An explicit-form solution to the plane elasticity and thermoelasticity problems for anisotropic and inhomogeneous solids

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This paper presents an analytical approach to solve the plane elasticity and thermoelasticity problems for inhomogeneous, orthotropic planes, half-planes, and strips. Solution of the problems is reduced to the governing Volterra integral equation formulated for the key function and accompanied by the corresponding integral conditions. By making use of the resolvent-kernel technique, the governing equation is solved and the solution to the original problem is presented in explicit form.

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1. Introduction

It is well-known that assuming the possibility that material properties depend on spatial coordinates can be a major challenge for analytical treatment of the elasticity and thermoelasticity problems for certain kinds of materials, such as FGM (Birman and Byrd, 2007). The main difficulty lies in the need to solve the governing differential equations with variable coefficients which often are not predefined. This fact makes it impossible, except for in a very few particular cases, to solve the problems analytically (Tanigawa, 1995). In the existing literature, this difficulty has been obviated in several ways. One way around this problem is to employ approximate solution methods, most of which are based upon the variational principles (Han et al., 2001; Luciano and Sacco, 1998; Zhang and Wang, 2006). This provides the possibility of simplifying the solution procedure. The other way is to employ linearization techniques (Kushnir and Popovych, 2006; Tanigawa, 1995) and thereby to simplify the problem formulation in a certain sense. Alternatively, analytical methods are developed for the materials with preliminary known material properties (in the form of linear, power, or exponential functions, etc.) those present no difficulties for the construction of solutions to the governing equations due to the fact that the variable coefficients of these equations are given in appropriate form. One more technique circumvents the problem by representation of a continuously inhomogeneous solid as a composite consisting of a number of tailored homogeneous layers. The overview of the relevant literature in these two directions has been provided in our previous publications (Tokovyy and Ma, 2008, 2009a,b). To optimize the convergence of the solution for a piecewise-homogeneous solid to one for a continuously inhomogeneous solid, Plevako (2002) proposed the consideration of piecewise-inhomogeneous solids with an each-layer inhomogeneity that enables comparatively simple construction of the solution. Through this approach, the elastic characteristics can be approximated by continuous polylines instead of piecewise constant functions, improving the approximation towards an exact solution. Analogous approximation of an inhomogeneous strip by a piecewise-exponential composite has been considered, e.g., by Guo and Noda (2007).

Despite the numerous ways around the problem of solving the governing equations of elasticity and thermoelasticity for inhomogeneous materials, there is a strong need in explicit analytical solutions for an arbitrary kind of inhomogeneity. In Tokovyy and Ma (2009b), we developed the method for analysis of the two-dimensional problems of elasticity and thermoelasticity for radially inhomogeneous hollow cylinders and disks, whose material properties can be regarded as arbitrary functions. Based upon the direct integration of the equilibrium equations, this method allows us to adopt the basic relations and conditions for the stress-tensor components obtained for the case of homogeneous solid for use with an inhomogeneous solid of the same shape. The central goal of this method is to reduce the original problems to the governing equation, which can be obtained on the...
basis of the compatibility equation in terms of stresses and accompanied with the boundary and integral conditions established by integration of the equilibrium equations. For inhomogeneous solids, the mentioned governing equation appears as the Volterra integral equation of the second kind (Goldberg, 1979), which has been treated by means of the simple iterations technique. An analogous method has been developed for treatment of isotropic, inhomogeneous strips, half-planes, and planes (Tokovyy and Rychahivsky, 2005; Tokovyy and Ma, 2009a). In Tokovyy and Ma (2008), the governing integral equation for the thermoelasticity problem in radially inhomogeneous and orthotropic cylinders and disks has been solved by means of the resolvent-kernel method (Pogorzelski, 1966, p. 13; Porter and Stirling, 1990, pp. 131 and 132) which readily results from Picard’s process of successive approximations (Tricomi, 1957, pp. 6–8) or from the series-form solution representation (Verlan’ and Sizikov, 1986, pp. 24–26). As for the analytical solution, the resolvent-kernel method is found to be more efficient in comparison to the iterative method due to the following advantages. First of all, the resolvent solution is a closed-form analytical solution. It appears in explicit functional form, which presents the stress-tensor components in terms of the given force loadings and temperature. This fact presents an inestimable advantage for further applications and analysis of the solution. Next, the resolvent-kernel is expressed through the Volterra-kernel (‘Intrinsic’ properties of the integral equation) and is independent of a free term of the equation (‘External’ properties). Since in our case the Volterra-kernel is a function of the material properties only, the resolvent-kernel is also the function of the material properties only, and it does not depend on any type of loading. Consequently, having been computed once for certain Volterra-kernel (for certain material properties, obviously), the resolvent-kernel can be employed for different kinds of force and thermal loadings. Finally, application of the resolvent-kernel method gives a large dividend in accuracy of computation.

