Recent developments of the Sinc numerical methods∗

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Abstract

This paper gives a survey of recent developments of the Sinc numerical methods. A variety of Sinc numerical methods have been developed by Stenger and his school. For a certain class of problems, the Sinc numerical methods have the convergence rates of $O(\exp(-\kappa \sqrt{n}))$ with some $\kappa > 0$, where $n$ is the number of nodes or bases used in the methods. Recently it has turned out that the Sinc numerical methods can achieve convergence rates of $O(\exp(-\kappa' n/\log n))$ with some $\kappa' > 0$ for a smaller but still practically meaningful class of problems, and that these convergence rates are best possible. The present paper demonstrates these facts for two Sinc numerical methods: the Sinc approximation and the Sinc-collocation method for two-point boundary value problems.

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1. Introduction

The Sinc approximation for a function $f$ defined on the real line $\mathbb{R}$ is given by

$$f(x) \approx \sum_{j=-N}^{N} f(jh)S(j,h)(x),$$

where $S(j,h)(x)$ is the Sinc function.

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where $S(j,h)(x)$ is the so-called Sinc function defined by

$$S(j,h)(x) = \frac{\sin[(\pi/h)(x - jh)]}{(\pi/h)(x - jh)}$$

and the step size $h$ is suitably chosen for a given positive integer $n = 2N + 1$.

During the last three decades there have been developed a variety of numerical methods based on the Sinc approximation, which are now collectively referred to as Sinc numerical methods (see [8,10,15,16]). The Sinc numerical methods cover

- function approximation,
- approximation of derivatives,
- approximate definite and indefinite integration,
- approximate solution of initial and boundary value ordinary differential equation (ODE) problems,
- approximation and inversion of Fourier and Laplace transforms,
- approximation of Hilbert transforms,
- approximation of definite and indefinite convolution,
- approximate solution of PDEs,
- approximate solution of integral equations,
- construction of conformal maps.

In the standard setup of the Sinc numerical methods, the errors are known to be $O(\exp(-\kappa\sqrt{n}))$ with some $\kappa > 0$, where $n$ is the number of nodes or bases used in the methods. However, Sugihara [18,19] has recently found that the errors in the Sinc numerical methods are $O(\exp(-\kappa'n/\log n))$ with some $\kappa' > 0$, in another setup that is also meaningful both theoretically and practically. It has also been found that the error bounds of $O(\exp(-\kappa'n/\log n))$ are best possible in a certain mathematical sense.

In the present paper we give detailed account of the above findings for two Sinc methods: (1) the Sinc method for function approximation (=the Sinc approximation) and (2) the Sinc-collocation method for linear two-point boundary value problems for second-order ODE. First, we review the Sinc approximation and the Sinc-collocation method for the two-point boundary problem in Sections 2 and 3, respectively. We describe the methods together with the standard convergence analyses, which show the convergence rates of $O(\exp(-\kappa\sqrt{n}))$. Next, in Section 4 we describe the recent developments of the Sinc approximation and the Sinc-collocation method with the improved convergence rates of $O(\exp(-\kappa'n/\log n))$. It is also shown that these convergence rates of $O(\exp(-\kappa'n/\log n))$ are best possible. Finally, we make some remarks in Section 5.

2. Review of the Sinc approximation

Following the standard treatises on the Sinc numerical methods [10,15], we first treat the Sinc approximation on the entire interval $(-\infty, \infty)$, and then proceed to that on a general interval $[a, b]$. 
2.1. Sinc approximation on the entire interval \((-\infty, \infty)\)

The Sinc approximation on the entire interval \((-\infty, \infty)\) is nothing but the Sinc approximation (1) itself given in Introduction.

To state convergence theorems, we introduce notation and definitions.

**Definition 1.** Let \(D_d\) denote the infinite strip region of width \(2d\) \((d > 0)\) in the complex plane:

\[ D_d \equiv \{ z \in \mathbb{C} \mid |\text{Im} z| < d \}. \]

For \(0 < \varepsilon < 1\), let \(D_d(\varepsilon)\) be defined by

\[ D_d(\varepsilon) = \{ z \in \mathbb{C} \mid |\text{Re} z| < 1/\varepsilon, |\text{Im} z| < d(1 - \varepsilon) \}. \]

Let \(H^1(D_d)\) be the Hardy space over the region \(D_d\), i.e., the set of functions \(f\) analytic in \(D_d\) such that

\[ \lim_{\varepsilon \to 0} \int_{\partial D_d(\varepsilon)} |f(z)| |dz| < \infty. \]

The following theorem, due to Stenger [15], presents the convergence result on the Sinc approximation, which shows that the convergence rate is given by \(O(\exp(-\pi d/\sqrt{N}))\) if the function to be approximated belongs to \(H^1(D_d)\) and decays exponentially on the real line.

