Journal of Computational and Applied Mathematics 234 (2010) 2578–2586

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Maximum norm error estimates of the Crank–Nicolson scheme for solving a linear moving boundary problem[☆]

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ARTICLE INFO

Article history:
Received 19 November 2009
Received in revised form 15 March 2010

Keywords:
Moving boundaries
Crank–Nicolson scheme
Energy analysis
Stability
Convergence

ABSTRACT

The Crank–Nicolson scheme is considered for solving a linear convection–diffusion equation with moving boundaries. The original problem is transformed into an equivalent system defined on a rectangular region by a linear transformation. Using energy techniques we show that the numerical solutions of the Crank–Nicolson scheme are unconditionally stable and convergent in the maximum norm. Numerical experiments are presented to support our theoretical results.

1. Introduction

Moving boundary problems occur in the mathematical modelling of many physical processes involving diffusion, such as the movement of the shoreline in a sedimentary ocean [1], the drift and collection of oil [2], heat conduction across the solid from a liquid–solid interface to the cooled surface [3] etc. Moving boundary problems also exist in the swelling of biological tissues [4,5] and the swelling of polymers [6].

Due to the difficulties in obtaining analytical solutions, it is important to develop numerical methods for moving boundary problems. Recently, more finite difference schemes have been used for dealing with moving boundary problems [7–10], but there are no analyses of the convergence and stability of difference schemes. In addition, Baines and Hubbard [11] established a moving mesh finite element algorithm for moving boundary problems. Immersed interface methods and immersed boundary methods also have been used to deal with moving boundary problems [12–14].

The linear convection–diffusion equation

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(au-\kappa\frac{\partial u}{\partial x}\right)=g(x,t),\quad (x,t)\in Q_T \tag{1.1}$$

along with the initial value condition

$$u(x,0)=u_0(x),\quad x\in\Omega_0, \tag{1.2}$$

and the moving boundary value conditions

$$u(x,t)=\Phi(x,t),\quad x\in\partial\Omega_t,\ 0<t\le T \tag{1.3}$$

☆ This work is supported by the NSF of China (No. 10871044) and by China Postdoctoral Science Foundation (No. 20070420956).

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0377-0427/$ – see front matter
doi:10.1016/j.cam.2010.03.024

as a mathematical model is widely used in various applications, where $Q_T = \left\{(x,t) \in R^2, x \in \Omega_t, t \in [0,T]\right\}$, Ω_t is an interval in R for each $t \in [0,T]$, a and κ are two positive constants.

Moving mesh method is very effective for dealing with moving boundary problems. Mackenzie and Mekwi [15] discussed the stability and convergence of time integration schemes for the solution of (1.1)–(1.3) (they took $g(x,t) = \Phi(x,t) = 0$). Using variable mesh method and energy techniques they showed that the backward Euler scheme is unconditionally stable in a mesh-dependent L_2 norm, but the Crank–Nicolson scheme is only conditionally stable.

Sun [2] gave a three-level linearized and weak coupled difference scheme using a moving mesh for the model of oil drift and collection with moving boundary value, and analyzed the solvability and convergence of the difference scheme. He proved that the convergence order of the difference scheme is $O(\tau^2 + h^2)$.

In this article, the Crank–Nicolson scheme for the linear convection–diffusion equation with moving boundaries (1.1)–(1.3) is analyzed. A linear transformation is introduced in our analysis to transform (1.1) to an equivalent equation defined on a rectangle region. It is proved that the Crank–Nicolson scheme for (1.1)–(1.3) is unconditionally stable and convergent in the maximum norm. The convergence order is $O(\tau^2 + h^2)$.

The contents will be organized as follows. In the next section, an equivalent system defined on a rectangular region is achieved by making a linear transformation to Eq. (1.1). Mesh generation and some notations are also introduced in this section. The Crank–Nicolson scheme is constructed for the equivalent system in Section 3. Section 4 presents the energy analysis for the Crank–Nicolson scheme and gives the main results of the article. Numerical experiments are provided to support our theoretical results in Section 5.

2. A linear transformation and mesh generation

Assume that the initial value u_0 and exterior force g are regular enough in (1.1)–(1.3), the boundary value Φ is piecewise smooth, $u_0(x) = \Phi(x,0)$, $x \in \partial\Omega_0$, and the domain Ω_t can be defined as $\Omega_t = [x_l(t), x_r(t)]$, where the functions $x_l(t), x_r(t) \in C^1[0,T]$, and $x_l(t) < x_r(t)$ for every $t \in [0,T]$.