This paper aims to develop the resolvent-kernel solution method for treatment of the governing Volterra integral equation of the second kind for the plane elasticity and thermoelasticity problems in inhomogeneous and orthotropic planes, half-planes, and strips, which are subjected to external force loadings, body forces, and temperature distribution.

2. Problems formulation

We consider the plane elasticity and thermoelasticity problems in the dimensionless Cartesian coordinate system \((x, y)\). The problems are governed (Timoshenko and Goodier, 1970) by the equilibrium equations:

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y = 0, \quad (1)
\]

compatibility equation

\[
\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} + \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 0, \quad (2)
\]

and physical relations for orthotropic material (Ambartsumyan, 1970; Lekhnitskii, 1981), which read

\[
\varepsilon_x = \alpha_1 \sigma_x + \alpha_2 \sigma_y + \gamma_1 T, \quad \varepsilon_y = \alpha_2 \sigma_x + \alpha_2 \sigma_y + \gamma_2 T, \quad \sigma_x = \alpha_1 \varepsilon_x + \alpha_2 \varepsilon_y + \gamma_1 T, \quad \sigma_y = \alpha_2 \varepsilon_x + \alpha_2 \varepsilon_y + \gamma_2 T, \quad \gamma_{xy} = \gamma_{yx}, \quad (3)
\]

for the plane case (Sadd, 2005; Tokovyy and Ma, 2008). Here \(\sigma_x, \sigma_y, \) and \(\tau_{xy}\) are the in-plane stress-tensor components; \(\varepsilon_x, \varepsilon_y, \) and \(\gamma_{xy}\) denote the in-plane strain-tensor components; \(X(x, y)\) and \(Y(x, y)\) are the projections of body forces onto the Cartesian axes \(x\) and \(y\), respectively;

\[
a_{11} = \frac{a_{11}}{E_x}, \quad a_{22} = \frac{a_{22}}{E_y}, \quad a_{12} = \frac{a_{12}}{E_y} = -\frac{a_{21}}{E_x}, \quad a_{11} = \begin{cases} 1, & 1 - V_{yx} V_{zx}, \quad a_{22} = \begin{cases} 1, & 1 - V_{yx} V_{yz}, \end{cases} \end{cases}, \quad a_{12} = \begin{cases} V_{yx}, & V_{yx} + V_{zx} V_{yz}, \end{cases} \quad a_{33} = \begin{cases} V_{yx}, & V_{yx} + V_{zx} V_{yz}, \end{cases} \quad \gamma_{1} = \begin{cases} \varepsilon_x, & \varepsilon_x + \varepsilon_2 V_{yx}, \end{cases} \quad \gamma_{2} = \begin{cases} \varepsilon_y, \gamma_{xy} + \varepsilon_2 V_{yx}, \end{cases} \quad (4)
\]

(in the expressions with braces, the upper and lower lines correspond to the plane stress and plane strain hypothesis, respectively) \(E(x, y)\) and \(E(y, y)\) are the Young’s moduli in the directions \(x\) and \(y\), respectively; \(\nu_{ij}(y)\) denotes the Poisson’s ratio describing the contraction in the direction in tension in the \(j\)-direction, \(i \neq j\); \(G_{ij}(y)\) is the shear modulus for the \((x, y)\)-coordinate surface; \(\sigma_x(y), \sigma_y(y), \) and \(\sigma_r(y)\) stand for the linear thermal expansion coefficients in the directions \(x, y,\) and \(z\), respectively; \(T(x, y)\) is the temperature distribution, which can be previously given or obtained from the corresponding heat conduction problem (Carslaw and Jaeger, 1959; Hetnarski and Eslami, 2009). The elastic moduli are interdependent:

\[
E_{ij} = E_{ji}, \quad \{i, j\} = \{x, y, z\}, \quad i \neq j. \quad (5)
\]

In this study, we restrict our focus to analysis of the problems in terms of stresses for inhomogeneous and orthotropic plane, \(\Omega_1 = \{(x, y) \in (\infty, \infty) \times (\infty, \infty)\}, \) half-plane \(\Omega_2 = \{(x, y) \in (\infty, \infty) \times [0, \infty)\}, \) strip \(\Omega_3 = \{(x, y) \in (\infty, \infty) \times [-b, b]\}, \) where \(b > 0\) is a dimensionless parameter. We consider the stress-state induced in the plane \(\Omega_1\) by the body forces, \(X\) and \(Y,\) and temperature, \(T.\) As to the half-plane \(\Omega_2,\) two more conditions

\[
\sigma_x = -p_0(x), \quad \tau_{xy} = q_0(x), \quad y = 0, \quad |x| < \infty, \quad (6)
\]