**Theorem 1** (Stenger [15]). Assume, with positive constants \(\alpha, \beta\) and \(d\), that

1. \(f\) belongs to \(H^1(D_d)\);
2. \(f\) decays exponentially on the real line, that is,

\[ |f(x)| \leq \alpha \exp(-\beta|x|) \text{ for all } x \in \mathbb{R}. \]

Then we have

\[ \sup_{-\infty < x < \infty} \left| f(x) - \sum_{j=-N}^{N} f(jh)S(j, h)(x) \right| \leq CN^{1/2} \exp\left[ - \left( \pi d \beta N \right)^{1/2} \right] \]

for some \(C\), where the mesh size \(h\) is taken as

\[ h = \left( \frac{\pi d}{\beta N} \right)^{1/2}. \]

2.2. Sinc approximation on a general interval \([a, b]\)

The basic idea of the Sinc approximation of \(f(x)\) on the interval \([a, b]\) is first to transform the function \(f(x)\) to that on the entire interval \((-\infty, \infty)\) by means of a properly selected variable transformation \(x = \psi(\xi)\), and then to apply the Sinc approximation on the entire interval \((-\infty, \infty)\) to the transformed function \(f(\psi(\xi))\). A more formal description of the Sinc approximation on the interval \([a, b]\) follows.
Step 1: With a suitably selected variable transformation \( x = \psi(\xi) \) such that
\[
\psi: (\infty, \infty) \to (a, b), \quad \lim_{\xi \to -\infty} \psi(\xi) = a, \quad \lim_{\xi \to \infty} \psi(\xi) = b
\]
transform the function \( f(x) \) defined on \([a, b]\) to \( f(\psi(\xi)) \) on the entire interval \((-\infty, \infty)\).

Step 2: Apply the Sinc approximation on the entire interval \((-\infty, \infty)\) to the transformed function \( f(\psi(\xi)) \) to obtain
\[
f(\psi(\xi)) \approx \sum_{j=-N}^{N} f(\psi(jh))S(j, h)(\xi), \quad -\infty < \xi < \infty
\]
or equivalently,
\[
f(x) \approx \sum_{j=-N}^{N} f(\psi(jh))S(j, h)(\psi^{-1}(x)), \quad a \leq x \leq b.
\]

For the convergence of this approximation, the following theorem follows immediately from Theorem 1.

**Theorem 2** (Stenger [15]). Assume that, for a variable transformation \( z = \psi(\xi) \), the transformed function \( f(\psi(\xi)) \) satisfies assumptions 1 and 2 in Theorem 1 with some \( \alpha, \beta \) and \( d \). Then we have
\[
\sup_{a \leq x \leq b} \left| f(x) - \sum_{j=-N}^{N} f(\psi(jh))S(j, h)(\psi^{-1}(x)) \right| \leq CN^{1/2} \exp[-(\pi d \beta N)^{1/2}]
\]
for some \( C \), where the mesh size \( h \) is taken as
\[
h = \left( \frac{\pi d}{\beta N} \right)^{1/2}.
\]

The variable transformation \( \psi \) in the above theorem is at our disposal. This suggests the possibility that even a function \( f \) with end-point singularity can be approximated successfully by (2) with a suitable choice of the transformation \( \psi \). This is indeed the case, as it is demonstrated below.

**Example 1.** The Sinc approximation yields a good result even for the function that has end-point singularity. For the function
\[
f(x) = x^{1/2}(1-x)^{3/4}, \quad x \in [0, 1]
\]
which has algebraic singularities at \( x = 0 \) and 1, we employ the variable transformation:
\[
x = \psi(\xi) \equiv \frac{1}{2} \tanh \frac{\xi}{2} + \frac{1}{2}.
\]