Introduce a linear transformation

$$\begin{cases} x = (1-\xi)x_l(t) + \xi x_r(t), & 0 \le \xi \le 1 \\ t = t, & 0 \le t \le T \end{cases} \tag{2.1}$$

and denote $w(\xi,t) = u\left((1-\xi)x_l(t) + \xi x_r(t), t\right)$, $G(\xi,t) = g\left((1-\xi)x_l(t) + \xi x_r(t), t\right)$. Then we have

$$\begin{aligned} &\frac{\partial w}{\partial t} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial u}{\partial t}, \qquad \frac{\partial w}{\partial \xi} = \frac{\partial u}{\partial x} x_\xi(t), \\ &\frac{\partial^2 w}{\partial \xi^2} = \frac{\partial u}{\partial x} x_{\xi\xi}(t) + \frac{\partial^2 u}{\partial x^2} x_\xi^2(t). \end{aligned} \tag{2.2}$$

It is obvious that

$$x_\xi(t) = x_r(t) - x_l(t) > 0, \qquad x_{\xi\xi}(t) = 0. \tag{2.3}$$

Performing in (1.1)–(1.3) the substitution (2.1) and then using (2.2)–(2.3) we obtain

$$\frac{\partial w}{\partial t} - \frac{\kappa}{(x_\xi)^2}\frac{\partial^2 w}{\partial \xi^2} - \frac{1}{x_\xi}\left(\frac{\partial x}{\partial t} - a\right)\frac{\partial w}{\partial \xi} = G(\xi,t), \quad (\xi,t) \in Q_R, \tag{2.4}$$

along with the initial value condition

$$w(\xi,0) = w_0(\xi), \quad 0 \le \xi \le 1 \tag{2.5}$$

and the boundary value conditions

$$w(0,t) = \phi_1(t), \qquad w(1,t) = \phi_2(t), \quad 0 < t \le T \tag{2.6}$$

where $Q_R = \left\{(\xi,t) \in R^2,\ 0 \le \xi \le 1, 0 < t < T\right\}$, $w_0(\xi) = u_0\left((1-\xi)x_l(0) + \xi x_r(0)\right)$, $\phi_1(t) = \Phi\left(x_l(t), t\right)$, $\phi_2(t) = \Phi\left(x_r(t), t\right)$.

Summarizing above results, we obtain the following theorem.

Theorem 1. *Assume that the interval Ω_t can be defined as $\Omega_t = [x_l(t), x_r(t)]$, where the functions $x_l(t), x_r(t) \in C^1[0,T]$. If $x_l(t) < x_r(t)$ for every $t \in [0,T]$, then the problem (1.1)–(1.3) is equivalent to (2.4)–(2.6).*

Let

$$\Omega_h(t) \equiv \{x_i(t) \mid x_i(t) = ih(t),\ 0 \le i \le M\}$$

be a variable mesh of the interval $\Omega_t = [x_l(t), x_r(t)]$ with $h(t) = \frac{1}{M}[x_r(t) - x_l(t)]$, where $t \in [0,T]$ is fixed. Let

$$\widetilde{\Omega}_h \equiv \{\xi_i \mid \xi_i = ih,\ 0 \le i \le M\}$$

be a uniform mesh of the interval [0, 1] with $h = 1/M$ and

$$\Omega_\tau \equiv \{t_n \mid t_n = n\tau,\ 0 \le n \le N\},$$

where $\tau = T/N$. Further, we denote

$$\Omega_h^\tau = \Omega_h(t) \times \Omega_\tau = \{(x_i^n, t_n) \mid x_i^n = ih(t_n),\ t_n = n\tau,\ 0 \le i \le M,\ 0 \le n \le N\},$$
$$\widetilde{\Omega}_h^\tau = \widetilde{\Omega}_h \times \Omega_\tau = \{(\xi_i, t_n) \mid \xi_i = ih,\ t_n = n\tau,\ 0 \le i \le M,\ 0 \le n \le N\}.$$

In addition, let

$$t_{n+\frac{1}{2}} = \frac{1}{2}(t_n + t_{n+1}), \qquad \xi_{i+\frac{1}{2}} = \frac{1}{2}(\xi_i + \xi_{i+1}), \qquad x^n_{i+\frac{1}{2}} = \frac{1}{2}(x_i^n + x_{i+1}^n).$$

It is obvious that

$$x_i^n = (1 - \xi_i)x_l(t_n) + \xi_i x_r(t_n), \qquad h(t_n) = h[x_r(t_n) - x_l(t_n)]. \tag{2.7}$$

Suppose $w = \{w_i^n \mid 0 \le i \le M, 0 \le n \le N\}$ is a grid function on $\widetilde{\Omega}_h^\tau$. Introduce the following notations:

$$w_i^{n+1/2} = \frac{1}{2}(w_i^n + w_i^{n+1}), \qquad \delta_t w_i^{n+1/2} = \frac{1}{\tau}(w_i^{n+1} - w_i^n)$$
$$\delta_\xi w^n_{i+\frac{1}{2}} = \frac{w^n_{i+1} - w^n_i}{h}, \qquad \delta_\xi^2 w_i^n = \frac{w^n_{i+1} - 2w^n_i + w^n_{i-1}}{h^2},$$
$$\|w^n\|_\infty = \max_{0\le i\le M} |w_i^n|, \qquad |w^n|_1 = \sqrt{h\sum_{i=0}^{M-1}\left(\delta_\xi w^n_{i+\frac{1}{2}}\right)^2},$$
$$\|w^n\| = \sqrt{h\left[\frac{1}{2}(w_0^n)^2 + \sum_{i=1}^{M-1}(w_i^n)^2 + \frac{1}{2}(w_M^n)^2\right]}.$$

For deriving the maximum norm estimate of the Crank–Nicolson scheme, we need the following lemma [16].