are imposed at the boundary in addition to the aforementioned body-force and thermal loadings. The stresses occur in the strip \(\Omega_3\) due to the action of foregoing body forces and temperature, as well as the external normal and shear force loadings

\[
\sigma_x = -p_1(x), \quad \tau_{xy} = q_1(x), \quad y = b, \quad |x| < \infty, \quad (7)
\]

\[
\sigma_x = -p_2(x), \quad \tau_{xy} = q_2(x), \quad y = -b, \quad |x| < \infty, \quad (8)
\]

applied to the sides \(y = \pm b.\) In order to construct the correct solutions, the thermal and force loadings and, consequently, the stresses are assumed to be vanishing at points of infinity and satisfying the necessary conditions (Rychahivsky and Tokovyy, 2008).

By making use of Eqs. (1) and (3), the compatibility equation (2) can be equally written in terms of stresses as

\[
\Delta [a_{11} \sigma_x + a_{12} \sigma_y + \gamma_1 T] = (x_1 - x_2) \frac{\partial^2 \sigma_x}{\partial y^2} + \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{2}{\partial \varepsilon_x} \left( \frac{\partial^2 \sigma_x}{\partial y^2} + \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \right) + \beta_1 \frac{\partial^2 \sigma_x}{\partial y^2} + \beta_2 \frac{\partial^2 \sigma_x}{\partial x^2} \sigma_y, \quad (8)
\]

where \(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\) denotes the two-dimensional Laplace operator; \(\beta_1 = a_{11} - a_{12}, \beta_2 = a_{22} - a_{11} ;\) and \(\sigma\) is the total in-plane stress given by formula

\[
\sigma = \sigma_x + \sigma_y. \quad (9)
\]

In the following sections, we consider in detail the construction of an analytical solution to the elasticity and thermoelasticity problems (1), (7) and (8) for the inhomogeneous strip \(\Omega_3.\) The problems (1) and (8) for the plane \(\Omega_1\) and (1), (6) and (8) for the half-plane \(\Omega_2\) can be treated in a similar manner. The corresponding solutions of the above-mentioned problems are given in Appendices A and B.
3. Solution method in the case of inhomogeneous and orthotropic strip \( \Omega_2 \)

By following the solution scheme proposed by Tokovyy and Rychhailovsky (2005), we opt for \( \sigma_y \) and \( \sigma_z \) as the governing functions. To determine these functions, we shall use Eq. (8), expressing the total stress in terms of \( \sigma_z \), as well as the equilibrium equation (1), which connect all the stress-tensor components. By elimination of the shear stress from Eqs. (1) and by taking relation (9) into account, the equation

\[
\Delta \sigma_y = \frac{\partial^2 \sigma_y}{\partial x^2} + \frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y},
\]

(10)
can be derived to express \( \sigma_y \) through the total stress \( \sigma \). Note that Eq. (10) does not involve the material properties (4), and thus it fits any kind of material whose elastic equilibrium is governed by Eqs. (1).

By making use of the Fourier integral transform (Brychkov and Prudnikov, 1989)

\[
\tilde{f}(y) = \tilde{f}(y; s) = \int_{-\infty}^{\infty} f(x, y) \exp(-isx) \, dx,
\]

(11)
(where \( s \) stands for a parameter of the transform, \( i^2 = -1 \), and \( f(x, y) \) is an arbitrary function whose transformation exists in at least a generalized sense), the governing equations (8) and (10) take the form

\[
\mathcal{T} \sigma_y = -s^2 \sigma + isX - \frac{dY}{dy},
\]

(12a)

\[
\mathcal{T}(a_1 \sigma + \xi T) = -s^2(\sigma_1 - \sigma_2)T + \frac{d}{dy} \left[ \left( 2\beta_1 \sigma_1 - \frac{1}{C_{01}} \right) \frac{d\sigma_1}{dy} \right]
+ \left( 2\beta_1 \sigma_2 + s^2 \beta_2 \right) \sigma_2 + \beta_1 \left( \frac{dY}{dy} - isX \right)
- \frac{d}{dy} \left( \frac{Y}{C_{01}} \right).\]

(12b)

Here \( \mathcal{T} = \frac{C_{02}}{C_{01}} - s^2 \). As for the case of strip \( \Omega_3 \), Eqs. (12) should be accompanied with the boundary conditions

\[
\sigma_y(b) = -\tilde{p}_1, \quad \sigma_y(-b) = -\tilde{p}_2,
\]

(13a)

\[
\frac{d\sigma_y}{dy} \bigg|_{y=b} = -isq_1 - Y(b), \quad \frac{d\sigma_y}{dy} \bigg|_{y=-b} = -isq_2 - Y(-b).
\]

(13b)

Note that conditions (13a) are obtained by application of the integral transformation (11) to conditions (7) for the normal stress; meanwhile conditions (13b) are established by application of the same integral transformation to the second equation (1) with conditions (7) for the shear stress in view.