---

1 This variable transformation was originally proposed for numerical integration [6,13,14,20], and is widely used in the Sinc numerical methods.
With a little calculation, we can verify that the transformed function $f(\psi(\xi))$ satisfies all the assumptions in Theorem 1 with $\beta = \frac{1}{2}$ and $d < \pi$, and hence Theorem 2 applies. (We here omit the value of $\alpha$, for it is irrelevant to both the order of magnitude of the error estimate and the selection of the mesh size $h$. For the same reason we will do so in the following.) Using $\beta = \frac{1}{2}$, $d = \pi/2$, we apply the Sinc approximation on the interval $(-\infty, \infty)$ to the transformed function. Fig. 1 shows the error in the Sinc approximation, which is denoted by “Ordinary-Sinc”. (We estimate the error $\sup_{0 \leq x \leq 1} |f(x) - \sum_{j=-N}^{N} f(\psi(jh))S(j, h)(\psi^{-1}(x))|$ by computing the difference between $f(x)$ and $\sum_{j=-N}^{N} f(\psi(jh))S(j, h)(\psi^{-1}(x))$ at 2000 equally spaced points in [0, 1]. In the following examples, we estimate the errors in a similar way.) For comparison, we also show the error in the polynomial interpolation with the Chebyshev nodes [5], which is denoted by “Chebyshev”. In Fig. 1 we observe that the Sinc approximation yields a good result and that its error behaves like $O(\exp(-\kappa \sqrt{n}))$, as expected from Theorem 2. We also observe that the polynomial interpolation with the Chebyshev nodes yields a poor result.

3. Review of the Sinc-collocation method for two-point boundary value problems

In this section we review the Sinc-collocation method for solving the following linear two-point boundary value problems for second-order ODE with the zero Dirichlet boundary condition:

$$
y''(x) + \mu(x)y'(x) + v(x)y(x) = \sigma(x), \quad x \in (a, b),$$
$$y(a) = y(b) = 0. \quad (5)$$

As in the Sinc approximation, we first describe the Sinc-collocation method on the entire interval $(-\infty, \infty)$, and then proceed to that on a general interval $[a, b]$. 
3.1. Sinc-collocation method on the entire interval \((-\infty, \infty)\)

The problem we consider here is the following:

\[
Ly(x) \equiv y''(x) + \mu(x)y'(x) + v(x)y(x) = \sigma(x), \quad x \in (-\infty, \infty),
\]

\[
\lim_{x \to \pm \infty} y(x) = 0.
\]  \(6\)

We assume an approximate solution \(y_n(x)\) to \((6)\) of the form

\[
y_n(x) \equiv \sum_{j=-N}^{N} w_j S(j,h)(x), \quad n = 2N + 1.
\]

\(7\)

Note that \(y_n(x)\) satisfies the boundary condition in \((6)\) because

\[
\lim_{x \to \pm \infty} S(j,h)(x) = 0.
\]

The coefficients \(\{w_j\}\) are determined from the collocation condition

\[
Ly_n(kh) = \sigma(kh), \quad k = -N, -N + 1, \ldots, N - 1, N,
\]

\(8\)

which yields the following system of linear equations in \(\{w_j\}\):

\[
\sum_{j=-N}^{N} \{\delta_{jk}^{(2)}/h^2 + \mu(kh)\delta_{jk}^{(1)}/h + v(kh)\delta_{jk}^{(0)}\}w_j = \sigma(kh),
\]

\[
k = -N, -N + 1, \ldots, N - 1, N,
\]

\(9\)

where

\[
\delta_{jk}^{(0)} \equiv S(j,h)(kh) = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases}
\]

\[
\delta_{jk}^{(1)} \equiv hS'(j,h)(kh) = \begin{cases} 0 & \text{if } j = k, \\ \frac{(-1)^{k-j}}{(k-j)} & \text{if } j \neq k, \end{cases}
\]

\[
\delta_{jk}^{(2)} \equiv h^2S''(j,h)(kh) = \begin{cases} \frac{\pi^2}{3} & \text{if } j = k, \\ \frac{-2(-1)^{k-j}}{(k-j)^2} & \text{if } j \neq k. \end{cases}
\]

By solving \((9)\) by the Gaussian elimination, we obtain the approximate solution \(y_n(x)\).