Lemma 2. *If $w^n = (w_0^n, w_1^n, \ldots, w_M^n)$ is a grid function on $\widetilde{\Omega}_h$ and satisfies $w_0^n = w_M^n = 0$, then*

$$\|w^n\|_\infty \le \frac{1}{2}|w^n|_1. \tag{2.8}$$

3. Construction of the Crank–Nicolson scheme

We denote

$$L(t) = x_r(t) - x_l(t) \tag{3.1}$$

and

$$c_1 = \max_{0\le t\le T} |L(t)|, \qquad c_2 = \max_{0\le t\le T} \max\{|x_l'(t)|, |x_r'(t)|\}. \tag{3.2}$$

According to (2.3), we have

$$x_\xi(t) = L(t), \qquad x_{\xi\xi}(t) = 0. \tag{3.3}$$

We consider the difference discretization of the problem (2.4)–(2.6). Define the grid functions

$$W_i^n = w(\xi_i, t_n), \qquad G_i^{n+\frac{1}{2}} = G(\xi_i, t_{n+\frac{1}{2}}), \qquad L_{n+\frac{1}{2}} = L(t_{n+\frac{1}{2}}), \quad \xi_i \in \widetilde{\Omega}_h,\ t_n \in \Omega_\tau.$$

The construction of Crank–Nicolson method is standard.

Consider the Eq. (2.4) at the point $(\xi_i, t_{n+\frac{1}{2}})$, we have

$$\frac{\partial w(\xi_i, t_{n+\frac{1}{2}})}{\partial t} - \frac{\kappa}{L^2(t_{n+\frac{1}{2}})}\frac{\partial^2 w(\xi_i, t_{n+\frac{1}{2}})}{\partial \xi^2} - \frac{1}{L(t_{n+\frac{1}{2}})}\left(\frac{\partial x(\xi_i, t_{n+\frac{1}{2}})}{\partial t} - a\right)\frac{\partial w(\xi_i, t_{n+\frac{1}{2}})}{\partial \xi} = G_i^{n+\frac{1}{2}}. \tag{3.4}$$

Using the method of Taylor expansion, one can derive that

$$\delta_t W_i^{n+\frac{1}{2}} - \frac{\kappa}{(L_{n+\frac{1}{2}})^2}\delta_\xi^2 W_i^{n+\frac{1}{2}} - \frac{1}{2L_{n+\frac{1}{2}}}(\delta_t x_i^{n+\frac{1}{2}} - a)\left(\delta_\xi W^{n+\frac{1}{2}}_{i-\frac{1}{2}} + \delta_\xi W^{n+\frac{1}{2}}_{i+\frac{1}{2}}\right)$$
$$= G_i^{n+\frac{1}{2}} + R_i^{n+\frac{1}{2}}, \quad 1 \le i \le M-1,\ 0 \le n \le N-1, \tag{3.5}$$

and there exists a constant $c > 0$ such that

$$|R_i^{n+\frac{1}{2}}| \le c(h^2 + \tau^2), \tag{3.6}$$

where $\delta_t x_i^{n+\frac{1}{2}} = \frac{1}{\tau}(x_i^{n+1} - x_i^n)$. In addition, it follows from (2.5) and (2.6) that

$$W_i^0 = w_0(\xi_i), \quad 0 \le i \le M, \tag{3.7}$$

$$W_0^n = \phi_1(t_n), \qquad W_M^n = \phi_2(t_n), \quad 1 \le n \le N. \tag{3.8}$$

Omitting the small terms $R_i^{n+\frac{1}{2}}$ in (3.5), and replacing the grid function W_i^n with the numerical approximation w_i^n, the Crank–Nicolson scheme for the system (2.4)–(2.6) is obtained as follows

$$\delta_t w_i^{n+\frac{1}{2}} - \frac{\kappa}{(L_{n+\frac{1}{2}})^2}\delta_\xi^2 w_i^{n+\frac{1}{2}} - \frac{(\delta_t x_i^{n+\frac{1}{2}} - a)}{2L_{n+\frac{1}{2}}}\left(\delta_\xi w_{i-\frac{1}{2}}^{n+\frac{1}{2}} + \delta_\xi w_{i+\frac{1}{2}}^{n+\frac{1}{2}}\right) = G_i^{n+\frac{1}{2}}, \quad 1 \le i \le M-1,\ 0 \le n \le N-1, \tag{3.9}$$

$$w_i^0 = w_0(\xi_i), \quad 0 \le i \le M, \tag{3.10}$$

$$w_0^n = \phi_1(t_n), \qquad w_M^n = \phi_2(t_n), \quad 1 \le n \le N. \tag{3.11}$$

Define a grid function $u = \{u_i^n\}$ on Ω_h^τ by

$$u_i^n = w_i^n, \quad 0 \le i \le M,\ 0 \le n \le N \tag{3.12}$$

and let u_i^n be an approximation of $u(x_i^n, t_n)$.