Since Eqs. (12) are ODEs of the same differential operator, their solution can be presented in the same form:

\[
\psi(y) = A \cosh(sy) + B \sinh(sy) + \frac{1}{s} \int_{-b}^{b} F(\xi) \sinh(s(y - \xi)) \, d\xi,
\]

(14)

where \( A \) and \( B \) are the constants of integration; \( \psi \) stands for \( \sigma_y \), and \( a_1 \sigma + \xi T \) and \( T \) denotes the right-hand member of Eqs. (12a) and (12b), respectively. In view of Eq. (14), the solution to Eq. (12a) with boundary conditions (13a) appears as

\[
\sigma_y = \left( \tilde{p}_1 + \frac{1}{s} \int_{-b}^{b} \left( isX(y) - \frac{dY(y)}{dy} - s^2 \sigma_1 \right) \sinh(s(b - y)) \, dy \right)
- \frac{\sinh(s(b + y))}{\sinh(2sb)} \left( \tilde{p}_2 - \tilde{p}_1 \right)
+ \frac{1}{s} \int_{-b}^{b} \left( isX(\xi) - \frac{dY(\xi)}{d\xi} - s^2 \sigma_1 \right) \sinh(s(y - \xi)) \, d\xi.
\]

(15)

Since expression (15) should meet conditions (13b) two integral conditions,

\[
\int_{-b}^{b} \sigma_1(y) \sinh(sy) \, dy = Z_1, \quad \int_{-b}^{b} \sigma_2(y) \cosh(sy) \, dy = Z_2,
\]

(16)
can be obtained, denoted here as

\[
Z_1 = -\left( \tilde{p}_1 - \tilde{p}_2 \right) \frac{\cosh(sb)}{s} + \left( \tilde{q}_1 + \tilde{q}_2 \right) \frac{\sinh(sb)}{s} + \frac{Y(b) - Y(-b)}{s^2}
\]
\[
\times \sinh(sb) + \frac{1}{s^2} \int_{-b}^{b} \left( isX - \frac{dY}{dy} \right) \sinh(sy) \, dy.
\]

\[
Z_2 = -\left( \tilde{p}_1 + \tilde{p}_2 \right) \frac{\cosh(sb)}{s} + \left( \tilde{q}_1 - \tilde{q}_2 \right) \frac{\cosh(sb)}{s} + \frac{Y(b) - Y(-b)}{s^2}
\]
\[
\times \cosh(sb) + \frac{1}{s^2} \int_{-b}^{b} \left( isX - \frac{dY}{dy} \right) \cosh(sy) \, dy.
\]

By making use of conditions (16), the relation

\[
\frac{1}{s} \int_{-b}^{b} \left( isX(y) - \frac{dY(y)}{dy} - s^2 \sigma(y) \right) \sinh(s(y - \xi)) \, dy
- \tilde{p}_1 + \tilde{p}_2 \cosh(2sb) + \left( \tilde{q}_2 + \frac{Y(-b)}{s} \right) \sinh(2sb)
\]

is obtained from expression (15), and then the latter expression yields

\[
\sigma_y = -\tilde{p}_2 \cosh(s(y + b)) - \left( \tilde{q}_2 \right) \sinh(s(y + b)) + \frac{1}{s}
\]
\[
\times \int_{-b}^{b} \left( isX(\xi) - \frac{dY(\xi)}{d\xi} - s^2 \sigma(\xi) \right) \sinh(s(y - \xi)) \, d\xi.
\]

(17)

On the basis of Eq. (14), the solution to Eq. (12b) can be obtained as follows:

\[
\sigma = \frac{1}{a_1} \left[ A \cosh(sy) + B \sinh(sy) - \xi T \right]
- s \int_{-b}^{b} \left( \frac{dF(\xi)}{d\xi} \sinh(s(y - \xi)) \right) \, d\xi
+ \left( 2\beta_1 \sigma_1 - \frac{1}{C_{01}} \right) \left( \tilde{p}_1 \cosh(s(y + b)) \right)
+ \left( 2\beta_1 \sigma_2 + s^2 \beta_2 \right) \left( \tilde{p}_2 \cosh(s(y + b)) \right)
+ \left[ \left( \frac{Y(b)}{s} \right) - \frac{Y(-b)}{s} \right] \sinh(s(y + b))
+ \int_{-b}^{b} \sigma_y(\xi) \left( \frac{d^2 \beta_1}{d\xi^2} + s^2 \beta_2 \right) \sinh(s(y - \xi)) \, d\xi
- \frac{d}{d\xi} \left( \frac{Y(\xi)}{C_{01}(\xi)} \right) \sinh(s(y - \xi)) \, d\xi
+ \frac{1}{s} \int_{-b}^{b} \left( \frac{dY(\xi)}{d\xi} \right) \sinh(s(y - \xi)) \, d\xi
\]
\[
\frac{d}{d\xi} \left( \frac{Y(\xi)}{C_{01}(\xi)} \right) \sinh(s(y - \xi)) \, d\xi
\]
\[
= \frac{1}{s^2} \int_{-b}^{b} \left( isX(\xi) - \frac{dY(\xi)}{d\xi} - s^2 \sigma(\xi) \right) \sinh(s(y - \xi)) \, d\xi.
\]