For this Sinc-collocation method, the following convergence theorem holds good, which, roughly speaking, states that the Sinc-collocation method converges at the rate of \(O(\exp(-\kappa \sqrt{n}))\), if the solution of problem \((6)\) belongs to \(H^1(\mathcal{D}_d)\) and decays exponentially on the real line.
Theorem 3 (Bialecki [2] and Stenger [15]). Assume that problem (6) has a unique solution \( y(x) \), and that the solution \( y(x) \) is analytic on the real line. Furthermore assume, with positive constants \( \alpha, \beta \) and \( d \), that

1. \( \mu, \mu' \) and \( \nu \) are analytic and bounded in the strip region \( \mathcal{D}_d \); 
2. \( \mu \) takes real values on the real line; 
3. \( \Re\{2\nu(x) - \mu'(x)\} \leq 0 \) for all \( x \in \mathbb{R} \); 
4. \( \sigma \) belongs to \( H^1(\mathcal{D}_d) \) and decays exponentially on the real line, that is, 
   \[ |\sigma(x)| \leq \alpha \exp(-\beta|x|) \quad \text{for all } x \in \mathbb{R}; \]
5. \( y \) belongs to \( H^1(\mathcal{D}_d) \) and decays exponentially on the real line, that is, 
   \[ |y(x)| \leq \alpha \exp(-\beta|x|) \quad \text{for all } x \in \mathbb{R}; \]

Then we have

\[
\sup_{-\infty < x < \infty} |y(x) - y_n(x)| \leq C N^{5/2} \exp\left[-(\pi d \beta N)^{1/2}\right]
\]

for some \( C \), where the mesh size \( h \) in the Sinc-collocation method is taken as

\[
h = \left( \frac{\pi d}{\beta N} \right)^{1/2}.
\]

3.2. Sinc-collocation method on a general interval \([a, b] \)

The two-point boundary value problem (5) on a general interval \([a, b] \) is now considered. The basic idea is first to transform the problem, with a properly selected variable transformation, to that on the entire interval \((-\infty, \infty) \) and then to solve the transformed problem by the Sinc-collocation method described in Section 3.1. A more formal description of the Sinc-collocation method on the interval \([a, b] \) follows.

Step 1: With an adroitly selected variable transformation \( x = \psi(\xi) \) such that

\[
\psi : (-\infty, \infty) \to (a, b), \quad \lim_{\xi \to -\infty} \psi(\xi) = a, \quad \lim_{\xi \to \infty} \psi(\xi) = b,
\]

transform problem (5) to that on the entire interval \((-\infty, \infty) \); the resulting problem is given by

\[
\ddot{\tilde{y}}(\xi) + \tilde{\mu}(\xi) \dot{\tilde{y}}(\xi) + \tilde{\nu}(\xi) \tilde{y}(\xi) = \tilde{\sigma}(\xi), \quad \xi \in (-\infty, \infty),
\]

\[
\lim_{\xi \to \pm\infty} \tilde{y}(\xi) = 0,
\]

where \( \tilde{y}(\xi) \equiv y(\psi(\xi)) \), and

\[
\tilde{\mu}(\xi) \equiv \psi'(\xi) \cdot \mu(\psi(\xi)) - \psi''(\xi)/\psi'(\xi),
\]

\[
\tilde{\nu}(\xi) \equiv (\psi'(\xi))^2 \cdot \nu(\psi(\xi)),
\]

\[
\tilde{\sigma}(\xi) \equiv (\psi'(\xi))^2 \cdot \sigma(\psi(\xi)).
\]
Step 2: Apply the Sinc-collocation method on the entire interval \((-\infty, \infty)\) to the transformed problem (10).

For the convergence of the Sinc-collocation method on the interval \([a, b]\), the following theorem follows from Theorem 3.

**Theorem 4** (Bialecki [2] and Stenger [15]). Assume that problem (5) has a unique solution \(y(x)\), and that \(y(x)\) is analytic on the interval \((a, b)\). Furthermore assume that, for a variable transformation \(z = \psi(\xi)\), the transformed problem satisfies assumptions 1–5 in Theorem 3 with some \(\alpha, \beta\) and \(d\). Then we have

\[
\sup_{a \leq x \leq b} |y(x) - \tilde{y}_n(\psi^{-1}(x))| \leq CN^{5/2} \exp\left(-\frac{\pi d \beta N}{N}\right)
\]

for some \(C\), where \(\tilde{y}_n(\xi)\) is the approximate solution to the transformed problem obtained by the Sinc-collocation method on the interval \((-\infty, \infty)\) with mesh size \(h = \left(\frac{\pi d}{\beta N}\right)^{1/2}\).

This theorem affords a theoretical foundation for the feature of the Sinc-collocation method that the Sinc-collocation method works well even if the solution of the problem has end-point singularity.