Denote

$$u_i^{n+\frac{1}{2}} = \frac{1}{2}(u_i^n + u_i^{n+1}), \qquad \delta_x u_{i+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{h(t_{n+\frac{1}{2}})}\left(u_{i+1}^{n+\frac{1}{2}} - u_i^{n+\frac{1}{2}}\right),$$

$$\delta_t u_i^{n+\frac{1}{2}} = \frac{1}{\tau}\left(u_i^{n+1} - u_i^n\right), \qquad \delta_x^2 u_i^{n+\frac{1}{2}} = \frac{1}{\left[h(t_{n+\frac{1}{2}})\right]^2}\left(u_{i+1}^{n+\frac{1}{2}} - 2u_i^{n+\frac{1}{2}} + u_{i-1}^{n+\frac{1}{2}}\right).$$

Then the difference scheme (3.9)–(3.11) can be written as

$$\delta_t u_i^{n+\frac{1}{2}} - \left(\delta_t x_i^{n+\frac{1}{2}}\right)\cdot\frac{1}{2}\left(\delta_x u_{i-\frac{1}{2}}^{n+\frac{1}{2}} + \delta_x u_{i+\frac{1}{2}}^{n+\frac{1}{2}}\right) + a\cdot\frac{1}{2}\left(\delta_x u_{i-\frac{1}{2}}^{n+\frac{1}{2}} + \delta_x u_{i+\frac{1}{2}}^{n+\frac{1}{2}}\right) - \kappa\delta_x^2 u_i^{n+\frac{1}{2}}$$

$$= G_i^{n+\frac{1}{2}}, \quad 1 \le i \le M-1,\ 0 \le n \le N-1, \tag{3.13}$$

$$u_i^0 = u_0(x_i^0), \quad 0 \le i \le M, \tag{3.14}$$

$$u_0^n = \phi_1(t_n), \qquad u_M^n = \phi_2(t_n), \quad 1 \le n \le N. \tag{3.15}$$

We can interpret the difference equation (3.13) as an approximation of (1.1) by

$$\delta_t u_i^{n+\frac{1}{2}} - \left(\delta_t x_i^{n+\frac{1}{2}}\right)\frac{\delta_x u_{i-\frac{1}{2}}^{n+\frac{1}{2}} + \delta_x u_{i+\frac{1}{2}}^{n+\frac{1}{2}}}{2} \sim \frac{\partial u\left(x_i^{n+\frac{1}{2}}, t_{n+\frac{1}{2}}\right)}{\partial t},$$

$$a\frac{\delta_x u_{i-\frac{1}{2}}^{n+\frac{1}{2}} + \delta_x u_{i+\frac{1}{2}}^{n+\frac{1}{2}}}{2} \sim a\frac{\partial u\left(x_i^{n+\frac{1}{2}}, t_{n+\frac{1}{2}}\right)}{\partial x},$$

$$-\kappa\delta_x^2 u_i^{n+\frac{1}{2}} \sim -\kappa\frac{\partial^2 u\left(x_i^{n+\frac{1}{2}}, t_{n+\frac{1}{2}}\right)}{\partial x^2}.$$

4. Maximum norm error estimates for the Crank–Nicolson scheme

In the following, we prove the solvability, stability and convergence of the Crank–Nicolson scheme (3.9)–(3.11).

Theorem 3. *Let $\{w_i^n \mid 0 \le i \le M,\ 0 \le n \le N\}$ be the solution of the difference scheme (3.9)–(3.11). If $w_0^n = w_M^n = 0,\ 0 \le n \le N$, then we have*

$$\|w^n\|_\infty \le \frac{1}{2}e^{\frac{3}{2}Tc^*}\left(|w^0|_1^2 + \frac{3(c_1)^2}{2\kappa}\tau\sum_{k=0}^{n-1}\|G^{k+\frac{1}{2}}\|^2\right)^{\frac{1}{2}}, \quad 0 \le n \le N, \tag{4.1}$$

where

$$c^* = \frac{(c_2 + |a|)^2}{2\kappa}. \tag{4.2}$$

Proof. Multiplying (3.9) by $h\delta_t w_i^{n+\frac{1}{2}}$ and summing up for i from 1 to $M-1$, we obtain

$$h\sum_{i=1}^{M-1}\left(\delta_t w_i^{n+\frac{1}{2}}\right)^2-\frac{\kappa}{(L_{n+\frac{1}{2}})^2}h\sum_{i=1}^{M-1}\left(\delta_t w_i^{n+\frac{1}{2}}\right)\left(\delta_\xi^2 w_i^{n+\frac{1}{2}}\right)-\frac{1}{2L_{n+\frac{1}{2}}}h\sum_{i=1}^{M-1}(\delta_t x_i^{n+\frac{1}{2}}-a)$$
$$\times\left(\delta_\xi w_{i-\frac{1}{2}}^{n+\frac{1}{2}}+\delta_\xi w_{i+\frac{1}{2}}^{n+\frac{1}{2}}\right)\left(\delta_t w_i^{n+\frac{1}{2}}\right)=h\sum_{i=1}^{M-1}\left(\delta_t w_i^{n+\frac{1}{2}}\right)G_i^{n+\frac{1}{2}}. \tag{4.3}$$

