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Regular precession of a rigid body (gyrostat) acted upon by an irreducible combination of three classical fields

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We show that a heavy, magnetized and electrically charged asymmetric rigid body moving about a fixed point while carrying a rotor and acted upon by three uniform fields can perform a regular precession about a nonvertical axis, of the type described for the case of a single field by Grioli in 1947. This is the first, and the only known by now, non-equilibrium solution of the problem of motion of a body in the presence of three classical fields, which are irreducible to a less number of fields.

1. Introduction

Although the problem of motion of a rigid body has a long history, most of that history was devoted to the study of motion under uniform or approximate Newtonian gravity field and in some cases its generalization through attaching a gyrostatic moment. For a detailed account of those cases, see [1,2,3]. Some recent advances in the field concerned more complicated versions involving axisymmetric combinations of non-uniform fields [4,5,6].

In spite of its practical importance, the problem of motion of rigid body under three uniform classical fields has escaped attention for a long time. Known are only two integrable cases involving two uniform fields [7,8,9]. Problems of motion in the presence of three significant (irreducible to two) fields were very rarely considered. Only equilibrium positions were classified and some of them were investigated for stability in [10].

In the present note we investigate the regular precessional motion of an asymmetric rigid body-gyrostst about a nonvertical axis under the action of three fields. This type of motion was described for the body in a single uniform gravity field by Grioli in 1947 [11]. Grioli's result was generalized by Kharlamova [12], who added a rotor to the body moving under the action of a single gravity field. We show that the same motion is still possible in the presence of three fields. Conditions are determined and explicit solution of the equations of motion is given. This result generalizes the one obtained recently in [13] involving two coupled fields by the presence of the (independent) third field and the rotor.

It should be noted that precessional motions the rigid body about a vertical axis are common for the symmetric top [14][1],. They were considered in a more general setting in some recent works, e.g. [15]. The most exhaustive analysis of precessional motions about a tilted (non-vertical) axis in a combination of coaxial fields can be found in [16] (see also [17]). As far as we know, the precession about a tilted axis in the presence of skew fields was treated for the first time in [13] in the case of two fields. The case of three skew fields was not considered before in the literature.

1.1. Equations of motiont about a vertical axis

Let **i, j, k** be the unit vectors along the axes of the system *Oxyz*, fixed in the body and let $\omega = (p, q, r)$ be the angular velocity of the body,

$$\alpha = (\alpha_1, \alpha_2, \alpha_3), \beta = (\beta_1, \beta_2, \beta_3), \gamma = (\gamma_1, \gamma_2, \gamma_3)$$

be the unit vectors along the axes of the inertial system *OXYZ*, all being referred to the body system. The relative position of the two

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systems will be described by the Eulerian angles: ψ- the angle of precession around the Z-axis, θ -the angle of nutation between z and Z, and φ the angle of rotation of the body around the z-axis.

The system under consideration here is a gyrostat. It consists of a principal rigid body carrying a symmetric rotor, uniformly rotating about its axis of symmetry, which is fixed in the main body. In virtue of symmetry of the rotor, this rotation does not affect the distribution of mass in the system (the gyrostat). The presence of the rotor is characterized by a constant vector $\boldsymbol{\sigma}$ added to the total angular momentum of the system. The system of principal axes of inertia of the body is not the most suitable for describing the regular precessional motion of the Grioli type, so that we assume the inertia matrix $\mathbf{I}$ in the $Oxyz$ system in the general form:

$$I = \begin{pmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{pmatrix} \tag{1}$$