**Example 2.** To illustrate that the Sinc-collocation method works well even if the solution of the problem has end-point singularity, we consider the problem [9, Example 3.1]

\[
y''(x) - \frac{3}{4x^2} y(x) = -3\sqrt{x}, \quad x \in (0, 1),
\]

\[
y(0) = y(1) = 0,
\]

which has a regular singular point at \(x = 0\), and has the exact solution \(y(x) = x^{3/2}(1-x)\). We employ the variable transformation (4) used in Example 1. With some calculation we can prove that the transformed problem, which is given by

\[
\tilde{y}''(\xi) + \left(2\psi(\xi) - 1\right)\tilde{y}'(\xi) - \frac{3}{4}(1 - \psi(\xi))^2 \tilde{y}(\xi) = -3(\psi(\xi))^{5/2}(1 - \psi(\xi))^2, \quad \xi \in (-\infty, \infty),
\]

\[
\lim_{\xi \to \pm\infty} \tilde{y}(\xi) = 0,
\]

satisfies all the assumptions in Theorem 3 with \(\beta = 1\) and \(d < \pi\), and hence Theorem 4 applies. Using \(\beta = 1\), \(d = \pi/2\), we apply the Sinc-collocation method on the interval \((-\infty, \infty)\) to the transformed problem. Fig. 2 shows the error in the approximate solution of the Sinc-collocation method, which is denoted by “Ordinary-Sinc”. For comparison, we also show the error in the Chebyshev-collocation solution [3, pp. 161–166], which is denoted by “Chebyshev”. We observe that the Sinc-collocation method yields a good result and that the error behaves like \(O(\exp(-\kappa \sqrt{n}))\), as expected from Theorem 4. We also observe that the Chebyshev-collocation method gives a good result, though its convergence rate is lower than that of the Sinc-collocation method.
Fig. 2. Errors in the Sinc-collocation solution and in the Chebyshev-collocation solution of problem (11).

4. New developments of the Sinc approximation and the Sinc-collocation method

As seen in the preceding sections, the Sinc approximation and the Sinc-collocation method on the entire interval $(-\infty, \infty)$ converge at the rates of $O(\exp(-\kappa \sqrt{n}))$ under the assumption that the function or the solution belongs to $H^1(\mathcal{D}_d)$ and decays exponentially on the real line. A careful examination of the proofs of these results reveals the following fact:

*For the convergence analysis of the Sinc methods, the assumption that functions belong to $H^1(\mathcal{D}_d)$ is mandatory, whereas the assumption that functions decay exponentially on the real line is not mandatory, although some assumption on the decay rate of functions on the real line is required.*

This fact naturally leads us to the study of the Sinc methods for functions that enjoy another type of the decay rate on the real line, such as the double exponential, the triple exponential and so on.

In Sections 4.1 and 4.2, we consider functions that decay double exponentially on the real line and show that the Sinc approximation and the Sinc-collocation method achieve convergence rates of $O(\exp(-\kappa'n/\log n))$ for such functions. In Sections 4.3, we deny the possibility of more rapid decay rate than the double exponential, and establish the optimality of the convergence rates of $O(\exp(-\kappa'n/\log n))$.

4.1. An $O(\exp(-\kappa'n/\log n))$ convergence of the Sinc approximation

The Sinc approximation method applied to a function with double-exponential decay rate on the real line converges at the rate of $O(\exp(-\kappa'n/\log n))$.

**Theorem 5** (Sugihara [19]). *Assume, with positive constants $\alpha$, $\beta$, $\gamma$ and $d$, that*

1. $f$ belongs to $H^1(\mathcal{D}_d)$;
2. $f$ decays double exponentially on the real line, that is,

$$|f(x)| \leq \alpha \exp(-\beta \exp(\gamma|x|)) \quad \text{for all } x \in \mathbb{R}.$$
Then we have

\[ \sup_{-\infty < x < \infty} \left| f(x) - \sum_{j=-N}^{N} f(jh)S(j,h)(x) \right| \leq C \exp \left[ \frac{-\pi d\gamma N}{\log(\pi d\gamma N/\beta)} \right] \]  

(12)

for some C, where the mesh size h is taken as

\[ h = \frac{\log(\pi d\gamma N/\beta)}{\gamma N}. \]

For the convergence of the Sinc approximation on the interval \([a,b]\), the following theorem follows directly from Theorem 5.