According to the assumption $w_0^n=w_M^n=0,\ 0\le n\le N$, we get that the time difference quotient $\delta_t w_i^{n+\frac{1}{2}}$ vanishes at the boundary points. Applying the discrete Green formula, we have

$$-h\sum_{i=1}^{M-1}\left(\delta_t w_i^{n+\frac{1}{2}}\right)\left(\delta_\xi^2 w_i^{n+\frac{1}{2}}\right)=h\sum_{i=0}^{M-1}\left(\delta_\xi w_{i+\frac{1}{2}}^{n+\frac{1}{2}}\right)\left(\delta_t(\delta_\xi w_{i+\frac{1}{2}}^{n+\frac{1}{2}})\right)$$
$$=\frac{1}{2\tau}\sum_{i=0}^{M-1}\left[\left(\delta_\xi w_{i+\frac{1}{2}}^{n+1}\right)^2-\left(\delta_\xi w_{i+\frac{1}{2}}^{n}\right)^2\right]=\frac{1}{2\tau}\left(|w^{n+1}|_1^2-|w^n|_1^2\right).$$

By using the inequality $ab\le\frac{1}{2}(a^2+b^2)$, we obtain

$$\frac{1}{2L_{n+\frac{1}{2}}}h\sum_{i=1}^{M-1}(\delta_t x_i^{n+\frac{1}{2}}-a)\left[\delta_\xi w_{i+\frac{1}{2}}^{n+\frac{1}{2}}+\delta_\xi w_{i-\frac{1}{2}}^{n+\frac{1}{2}}\right]\left(\delta_t w_i^{n+\frac{1}{2}}\right)$$
$$\le\frac{1}{2}h\sum_{i=1}^{M-1}\left\{\left(\delta_t w_i^{n+\frac{1}{2}}\right)^2+\frac{(\delta_t x_i^{n+\frac{1}{2}}-a)^2}{2(L_{n+\frac{1}{2}})^2}\left[\left(\delta_\xi w_{i-\frac{1}{2}}^{n+\frac{1}{2}}\right)^2+\left(\delta_\xi w_{i+\frac{1}{2}}^{n+\frac{1}{2}}\right)^2\right]\right\} \tag{4.4}$$

and

$$h\sum_{i=1}^{M-1}\left(\delta_t w_i^{n+\frac{1}{2}}\right)G_i^{n+\frac{1}{2}}\le h\sum_{i=1}^{M-1}\left[\frac{1}{2}\left(\delta_t w_i^{n+\frac{1}{2}}\right)^2+\frac{1}{2}\left(G_i^{n+\frac{1}{2}}\right)^2\right]=\frac{1}{2}\|\delta_t w^{n+\frac{1}{2}}\|^2+\frac{1}{2}\|G^{n+\frac{1}{2}}\|^2. \tag{4.5}$$

Substituting (4.4)–(4.5) into (4.3), we get

$$\frac{1}{2\tau}\frac{\kappa}{(L_{n+\frac{1}{2}})^2}\left[|w^{n+1}|_1^2-|w^n|_1^2\right]\le\frac{1}{4}h\sum_{i=1}^{M-1}\frac{\left(\delta_t x_i^{n+\frac{1}{2}}-a\right)^2}{\left(L_{n+\frac{1}{2}}\right)^2}\left[\left(\delta_\xi w_{i-\frac{1}{2}}^{n+\frac{1}{2}}\right)^2+\left(\delta_\xi w_{i+\frac{1}{2}}^{n+\frac{1}{2}}\right)^2\right]+\frac{1}{2}\|G^{n+\frac{1}{2}}\|^2.$$

Using (3.2) and (4.2), we have

$$\frac{1}{\tau}\left(|w^{n+1}|_1^2-|w^n|_1^2\right)\le\frac{h}{2\kappa}\sum_{i=1}^{M-1}(\delta_t x_i^{n+\frac{1}{2}}-a)^2\left[\left(\delta_\xi w_{i-\frac{1}{2}}^{n+\frac{1}{2}}\right)^2+\left(\delta_\xi w_{i+\frac{1}{2}}^{n+\frac{1}{2}}\right)^2\right]+\frac{(L_{n+\frac{1}{2}})^2}{\kappa}\|G^{n+\frac{1}{2}}\|^2$$
$$\le\frac{(c_2+a)^2}{\kappa}|w^{n+\frac{1}{2}}|_1^2+\frac{c_1^2}{\kappa}\|G^{n+\frac{1}{2}}\|^2$$
$$\le\frac{(c_2+a)^2}{2\kappa}\left(|w^n|_1^2+|w^{n+1}|_1^2\right)+\frac{c_1^2}{\kappa}\|G^{n+\frac{1}{2}}\|^2$$

and

$$\left(1-\tau c^*\right)|w^{n+1}|_1^2\le\left(1+\tau c^*\right)|w^n|_1^2+\frac{(c_1)^2}{\kappa}\tau\|G^{n+\frac{1}{2}}\|^2,\quad 0\le n\le N-1.$$