The position of the body can be described using Euler's angles: $\psi,\ \theta,\ \varphi$. The space vector basis and the angular velocity of the body have the form (e.g. the review book of Leimanis [1]):

$$\begin{aligned} \alpha &= (\cos\psi\cos\varphi - \cos\theta\sin\psi\sin\varphi, -\cos\psi\sin\varphi \\ &\quad - \cos\theta\sin\psi\cos\varphi, \sin\theta\sin\psi), \beta = (\sin\psi\cos\varphi \\ &\quad + \cos\theta\cos\psi\sin\varphi, -\sin\psi\sin\varphi + \cos\theta\cos\psi\cos\varphi, \\ &\quad - \sin\theta\cos\psi), \\ \gamma &= (\sin\theta\sin\varphi, \sin\theta\cos\varphi, \cos\theta), \end{aligned} \tag{2}$$

$$\begin{aligned} \omega &= (\dot\psi\sin\theta\sin\varphi + \dot\theta\cos\varphi, \dot\psi\sin\theta\cos\varphi \\ &\quad - \dot\theta\sin\varphi, \dot\psi\cos\theta + \dot\varphi). \end{aligned} \tag{3}$$

Let the body be in motion about the fixed point O, while acted upon by forces derived from a potential

$$V = \mathbf{a}\cdot\alpha + \mathbf{b}\cdot\beta + \mathbf{c}\cdot\gamma. \tag{4}$$

This potential can be interpreted as due to three uniform fields: gravity, magnetic and electric fields acting on three types of *centers* in the body: centre of mass, magnetic moment and centre of electric charge. For the three centers to be irreducible to a less number of centers, it is necessary that the three centers do not lie in a plane passing through the fixed point. The determinant formed of the components of the centres in the body system of axes should not vanish. The same can be said about the three vectors representing the three fields in space.

Remark. We assume that during the motion of the electric charges carried by the body, both Lorentz forces excerted by the magnetic field and the radiation due to accelerated charges remain negligible.

The dynamical problem can be formulated in the form of Euler-Poisson equations (see e.g[1].)

$$\begin{aligned} \dot\omega\mathbf{I} + \omega\times(\omega\mathbf{I}+\sigma) &= \alpha\times\frac{\partial V}{\partial\alpha} + \beta\times\frac{\partial V}{\partial\beta} + \gamma\times\frac{\partial V}{\partial\gamma} \\ &= \alpha\times\mathbf{a} + \beta\times\mathbf{b} + \gamma\times\mathbf{c}, \\ \dot\alpha + \omega\times\alpha = \mathbf{0}, &\quad \dot\beta + \omega\times\beta = \mathbf{0}, \quad \dot\gamma + \omega\times\gamma = \mathbf{0}. \end{aligned} \tag{5}$$

2. The solution

The regular precession is most simply described as the proper rotation of the body with a uniform angular velocity $\dot\varphi = \Omega$ about its $z-$axis, which simultaneously precesses with the same angular velocity $\dot\psi = \Omega$ about the space axis Z keeping with it a fixed angle $\theta = \frac{\pi}{2}$. In this motion $\varphi = \psi = \Omega t$, which we denote by u. The described motion is periodic, with period $T = \frac{2\pi}{\Omega}$ in t and period 2π in u.

The solution of the Euler-Poisson Eqs. (5) corresponding to the above choices is

$$\begin{aligned} \alpha &= (\cos^2 u, -\sin u\cos u, \sin u), \\ \beta &= (\sin u\cos u, -\sin^2 u, -\cos u), \\ \gamma &= (\sin u, \cos u, 0), \end{aligned} \tag{6}$$

$$\omega = (\Omega\sin u, \Omega\cos u, \Omega). \tag{7}$$

The last expression can be written as

$$\omega = \Omega(\gamma + \mathbf{k}),$$

so that $\omega^2 = 2\Omega^2$. The momentary angular velocity $\boldsymbol{\omega}$ is constant in magnitude $\sqrt{2}\Omega$ and makes equal angles $\pi/4$ with the two axes Z and z, fixed in space and in the body, respectively. The motion can thus be given an alternative description in the following manner:

Let C_m and C_f be two identical right circular cones with vertices at the origin and with semi-vertical angle $\pi/4$. The first cone, with axis along the $z-$ axis, is fixed in the body and moving with it and the second is fixed in space with its axis coinciding with the axis. The precessional motion can be represented as rolling (without sliding) of the movable cone C_m on the fixed cone C_f with the angular velocity $\sqrt{2}\Omega$.