**Theorem 6** (Sugihara [19]). Assume that, for a variable transformation \( z = \psi(\zeta) \), the transformed function \( f(\psi(\zeta)) \) satisfies assumptions 1 and 2 in Theorem 5 with some \( \alpha, \beta, \gamma \) and \( d \). Then we have

\[ \sup_{a \leq x \leq b} \left| f(x) - \sum_{j=-N}^{N} f(\psi(jh))S(j,h)(\psi^{-1}(x)) \right| \leq C \exp \left[ \frac{-\pi d\gamma N}{\log(\pi d\gamma N/\beta)} \right] \]

for some C, where the mesh size h is taken as

\[ h = \frac{\log(\pi d\gamma N/\beta)}{\gamma N}. \]

**Example 3.** Function (3) treated in Example 1 is considered again. As a variable transformation, one of the so-called double-exponential transformations\(^2\)

\[ x = \psi_2(\xi) \equiv \frac{1}{2} \tanh \left( \frac{\pi}{2} \sinh \xi \right) + \frac{1}{2} \]  

(13)

is employed. With a little calculation it is shown that the transformed function \( f(\psi_2(\xi)) \) satisfies all the assumptions in Theorem 5 with \( \beta = \pi/4, \gamma = 1 \) and \( d < \pi/2 \), and hence Theorem 6 applies. With \( \beta = \pi/4, \gamma = 1, d = \pi/4 \), the Sinc approximation on the interval \((-\infty, \infty)\) is applied to the transformed function. Fig. 3 shows the error in the Sinc approximation employing the variable transformation (13), which is denoted by “DE-Sinc”. For comparison, the error in the Sinc approximation incorporated with the variable transformation (4) is shown again, which is denoted by “Ordinary-Sinc”. It is observed that the variable transformation (13) enhances the Sinc approximation method remarkably, and that the error in the Sinc approximation with the variable transformation (13) converges to zero at the rate of \( O(\exp(-\kappa n/\log n)) \), as expected from Theorem 6.

\(^2\) This variable transformation was originated for numerical integration in [21] (see also [11]). Recently, its usefulness in the Sinc numerical methods has been recognized [7,12].
4.2. An $O(\exp(-\kappa' n / \log n))$ convergence of the Sinc-collocation method

We here consider the Sinc-collocation method for the problem whose solution decays double exponentially on the real line. We can prove the following theorem, which shows that the convergence rate of the Sinc-collocation method is given by $O(\exp(-\kappa' n / \log n))$.

**Theorem 7** (Sugihara [18]). Assume that problem (6) has a unique solution $y(x)$, and that solution $y(x)$ is analytic on the real line. Furthermore assume, with positive constants $A$, $B$, $\alpha$, $\beta$, $\gamma$ and $\delta$, that

1. $\mu$, $\mu'$ and $\nu$ are analytic in the strip region $\mathcal{D}_d$, and their absolute values on the real line are bounded from above as follows:
   $|\mu(x)|, |\mu'(x)|, |\nu(x)| \leq A \exp(B|x|)$ for all $x \in \mathbb{R}$;

2. $\mu, \mu', \nu, \nu'$ and $\nu y$ belong to $H^1(\mathcal{D}_d)$;

3. $\mu$ takes real values on the real line;

4. $\Re\{2\nu(x) - \mu'(x)\} \leq 0$ for all $x \in \mathbb{R}$;

5. $\sigma$ belongs to $H^1(\mathcal{D}_d)$ and decays double exponentially on the real line, that is,
   $|\sigma(x)| \leq \alpha \exp(-\beta \exp(\gamma|x|))$ for all $x \in \mathbb{R}$;

6. $y$ belongs to $H^1(\mathcal{D}_d)$ and decays double exponentially on the real line, that is,
   $|y(x)| \leq \alpha \exp(-\beta \exp(\gamma|x|))$ for all $x \in \mathbb{R}$.

Then we have

$$\sup_{-\infty < x < \infty} |y(x) - y_n(x)| \leq C(\log N)^{B/\gamma + 3/2} \exp\left[-\pi d \gamma N \frac{\log(\pi d \gamma N/\beta)}{\log(\pi d \gamma N/\beta)}\right]$$
for some $C$, where the mesh size $h$ in the Sinc-collocation method is taken as
\[ h = \frac{\log(\pi d \gamma N / \beta)}{\gamma N}. \]

For the convergence of the Sinc-collocation method on the interval $[a, b]$, we obtain the following theorem from Theorem 7.