Supposing the time-step size $\tau\le 1/(3c^*)$, we obtain

$$|w^{n+1}|_1^2\le(1+3\tau c^*)|w^n|_1^2+\frac{3\tau(c_1)^2}{2\kappa}\|G^{n+\frac{1}{2}}\|^2,\quad 0\le n\le N-1.$$

Thus, the discrete Gronwall inequality yields

$$|w^n|_1^2\le e^{3c^*n\tau}\left[|w^0|_1^2+\frac{3(c_1)^2}{2\kappa}\tau\sum_{k=0}^{n-1}\|G^{k+\frac{1}{2}}\|^2\right],\quad 0\le n\le N.$$

Applying Lemma 2, we can obtain the estimate (4.1). □

Theorem 4. *The difference scheme (3.9)–(3.11) is uniquely solvable.*

Proof. Since (3.9)–(3.11) is a system of linear algebraic equations at each time level, it suffices to show that the corresponding homogeneous equations:

$$\delta_t w_i^{n+\frac{1}{2}} - \frac{\kappa}{(L_{n+\frac{1}{2}})^2}\delta_\xi^2 w_i^{n+\frac{1}{2}} = \frac{(\delta_t x_i^{n+\frac{1}{2}} - a)}{2L_{n+\frac{1}{2}}}\left(\delta_\xi w_{i+\frac{1}{2}}^{n+\frac{1}{2}} + \delta_\xi w_{i-\frac{1}{2}}^{n+\frac{1}{2}}\right), \quad 1 \le i \le M-1,\ 0 \le n \le N-1 \tag{4.6}$$

$$w_i^0 = 0, \quad 0 \le i \le M, \tag{4.7}$$

$$w_0^n = w_M^n = 0, \quad 1 \le n \le N \tag{4.8}$$

have only a trivial solution. Using Theorem 3, one gets

$$\|w^n\|_\infty = 0, \quad 0 \le n \le N.$$

This implies that $w_i^n = 0,\ 0 \le i \le M,\ 0 \le n \le N$. The proof is complete. □

Theorem 5. *Let the system (1.1)–(1.3) have the solution $u(x,t) \in C_{x,t}^{4,3}(Q_T)$ and $\{w_i^n \mid 0 \le i \le M,\ 0 \le n \le N\}$ be the solution of the difference scheme (3.9)–(3.11). Let*

$$u_i^n = w_i^n, \quad 0 \le i \le M,\ 0 \le n \le N.$$

Then the estimate

$$\max_{0\le i\le M} |u(x_i^n, t_n) - u_i^n| \le \frac{1}{2}\mathrm{e}^{\frac{3}{2}Tc^*}\sqrt{\frac{3T}{2\kappa}}c_1 c(h^2 + \tau^2) \tag{4.9}$$

holds for $n\tau \le T$.

Proof. Denote

$$\tilde{w}_i^n = W_i^n - w_i^n, \quad 0 \le i \le M,\ 0 \le n \le N.$$

Subtracting (3.9)–(3.11) from (3.5), (3.7) and (3.8), we obtain the error system of equations

$$\delta_t \tilde{w}_i^{n+\frac{1}{2}} - \frac{\kappa}{(L_{n+\frac{1}{2}})^2}\delta_\xi^2 \tilde{w}_i^{n+\frac{1}{2}} - \frac{(\delta_t x_i^{n+\frac{1}{2}} - a)}{2L_{n+\frac{1}{2}}}\left(\delta_\xi \tilde{w}_{i-\frac{1}{2}}^{n+\frac{1}{2}} + \delta_\xi \tilde{w}_{i+\frac{1}{2}}^{n+\frac{1}{2}}\right) = R_i^{n+\frac{1}{2}}, \quad 1 \le i \le M-1,\ 0 \le n \le N-1,$$

$$\tilde{w}_i^0 = 0, \quad 0 \le i \le M,$$
$$\tilde{w}_0^n = \tilde{w}_M^n = 0, \quad 1 \le n \le N.$$

According to Theorem 3, we have

$$\|\tilde{w}^n\|_\infty \le \frac{1}{2}\mathrm{e}^{\frac{3}{2}c^*T}\left\{\frac{3(c_1)^2}{2\kappa}\tau\sum_{k=0}^{n-1}\left[h\sum_{i=1}^{M-1}(R_i^{k+\frac{1}{2}})^2\right]\right\}^{\frac{1}{2}}, \quad 0 \le n \le N.$$