(18)

after uncomplicated mathematical derivation, which involves integration by parts and application of conditions (13). Here the constants of integration, \( A \) and \( B \), are to be determined from integral conditions (16). By submitting Eq. (17) into Eq. (18), the latter formula takes on the appearance of the Volterra integral equation:
\[\sigma = A \cosh(y) + B \sinh(y) + \Theta(y) + P(y) + Q(y) + \Psi(y)\]

where

\[\Theta(y) = -\frac{1}{\alpha_1(y)} \left[ a_1(y) T(y) + s \int_b^y (a_1(\xi) - a_2(\xi)) T(\xi) \sinh(s(y - \xi)) d\xi \right].\]

\[P(y) = -\frac{p_2}{\alpha_1(y)} \int_b^y \left( \frac{d^2 \beta_1(y)}{dy^2} + s^2 \beta_2(y) \right) \frac{\sinh(s(y - \eta))}{s} \cosh(s(\eta + b)) \sinh(s(\eta + b)) d\eta,
\]

\[Q(y) = \frac{i q_2}{\alpha_1(y)} \left( 2 \beta_1(y) - \frac{1}{c_{\gamma\gamma}(y)} \right) \cosh(s(y + b)) \sinh(s(y + b)) - \int_b^y \left[ \frac{d \beta_1(y)}{dy} + s \beta_2(y) \right] \frac{\sinh(s(y - \eta))}{s} \cosh(s(\eta + b)) \sinh(s(\eta + b)) d\eta + s \left( 2 \beta_1(y) - \frac{1}{c_{\gamma\gamma}(y)} \right) \cosh(s(y - \eta)) \sinh(s(\eta + b)) d\eta,
\]

\[\Psi(y) = \frac{1}{s \alpha_1(y)} \left[ -a_1(y) Q(y) + \frac{\partial}{\partial y} \left( \frac{\gamma}{b_1(y)} \right) \sinh(s(y - \zeta)) d\zeta + \int_b^y \left( \frac{d \gamma}{d\zeta} \right) \frac{\sinh(s(y - \zeta))}{s} \cosh(s(\eta + b)) \sinh(s(\eta + b)) d\eta,
\]

\[K(y, \zeta) = -\frac{s}{\alpha_1(y)} \left[ 2 \beta_1(y) - \frac{1}{c_{\gamma\gamma}(y)} \right] \cosh(s(y - \zeta)) \sinh(s(\eta + b)) d\eta + \int_b^y \Phi(y, \eta) \sinh(s(y - \zeta)) d\eta,
\]

\[\Phi(y) = \Theta(y) + P(y) + Q(y) + \Psi(y) + \int_b^y (\Theta(\zeta) + P(\zeta) + Q(\zeta) + \Psi(\zeta)) R(y, \zeta) d\zeta,
\]

\[A = \frac{F_1 I_{12} - F_2 I_{12}}{I_{11} I_{12} - I_{12} I_{12}} \quad B = \frac{F_1 I_{11} - F_2 I_{11}}{I_{11} I_{12} - I_{12} I_{12}}.
\]

\[F_1 = Z_1 - \int_b^y \Phi(y) \sinh(s(y + b)) d\eta, \quad F_2 = Z_2 - \int_b^y \Phi(y) \cosh(s(y + b)) d\eta,
\]

\[I_{11} = \int_b^y f_1(y) \sinh(s(y + b)) d\eta, \quad I_{21} = \int_b^y f_2(y) \cosh(s(y + b)) d\eta,
\]

and \(R(y, \zeta)\) is the resolvent-kernel given in the form of series

\[R(y, \zeta) = \sum_{n=0}^\infty K_{n+1}(y, \zeta),
\]

by the requiring (Verlan' and Sizikov, 1986), or iterated (Trikomi, 1957), kernels

\[K_n(y, \zeta) = K(y, \zeta) - \int_\zeta^y K(y, t) K_n(t, \xi) d\xi, \quad n = 1, 2, \ldots
\]

Since the iterated kernels decrease with increasing \(n\), the series (21) can be truncated

\[R(x, y) \approx R_0(y, \zeta) = \sum_{n=0}^N K_{n+1}(y, \zeta),
\]

for practical computations.