Now, substituting the solution (6) and (7) in (5) we note that the three vector Poisson equations are identically satisfied, so that we obtain from the remaining Euler equation only three scalar equations involving powers of trigonometric functions of Ωt. The conditions that each coefficient of the independent trigonometric terms must vanish lead in a simple way to the following set of conditions on the values of parameters:

$$A = B, F = 0, \tag{8a}$$

$$c_1 = c_2 = \sigma_1 = \sigma_2 = 0, \tag{8b}$$

$$a_2 = b_1 = 0, b_2 = -a_1, \tag{8c}$$

$$b_3 + \Omega^2 D = 0, \tag{8d}$$

$$a_3 - \Omega^2 E = 0 \tag{9}$$

$$C\Omega^2 + \sigma_3\Omega + a_1 - c_3 = 0 \tag{10}$$

Now we note that:

(1) The first condition (8a) means that the $x, y-$ plane is a principal plane containing one of the two circular cross-sections of the ellipsoid of inertia of the body.
(2) From (8b), both the centre of mass and the gyrostatic momentum lie on the $z-$axis, i.e. the line passing through the fixed point and perpendicular to that cross-section.
(3) Till now, we have fixed the choice of $z-$ axis but we still have the freedom to choose any two orthogonal axes in the plane of the circular cross-section as x, y axes. We shall fix this freedom by choosing $y-$ axis to be the line of intersection of the two circular cross-sections, which is a principal axis(namely, the medium axis). This adds to (8) the condition

$$D = 0 \tag{11}$$

and then from (8d) we obtain

$$b_3 = 0 \tag{12}$$

Thus, the inertia matrix becomes

$$I = \begin{pmatrix} B & 0 & -E \\ 0 & B & 0 \\ -E & 0 & C \end{pmatrix} \tag{13}$$