Using (3.6), we arrive at

$$\|\tilde{w}^n\|_\infty \le \frac{1}{2}\mathrm{e}^{\frac{3}{2}Tc^*}\sqrt{\frac{3T}{2\kappa}}c_1 c(h^2 + \tau^2), \quad 0 \le n \le N.$$

Since $u(x_i^n, t_n) - u_i^n = w(\xi_i, t_n) - w_i^n = \tilde{w}_i^n$, this completes the proof. □

5. Numerical experiments

To verify our discussions in the previous sections, we solve the problem (1.1)–(1.3) numerically by the Crank–Nicolson scheme (3.9)–(3.11). Denote

$$e(h,\tau) = \max_{1\le n\le N}\max_{0\le i\le M}|u(x_i^n, t_n) - w_i^n|.$$

Taking the domain $\Omega_t = [x_l(t), x_r(t)]$ and $a = 0,\ \kappa = 0.05$, we consider the following cases.

Case 1. $x_l(t) = 1 - \mathrm{e}^{\frac{t}{2}},\ x_r(t) = \mathrm{e}^{\frac{t}{2}}$, initial data $u_0(x) = x\sin(x-1)$, boundary values $\phi_1(t) = \phi_2(t) = 0$ and forcing function

$$g(x,t) = \sin(x - \mathrm{e}^{\frac{t}{2}})\left(0.5\mathrm{e}^{\frac{t}{2}} + 0.05(x - 1 + \mathrm{e}^{\frac{t}{2}})\right) - \cos(x - \mathrm{e}^{\frac{t}{2}})\left(0.5\mathrm{e}^{\frac{t}{2}}(x - 1 + \mathrm{e}^{\frac{t}{2}}) + 0.1\right).$$

In this case, the analytic solution of the system (1.1)–(1.3) is $u(x,t) = (x - 1 + \mathrm{e}^{\frac{t}{2}})\sin(x - \mathrm{e}^{\frac{t}{2}})$.

Table 1
Convergence of Crank–Nicolson scheme in maximum norm for Case 1.

h	τ	$e(h, \tau)$	$\log_2 \frac{e(h,\tau)}{e(2h,2\tau)}$
1/64	1/64	1.2583e−4	*
1/128	1/128	3.1515e−5	1.9974
1/256	1/256	7.8798e−6	1.9998
1/512	1/512	1.9700e−6	2.0000
1/1024	1/1024	4.9251e−7	2.0000

Table 2
Convergence of Crank–Nicolson scheme in maximum norm for Case 2.

h	τ	$e(h, \tau)$	$\log_2 \frac{e(h,\tau)}{e(2h,2\tau)}$
1/64	1/64	2.1771e−5	*
1/128	1/128	5.4436e−6	1.9998
1/256	1/256	1.3609e−6	2.0000
1/512	1/512	3.4019e−7	2.0001
1/1024	1/1024	8.5047e−8	2.0000
1/2048	1/2048	2.1262e−8	2.0000

Table 3
Convergence of Crank–Nicolson scheme in maximum norm for Case 3.

h	τ	$e(h, \tau)$	$\log_2 \frac{e(h,\tau)}{e(2h,2\tau)}$
1/64	1/64	7.9416e−5	*
1/128	1/128	1.9860e−5	1.9996
1/256	1/256	4.9655e−6	1.9999
1/512	1/512	1.2414e−6	2.0000
1/1024	1/1024	3.1035e−7	2.0000

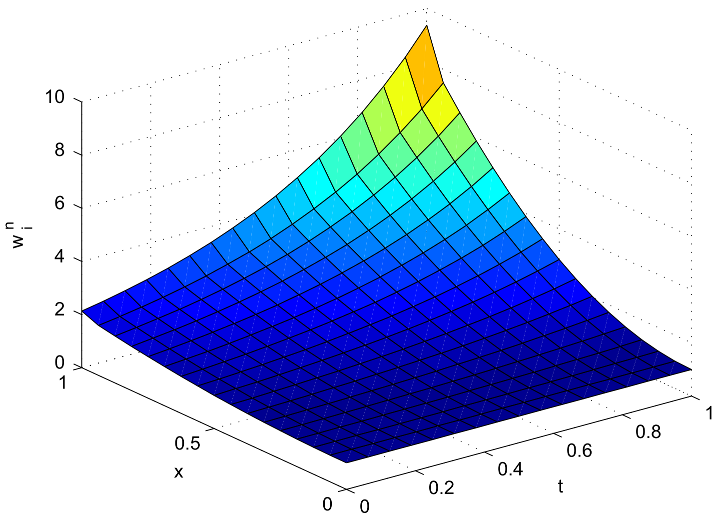

Fig. 1. Numerical solution of Crank–Nicolson scheme with $h = \tau = 1/16$ for Case 3.