After the total in-plane stress is found in the form (20), the normal stress \(\sigma_y\) can be computed by means of formula (17), which becomes:

\[\sigma_y = -p_2 \cosh(s(y + b)) \left( \frac{i q_2 + \frac{\gamma}{b_1(y)} \left( \frac{d \gamma}{b_1(y)} \right)}{s} \cosh(s(y + b)) + \frac{1}{s} \right) \sinh(s(y + b)) \sinh(s(y + b)) d\eta + s \left( 2 \beta_1(y) - \frac{1}{c_{\gamma\gamma}(y)} \right) \cosh(s(y - \eta)) \sinh(s(\eta + b)) d\eta,
\]

The normal stress \(\sigma_x\) can be then found from Eq. (9) as

\[\sigma_x = \sigma - \sigma_y.
\]

By making use of the second equation (1) and formula (11), the shearing stress \(\tau_{xy}\) assumes the form

\[\tau_{xy} = \frac{i}{s} \left( \frac{d \sigma_y}{dy} + \frac{\gamma}{b_1(y)} \right),
\]

which yields

\[\tau_{xy} = \int_b^y \left[ \frac{d \sigma_y}{dy} \left( \frac{i q_2 + \frac{\gamma}{b_1(y)} \left( \frac{d \gamma}{b_1(y)} \right)}{s} \cosh(s(y + b)) + \frac{1}{s} \right) \sinh(s(y + b)) \sinh(s(y + b)) d\eta + s \left( 2 \beta_1(y) - \frac{1}{c_{\gamma\gamma}(y)} \right) \cosh(s(y - \eta)) \sinh(s(\eta + b)) d\eta,
\]

with substitution of Eq. (23).

After all the stress-tensor components (23)–(25) as the Fourier-transform have been found, they can be inverted by means of the formula (Brychkov and Prudnikov, 1989)
\[ f(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \zeta(y) \exp(isx) \, ds. \]

Note that despite the different shape of domains \( \Omega_1, \Omega_2, \) and \( \Omega_3, \) and thus different boundary conditions, the governing equations (12) are the same for all the regions mentioned. This allows employment of the same solution strategy to treat the two remaining cases of \( \Omega_1 \) and \( \Omega_2 \) (see Appendices A and B).

4. Numerical examples and discussion

4.1. The inhomogeneous strip subjected to the external force loadings

Herein, we illustrate our approach with the strip \( \Omega_3 (b = 1) \) subjected to the normal tractions

\[ p_1 = p_2 = p_0(x), \quad \zeta(x) = e^{-ax^2}, \tag{26} \]

while

\[ q_1 = q_0 = X = Y = T = 0, \]

under the plane stress hypothesis. Here \( p \) and \( a \) are the constant parameters of loading and \( a > 0. \) Distribution of the function \( \zeta(x) \) is depicted in Fig. 1 for \( a = 1. \) As we can see, the loading (26) is smooth and continuous, and rapidly (depending on the value of parameter \( a \)) vanishes with \( x \to \pm \infty. \) Such properties make the condition (26) convenient to verify the analytical solution.

For the first example of an inhomogeneous material, the strip is assumed to be orthotropic and exponentially graded as

\[ E_x = E_0^{(1)} \omega(y), \quad E_y = E_0^{(2)} \omega(y), \quad G_{xy} = C_{xy}^{(1)} \omega(y), \quad \omega(y) = e^{ky}, \tag{27} \]

where \( k = \text{const.}; E_0^{(1)}, E_0^{(2)}, \) and \( C_{xy}^{(1)} \) are the constant values of the elastic moduli \( E_x, E_y, \) and \( G_{xy}, \) respectively, at the center line \( y = 0, \) and \( C_{xy}^{(1)} = E_0^{(2)}/(2(1 + \nu_{xy})). \) The Poisson’s ratio \( \nu_{xy} \) is assumed to be constant, and \( \nu_{xy} = 0.2 \) for all the following computations. We introduce the parameter

\[ \lambda = \frac{E_0^{(1)}}{E_0^{(2)}}, \tag{28} \]

characterizing the ratio of the orthotropic moduli. In light of Eqs. (5) and (27), we can also obtain \( \nu_{xy} = \lambda^{-1} \nu_{xy} \) and \( C_{xy}^{(1)} = \lambda E_0^{(2)}/(2(1 + \nu_{xy})). \) Obviously, the case \( \lambda = 1 \) corresponds to the isotropic material. As follows from Eq. (4), the stresses are independent of the engineering characteristics with the subscript \( z \) for the plane stress case.

