in the univariate and multivariate cases, the matrix-variate BS distribution becomes a particular case of the matrix-variate GBS distribution.

6. Concluding remarks, future research and open problems

In this paper, we have established a connection between the Hadamard and matrix multiplications. We have also studied some new properties of the Hadamard product and explored the inverse problem associated with the established connection, which is useful in several statistical applications. In addition, we have proposed a matrix-variate GBS distribution, which includes the matrix-variate classical Birnbaum–Saunders distribution as a special case. We have provided three representations for the matrix-variate GBS density, one of them based on the established connection between the Hadamard and matrix products. We have represented this density in terms of the original matrices by means of the Hadamard product because this facilitates matrix transformations in a natural manner, which are useful for some statistical procedures, such as multivariate data analysis and statistical shape theory.

This work has raised many challenging problems that we are currently working on. For example, based on the matrix representation, we have found the following issues and questions to be of interest:

1. In the context of the first matrix representation of the GBS density presented in Section 5.1.2, we note the following:

(a) This representation involves a special algebra that studies the diagonalization in (16) and its relationship with the well-known matrix differential calculus; see [29]. This algebra is completely unexplored, and its study is of great interest.
(b) Another problem we can explore is about the connections between the diagonalization operation and the classical products, such as the usual, Hadamard and Kronecker multiplications.

2. In the context of the second matrix representation of the GBS density in Section 5.1.3, we note the following:
(a) Classical techniques of the matrix-variate theory applied to a Gaussian random matrix (see [30]) are a source of a number of extensions to, for example, elliptic random matrices. Therefore, to transfer the matrix-variate theory to GBS random matrices poses an interesting problem.
(b) The above strongly depends on matrix transformations (Euclidean or affine, say) of the GBS random matrix and the leading integration over the Stiefel manifold. Hence, another open problem is concerning the integration theory associated with functions expressed in terms of Hadamard products.
(c) One of the key applications of the two issues mentioned above in (a) and (b) arises in the context of statistical theory of shape based on asymmetric distributions, instead of the classical theory that, until now, is only based on symmetric distributions. Thus, it is of great interest to derive the shape densities under GBS random matrices.
(d) Perhaps, the most attractive open problem arising from this study corresponds to expressing the Hadamard product of two matrices as a function of the usual product of the involved matrices. As noted in Theorem 4, the case of dimension 2 is extremely cumbersome and its solution requires some tricks including a new expansion of the determinant. In fact, the case of dimension 3 seems difficult and a generalization to any order is indeed a very complicated and challenging problem.
(e) A new matrix representation of the GBS density can be inferred from Section 4.3. In this case, the open problem reduces to the task of finding a representation of a diagonal matrix in terms of the original and identity matrices and of the usual matrix product. This representation involving conjugate cyclic permutations can generate a new study of the GBS density by noting that, for a $k \times k$ matrix $\boldsymbol{X} = (Xij)$ and a function g,
$$g\left(\sum_{i=1}^{k}\sum_{j=1}^{k} X_{ij}\right) = g\left(\sum_{p,\,p'} \mathrm{tr}((\boldsymbol{X} \odot \boldsymbol{I}_p) \cdot \boldsymbol{I}_{p'})\right),$$
where the summation is over all the pairs of conjugate cyclic permutations p and p' of $P_{12\cdots k}$.

Some of these problems are currently under investigation, and we hope to report these findings in the future.

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