**Theorem 8** (Sugihara [18]). Assume that problem (5) has a unique solution $y(x)$, and that $y(x)$ is analytic on the interval $(a, b)$. Furthermore assume that, for a variable transformation $z = \psi(\zeta)$, the transformed problem satisfies assumptions 1–6 in Theorem 7 with some $A, B, \alpha, \beta, \gamma$ and $d$. Then we have
\[ \sup_{a \leq x \leq b} |y(x) - \tilde{y}_n(\psi^{-1}(x))| \leq C(\log N)N^{B/\gamma + 3/2} \exp \left[ -\frac{\pi d \gamma N}{\log(\pi d \gamma N / \beta)} \right] \]
for some $C$, where $\tilde{y}_n(\zeta)$ is the approximate solution to the transformed problem obtained by the Sinc-collocation method on the interval $(-\infty, \infty)$ with mesh size
\[ h = \frac{\log(\pi d \gamma N / \beta)}{\gamma N}. \]

**Example 4.** Problem (11) treated in Example 2 is considered again. Variable transformation (13) used in Example 3 is employed. The transformed problem is given by
\[
\begin{align*}
\tilde{y}''(\zeta) + \{ & (\pi \cosh \zeta) (2\psi(\zeta) - 1) - \tanh \zeta \} \tilde{y}'(\zeta) \\
- & \frac{3}{4} (\pi \cosh \zeta)^2 (1 - \psi(\zeta))^2 \tilde{y}(\zeta) = -3(\pi \cosh \zeta)^2 (\psi(\zeta))^{5/2} (1 - \psi(\zeta))^2, \\
\zeta \in & (-\infty, \infty), \\
\lim_{\zeta \to \pm \infty} \tilde{y}(\zeta) & = 0.
\end{align*}
\]

With lengthy calculation, it can be proved that this problem satisfies all the assumptions in Theorem 7 with $B = 2$, $\beta = \pi/2$, $\gamma = 1$ and $d < \pi/2$, and hence Theorem 8 applies. With $\beta = \pi/2$, $\gamma = 1$, $d = \pi/4$, the Sinc-collocation method on the interval $(-\infty, \infty)$ is applied to the transformed problem. Fig. 4 shows the error in the approximate solution of the Sinc-collocation method employing the variable transformation (13), which is denoted by “DE-Sinc”. For comparison, the error by the Sinc-collocation method with the variable transformation (4) is shown again, which is denoted by “Ordinary-Sinc”. It is observed that the variable transformation (13) enhances the Sinc-collocation method considerably to the convergence rate of $O(\exp(-\kappa'n/\log n))$, as expected from Theorem 8.

4.3. Optimality of the $O(\exp(-\kappa'n/\log n))$ convergence rate

We may naturally be tempted to proceed to the case where the decay rate is more rapid than the double exponential. However, the following astonishing theorem denies this possibility.
Theorem 9 (Sugihara [17]). If a function $f$ satisfies the following two conditions:

1. $f$ belongs to $H^1(D_0)$;
2. the decay rate on the real line satisfies

$$f(x) = O\left(\exp(-\beta \exp(\gamma |x|))\right) \quad \text{as} \quad |x| \to \infty,$$

where $\beta > 0$ and $\gamma > \pi/(2d)$,

then $f \equiv 0$.

This theorem implies that the double exponential decay treated in the preceding sections is extremal and that the convergence rates of $O(\exp(-\kappa n/\log n))$ are best possible.

5. Concluding remarks

5.1. More examples

In Examples 1–4, we saw that the “double-exponential” Sinc methods, i.e., the Sinc numerical methods incorporated with double-exponential transformations, outperform both the “ordinary” Sinc methods, i.e., the Sinc numerical methods combined with ordinary transformations, and the polynomial-based methods. It should be noted, however, that there exist some functions for which this is not the case. Three examples are shown below.

Example 5. Consider the function

$$f(x) = \frac{x(1-x)e^{-x}}{(1/2)^2 + (x - 1/2)^2}, \quad x \in [0, 1]$$

(14)
Fig. 5. Errors for the function $x(1-x)e^{-x}/((1/2)^2 + (x - 1/2)^2)$ in the polynomial interpolation with the Chebyshev nodes and in the Sinc approximation using the variable transformations (4) and (13).