Because of the symmetry in boundary condition (26), the change in sign of the parameter \( k \) will change the distribution of stresses symmetrically with respect to the center line of the strip. Thus, we restrict our focus to negative values of \( k \) only. The case \( k = 0 \) corresponds to the homogeneous material properties. Behavior of the function \( \omega(y), \) which appears in Eq. (27), is shown in Fig. 2 for \( k = 0 \) and \( k = -1. \)

It is worth noting that in the case of isotropic and homogeneous material \( (\lambda = 1, k = 0), \) the governing equation (8) is homogeneous, and thus its solution can be found easily by means of Eq. (20) with respect-to-containing members omitted. Full-field analysis of this case is depicted in Fig. 3. In contrast to the isotropic material, the right-hand member of Eq. (8) remains, at least in general, even for the homogeneous \( (k = 0) \) but orthotropic \( (\lambda \neq 1) \) material. Thus, the resolvent-kernel formula should be employed in that case. The relative error in computation of solution (20) for \( \lambda = 1 \) and \( k = -1 \) by keeping \( N = 1 \) in Eq. (22) lies within 10%; for \( N = 2, \) it falls within 0.2%. Therefore, in this case, the employment of \( R_N, \) which is described by formula (22) instead of \( R \) in the form (21), is good enough by keeping the first two constituents in (22) only. For the cases \( \lambda = 1/2 \) and \( k = 0; -1, \) one needs five constituents for achievement of similar accuracy; for \( \lambda = 2 \) and \( k = 0; -1, \) one needs six constituents.

Distribution of the transversal stress \( \sigma_y \) in the cross-section \( x = 0 \) is depicted in Fig. 4 for different cases of the orthotropic ratio (28) and parameter of inhomogeneity \( k. \) As was to be expected, the curves for \( k = 0 \) are symmetrical with respect to the middle point \( y = 0, \) and the curves for \( k = -1 \) are not symmetric due to the effect of inhomogeneity. Thus, for \( k = 0, \) the stress \( \sigma_y \) assumes the maximum values at \( x = 0, \) and, naturally, the maximum values of this stress are shifted in the direction of inhomogeneity increase. For the considered parameters, the effect of orthotropy is more essential in comparison to the influence of inhomogeneity. As we can observe in both cases \( k = 0 \) and \( k = -1, \) the values of the stress \( \sigma_y \) for \( \lambda = 2 \) exceed the corresponding values for \( \lambda = 1 \) and \( \lambda = 1/2 \) while moving away from the boundaries. This is in a good agreement with the results presented by Tokovsky and Ma (2008). Analogous conclusions hold for the longitudinal stress \( \sigma_x \) shown in Fig. 5.

Let us consider now the material inhomogeneity in the form of polynomial functions

\[ E_x = E_0^{(1)}w(y), \quad E_y = E_0^{(2)}w(y), \quad G_{xy} = C_{xy}^{(1)}w(y), \quad w(y) = (c_1 + c_2y)^k. \tag{29} \]
where \( c_1, c_2, k = \text{const.}; E_0^x, E_0^y, \) and \( G_0^{xy} \) are the constant values of the elastic modulii \( E_x, E_y, \) and \( G_{xy}, \) respectively, and \( G_0^{xy} = E_0^x/(2(1 + \nu_{xy})). \) As above, \( \nu_{xy} = 0.2. \) In the case of \( c_2 = 1 \) and \( c_1 = 2, \) the values \( E_0^x, E_0^y, \) and \( G_0^{xy} \) can be reached by \( E_x, E_y, \) and \( G_{xy}, \) respectively, at the line \( y = -1 \) (Fig. 6).

The cross-wise distributions of the normal stresses at the cross-section \( x = 0 \) of the strip made of the material (29) and subjected to the force loadings (26) are shown in Fig. 7. As we can see, the inhomogeneity in the material properties (29) causes a similar effect on distribution of the stress-tensor components.

**4.2. Thermal stresses in the inhomogeneous strip**

In absence of the force loadings, the stresses in the inhomogeneous strip are induced by only the non-uniform temperature distribution. For example, let us consider the strip \( \{y = 1 \} \) subjected to the temperature whose Fourier-transformation is found in the form

\[
T = T_0 \frac{\sinh(s(1 + y))}{\sinh(2s)},
\]

\[(30)\]

from the classic heat conduction problem (Carslaw and Jaeger, 1959) under constant thermo-physical properties. Here, \( T_0 \) is the temperature of the side \( y = 1 \) and \( T_0 = \tau_0(x^2 - a^2)e^{-x^2}, \) \( a = 1/\sqrt{2} \) (Rychahivskyy and Tokovy, 2008); \( \tau_0 = \text{const.} \) At the side \( y = -1, \) the temperature is equivalent to zero. Distribution of the temperature with Fourier-transformation (30) is displayed in Fig. 8. As for computation of the stress-tensor components, the material properties are assumed in the form