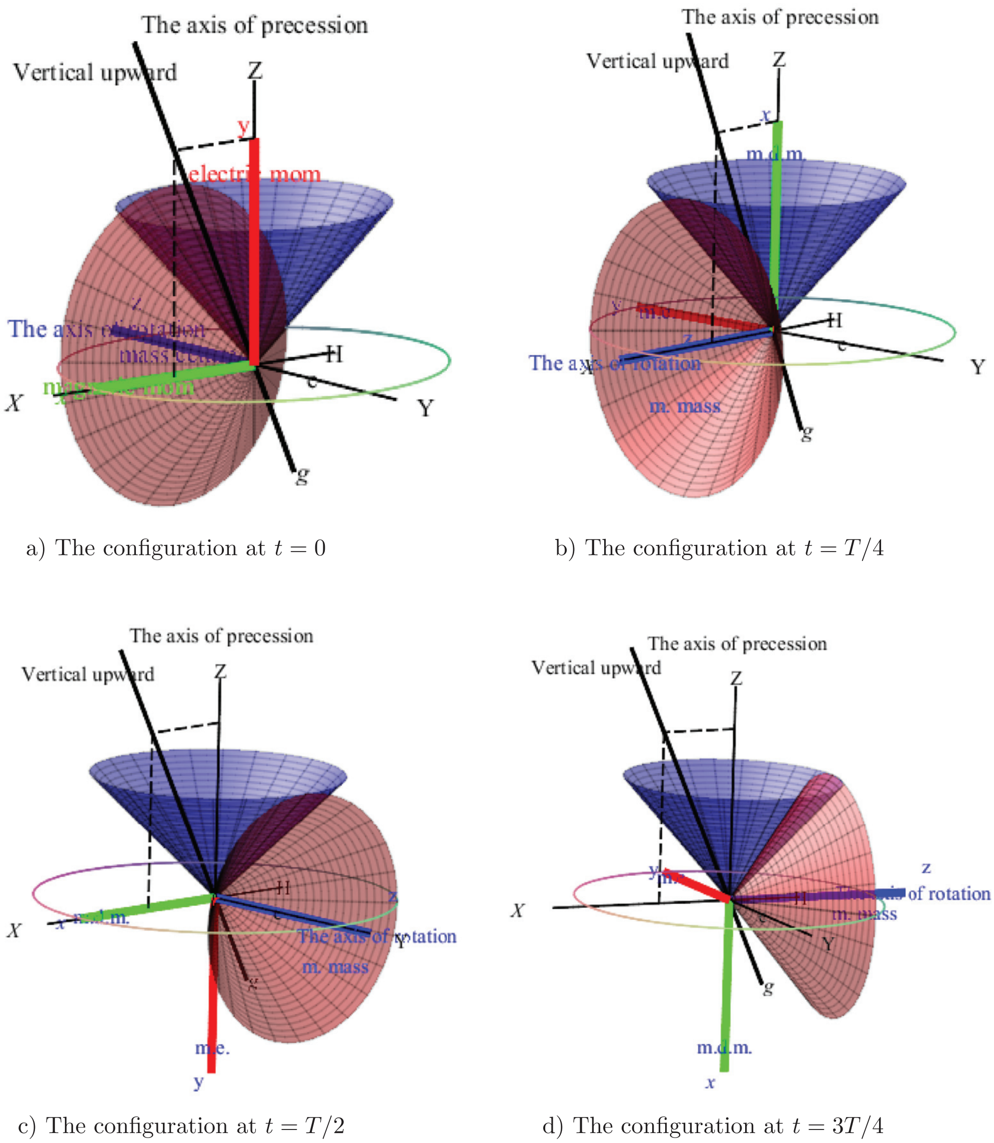

Fig. 1. The configuration of motion at four equal intervals. The $x-$ axis carrying the magnetic moment is represented by thick green line segment. The $y-$ axis carrying the electric moment is represented by thick red line segment. The $z-$axis carrying the centre of mass is represented by thick blue line segment. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

and the potential of the three force fields can be written as

$$V = a_1(\alpha_1 - \beta_2) + a_3\alpha_3 + c_3\gamma_3 \tag{14}$$

where we still have only two equations to be satisfied, namely, (9) and (10). From now on, we descriminate between two possible cases:

2.1. The first case - the case of a simple rigid body $\sigma_3 = 0$

When the gyrostatic momentum vanishes, (9) and (10) give

$$\Omega = \pm\sqrt{\frac{c_3 - a_1}{C}} \tag{15}$$

$$a_3 = \frac{E(c_3 - a_1)}{C} \tag{16}$$

We note that:

1. The angular velocity Ω is real only under the condition $c_3 - a_1 \geqq 0$. The generic motion is periodic of time period $2\pi/\Omega$. When $c_3 = a_1$ we have $\Omega = 0$ and also $a_3 = 0$. The solution (7) in that case becomes

$$\alpha = (1, 0, 0), \beta = (0, 0, -1), \gamma = (0, 1, 0) \tag{17}$$

and describes one of the possible equilibrium positions Thus, regular precession is possible only when $c_3 > a_1$.
2. The motion is time reversible, i.e. the change of sign of Ω is equivalent to changing the sign of time.

In virtue of (16) the potential (14) becomes

$$V = a_1(\alpha_1 - \beta_2) + c_3\gamma_3 + \frac{E(c_3 - a_1)}{C}\alpha_3 \tag{18}$$

Physical interpretation of this potential can be performed in different manners. An example is given in [13] for the case of two coupled fields. Here we have only two parameters a_1 and c_3, which characterize three centres of charge. The matrix formed of the coefficients in the potential is

$$\begin{pmatrix} a_1 & 0 & \frac{E}{C}(c_3 - a_1) \\ 0 & -a_1 & 0 \\ 0 & 0 & c_3 \end{pmatrix}$$

It has rank 3, so that the three vectors describing the centres are not lying in a plane and we are dealing with three physically independent effects.