which is analytic on the interval $[0, 1]$, but has poles at $1/2 \pm 1/2i$. To this function we first apply the Sinc approximation with $x = \psi_1(\xi)$ in (4). We take the parameter values $\beta = 1$ and $d = \pi/4$, since the transformed function $f(\psi_1(\xi)))$ satisfies the assumptions in Theorem 1 with $\beta = 1$ and $d < \pi/2$. In Fig. 5 we observe that the error, denoted by “Ordinary-Sinc”, behaves like $O(\exp(-\kappa\sqrt{n}))$, as expected from Theorem 1. Secondly, we apply the Sinc approximation using the double-exponential transformations $x = \psi_2(\xi)$ in (13). We take the parameter values $\beta = \pi/2$, $\gamma = 1$, $d = \pi/12$, because the transformed function $f(\psi_2(\xi))$ satisfies the assumptions in Theorem 5 with $\beta = \pi/2$, $\gamma = 1$ and $d < \pi/6$. Fig. 5 shows that the error in the double-exponential Sinc approximation, denoted by “DE-Sinc”, converges to zero at the rate of $O(\exp(-\kappa n/\log n))$, as expected from Theorem 5. Thirdly, we apply the polynomial interpolation with the Chebyshev nodes, whose error is also shown in Fig. 5, denoted by “Chebyshev”. We observe that the polynomial-based method yields an extremely good result. The theory of polynomial interpolation [4] tells us that the error is $O((1 + \sqrt{2})^{-n})$, which accords with the behavior of the observed error. In general, for functions that are analytic on the underlying interval the error in the polynomial interpolation with the Chebyshev nodes converges to zero at the rate of $O(R^{-n})$ with $R > 1$. Hence in these cases the polynomial-based approximation outdoes both the ordinary Sinc approximation and the double-exponential Sinc approximation.

Example 6. Consider approximating the function

$$f(x) = x^{1/2}(1-x)^{3/4}\sin((log(x/(1-x)))/(2)^{1/2}), \quad x \in [0, 1],$$

where $\sin((x/(2)^{1/2})$ is the Jacobi elliptic function, which has singularities at $K(2m + (2n + 1)i)$ where $K=1.85407467730137\ldots$ and $m$ and $n$ are integers [1]. We first apply the Sinc approximation with the ordinary transformation $x = \psi_1(\xi)$ in (4). Since the assumptions in Theorem 1 with $\beta = 1/2$ and $d < K$ are met, we choose the parameter values $\beta = 1/2$ and $d = K/2$. The result is shown in Fig. 6 by “Ordinary-Sinc”. We observe that the error behaves like $O(\exp(-\kappa n))$, as expected from Theorem 1. Note that the error oscillates due to the oscillation of the approximated function. Secondly, we apply the Sinc approximation with the double-exponential transformations $x = \psi_2(\xi)$ in
(13). In this case Theorem 5, giving a theoretical foundation of the use of the double-exponential Sinc approximation, is not applicable, since assumption 1 is not satisfied for any $d$ because of the singularities of the transformed function lying arbitrarily near the real axis. Though without theoretical justifications we used the same parameter values $\beta = \pi/4, \gamma = 1, d = \pi/4$ as for the double-exponential Sinc approximation in Example 3. The result is shown in Fig. 6 by “DE-Sinc”. The performance of the double-exponential Sinc approximation is comparable to that of the ordinary Sinc approximation, although no theoretical analysis explains this. Thirdly, we apply the polynomial interpolation with the Chebyshev nodes, whose error is also shown in Fig. 6 by “Chebyshev”. Not surprisingly, this gives only a poor result.

**Example 7.** Consider the function

$$f(x) = x^{1/2}(1 - x)^{3/4}\text{sn}(\log(x/(1 - x))), \left(\frac{1}{2}\right)^{1/2}$$

$$= x^{1/2}(1 - x)^{3/4}\text{sn}\left((1/(1 - x) - 1/x)/2, \left(\frac{1}{2}\right)^{1/2}\right), \quad x \in [0, 1],$$

to which we apply the three approximation schemes as above. For the ordinary Sinc approximation we use $\beta = \pi/2$ and $d = \pi/4$, the same values as in Example 1, and for the double-exponential Sinc approximation we use $\beta = \pi/4, \gamma = 1$ and $d = \pi/4$, the same values as in Example 3. Note that these parameter values are not justified theoretically. The results are shown in Fig. 7, which are desperately poor. Approximating this function seems to be beyond the power of the Sinc methods.

**5.2. Near-optimality of the Sinc-collocation method**

Recently, near-optimality of the Sinc approximation has been established in certain function spaces [19]. We suspect that the near-optimality of the Sinc-collocation method can be formulated with some proper setup of function spaces.
5.3. Tractability of the assumptions in Theorems 3 and 7

Assumption 5 in Theorem 3 for the convergence of the ordinary Sinc-collocation method and assumptions 2 and 6 in Theorem 7 for the convergence of the double-exponential Sinc-collocation method seem less tractable, because to check the assumptions requires a knowledge of the true solution. Hence it is desired to replace the assumptions with those which are much more tractable, which is left as a future work.

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References