Case 2. $x_l(t) = \frac{1}{2}(1 - e^{-2t}),\ x_r(t) = \frac{1}{2}(1 + e^{-2t})$, initial data $u_0(x) = (x - 1)\sin x$, boundary values $\phi_1(t) = \phi_2(t) = 0$ and forcing function

$$g(x, t) = \sin(x + 0.5e^{-2t} - 0.5)\left(e^{-2t} + 0.05(x - 0.5e^{-2t} - 0.5)\right) - \cos(x + 0.5e^{-2t} - 0.5)\left(e^{-2t}(x - 0.5e^{-2t} - 0.5) + 0.1\right).$$

In this case, the analytic solution of the system (1.1)–(1.3) is $u(x, t) = (x - 0.5e^{-2t} - 0.5)\sin(x + 0.5e^{-2t} - 0.5)$.

Case 3. $x_l(t) = 0,\quad x_r(t) = e^{\frac{t}{2}}$, initial data $u_0(x) = x^2 + 1$, boundary values $\phi_1(t) = 1,\ \phi_2(t) = e^{2t} + 1$ and forcing function

$$g(x, t) = (x^2 - 0.1)e^t.$$

In this case, the analytic solution of the system (1.1)–(1.3) is $u(x, t) = x^2e^t + 1$.

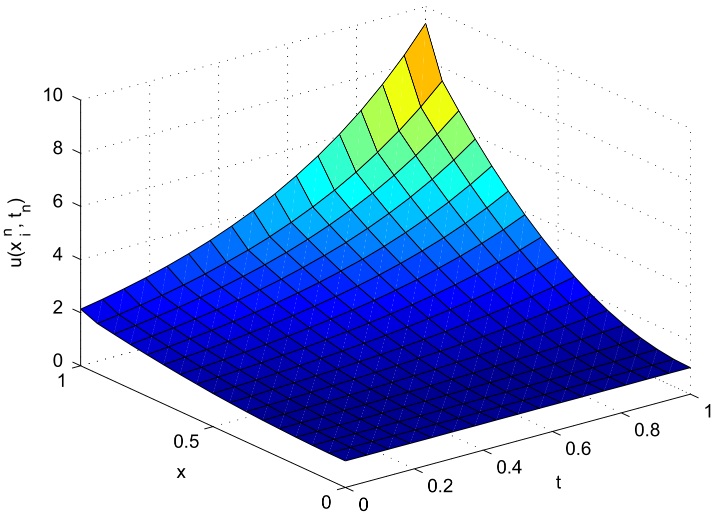

Fig. 2. Analytic solution for Case 3.

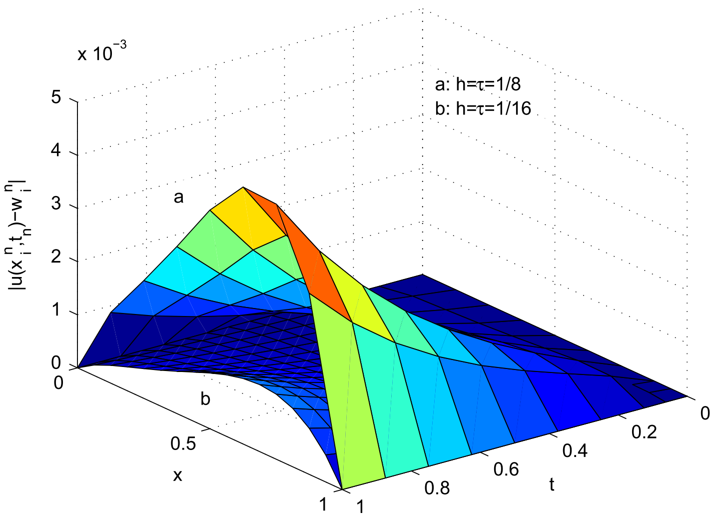

Fig. 3. Error surfaces of numerical solutions with different stepsize for Case 3.

In Tables 1–3, the maximum norm errors between $u(x_i^n, t_n)$ and w_i^n, $1 \le i \le M-1$, $1 \le n \le N$ are given by $e(h, \tau)$ and the numerical order of convergence is computed by $\log_2 (e(h, \tau)/e(2h, 2\tau))$. From these tables we observe that the numerical results are in accordance with the theoretical results.

Further, some figures are drawn for Case 3. It is shown from Figs. 1 and 2 that the numerical solution of the Crank–Nicolson scheme with $h = \tau = 1/16$ and the analytic solution are much the same. Fig. 3 shows the trend of error $|u(x_i^n, t_n) - w_i^n|$ when the stepsize h and τ are changed.

6. Conclusion

In this article, for a linear moving boundary value problem, we construct a Crank–Nicolson type difference scheme and prove that the scheme is unconditionally stable and convergent in the maximum norm. Introduction of a linear transformation is an effective method to avoid discussing the Crank–Nicolson scheme on moving boundaries directly.

Acknowledgements

The author would like to thank the reviewers for their invaluable critical comments and helpful suggestions which improve this paper significantly.

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