Let us write the potential (18) in the form

$$V = a_1 \mathbf{i} \cdot \alpha - a_1 \mathbf{j} \cdot \beta + \mathbf{k} \cdot (c_3 \gamma + \frac{E(c_3 - a_1)}{C} \alpha) \quad (19)$$

The first two terms of (19) are coupled by the constant a_1, so that they can vanish only simultaneously. They may be understood as due to a system of charges with moment a_1, whose center lies on the x-axis and a magnetized part of the body with magnetic moment a_1 directed along the negative $y-$axis subject to two uniform electric and magnetic fields $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, orthogonal to each other and of equal intensities.

The third term of (19) can be interpreted as the gravity effect on the body with centre of mass on the $z-$axis (normalized so that the product of mass and its distance from the fixed point equals unity) by a uniform gravity field $\mathbf{g} = c_3 \gamma + \frac{E(c_3 - a_1)}{C} \alpha$. The vector $\mathbf{g}$ is pointing vertically upwards. The axis of precession along the vector $\boldsymbol{\gamma}$ is inclined to the vertical vector $\mathbf{g}$ at an angle

$$\delta = \tan^{-1} \frac{E(c_3 - a_1)}{C c_3} \quad (20)$$

Note here that the axis of precession can take a vertical position only when $E = 0$, i.e. when the $z-$axis is an axis of dynamical symmetry.

To compare with the known result of one gravity field due to Grioli (see e.g[1].), we express this angle in terms of the principal moments of inertia at the fixed point. The inertia matrix (13) has one eigenvalue B (the medium moment of inertia). Let the other two principal moments be A_0 and C_0. We have the relations

$$A_0 + C_0 = B + C, A_0 C_0 = BC - E^2 \quad (21)$$

from which we find

$$\frac{E}{C} = \frac{\sqrt{(A_0 - B)(B - C_0)}}{A_0 - B + C_0}$$

and thus the angle (20) becomes

$$\delta = \tan^{-1} [\frac{\sqrt{(A_0 - B)(B - C_0)}}{A_0 - B + C_0} (1 - \frac{a_1}{c_3})] \quad (22)$$

In the case of single gravity field $a_1 = 0$ (Grioli's case) we obtain the same angle as obtained for that case by Guliaev (see e.g[1]., [6]).

Fig. 1 illustrates the picture of the motion of the body axes relative to the space axes at times 0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$ of its time period.

2.2. The second case - the case of a gyrostat $\sigma_3 \neq 0$

In this case the potential is given by (14) and Ω is subject to two conditions (9) and (10). Their compatibility (the resultant) condition is

$$[E(a_1 - c_3) + C a_3]^2 = E a_3 \sigma_3^2 \quad (23)$$

If we take parameters satisfying this condition then the angular velocity is given by

$$\Omega = \frac{1}{\sigma_3} \left(c_3 - a_1 - \frac{C}{E} a_3 \right) = \pm \sqrt{\frac{a_3}{E}} \quad (24)$$

Here we note that:

1. The motion for a set of parameters is not time reversible. It becomes time reversible only after simultaneous change of signs of Ω and σ_3.
2. From (23) the values of E and a_3 must have the same sign.
3. The angular velocity Ω from (10) is real only when $\sigma_3^2 + 4C(c_3 - a_1) > 0$. In the generic case there are two values of Ω corresponding to every combination of parameters. In the case of vanishing gyrostatic momentum ($\sigma_3 = 0$) the two values are equal in magnitude and different in sign.
4. The physical interpretation is almost the same as in the previous section, except that the angle between the axis of precession and the vertical

   $$\delta = \tan^{-1} \frac{a_3}{c_3},$$

   as well as the angular velocity, depends on the gyrostatic momentum in virtue of (23).
5. When $a_1 = 0$ our case becomes equivalent to the case of [12], which adds the rotor to Grioli's classical result.

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