\[
E_x = E_0^x, \quad E_y = E_0^y, \quad G_{xy} = G_0^{xy}, \quad \alpha_x = \alpha_x^0 v(y),
\]

\[
\alpha_y = \alpha_y^0 v(y), \quad v(y) = (2 + y)^\lambda,
\]

where \( \lambda = \frac{1}{2}, 1, 2 \) in the case of homogeneous, \( \lambda = 0 \) (solid lines), and inhomogeneous, \( \lambda = 1 \) (dashed lines), material.
Fig. 5. Distribution of the dimensionless stress $\sigma_x/p$ at $x = 0$ for the cases $k = 0$ (a) and $k = -1$ (b) while $\lambda = \frac{1}{2}, 1, 2$.

Fig. 6. Distribution of the function $w(y) = (c_1 + c_2 y)^k$ for $c_1 = 2, c_2 = 1, k = 0; -1$.

Fig. 7. $x = 0$-distributions of the dimensionless stresses $\sigma_y/p$ (a) and $\sigma_x/p$ (b) in the strip with material properties (29) for $\lambda = 1, c_1 = 2, c_2 = 1, k = 0$ (solid lines), and $k = -1$ (dashed lines).

Fig. 8. Full-field distribution of the temperature $T/T_0$ computed by means of the inverse Fourier-transformation applied to Eq. (30).
where $E_0^0, x_0^0, x_0^0, k = \text{const}$; $C_{xy}^0 = E_0^0/(2 + 2\nu_y)$; $\nu_y = 0.2$. Distribution of the function $v(y)$ for $k = 0; 1; 2$ is shown in Fig. 9. Considering the plane stress case, we introduce the parameter

$$\gamma = \frac{x_0^0}{x_0^0}.$$

For $k = 0$ and $\lambda = 1$, we obtain the exact analytical solution, since $K(y, \zeta) \equiv 0$ in this case. Moreover, for homogeneous ($k = 0$) and isotropic ($\delta = \gamma = 1$) material, there will be no stresses occurring in the strip due to temperature (30), as the strip is a simply-connected domain with a force-free boundary. Full-field analysis of case $k = \lambda = \gamma = 1$ is shown in Fig. 10. As we can see, the transversal stress $\sigma_y (a)$ and the shear stress $\sigma_{xy} (c)$ satisfy the homogeneous boundary conditions. Moreover, all the stress-tensor components are self-equilibrated.

Analysis of the normal stresses due to temperature (30) is shown in Fig. 11 for different values of the ratios $\lambda$ and $\gamma$, and parameter $k$. For $\lambda = \gamma = 1$ (cases 1, 5, and 9), the stress intensity is as higher as the parameter $k$ is greater. The same conclusion holds for $\lambda = 2$ and $\gamma = 1$ (cases 2 and 6) as well as for $\lambda = 2$ and $\gamma = 2$ (cases 4 and 8). For $\lambda = 1$ and $\gamma = 2$ (cases 3, 7 and 10), the variation of parameter $k$ causes the change in qualitative behavior of the stresses.

5. Conclusions

An efficient technique for analysis of the plane elasticity and thermoelasticity problems for inhomogeneous and orthotropic infinite solids is presented. The original problems are reduced to solution of the Volterra integral equations. From the mathematical point of view, the proposed technique can be employed without any restrictions for the functions prescribing the material properties (besides the existence of corresponding derivatives, at least in a generalized sense). From the mechanical standpoint, the material properties should be accepted in the form which does not lead out of the model of continua mechanics, assuring strain-energy function within the necessary restrictions.

Application of the resolvent-kernel formula to solve the governing integral equation shows advantages over the simple-iteration solution, as mentioned in the Introduction and verified later.
In the case of isotropic material with constant engineering characteristics, the compatibility equation (8) becomes homogeneous and thus it involves the total in-plane stress as the only question for function (Tokovyy and Ma, 2009a). Therefore, Eqs. (12) are uncoupled, and their solutions can be found easily by means of well-known procedures (Timoshenko and Goodier, 1970; Vihak and Rychahuksky, 2001; Vigak, 2004). In spite of the isotropic case, the right-hand member of Eq. (8) will involve, in general, the stress \( \sigma_r \) for the orthotropic material, even though the elastic compliances (4) are constant. For this case, Eq. (8) takes the form

\[
\sigma_y(y) = \frac{i}{s} \left( \frac{d\sigma_y}{dy} - \gamma \right), \quad \sigma_x = \sigma - \sigma_y,
\]

\[
\tilde{\sigma}(y) = \tilde{\sigma}_c(y) + \int_0^\infty R_2(y, \xi)F_2(\xi) d\xi,
\]

where

\[
f_c(y) = e^{-i\gamma} \frac{a_{11}(y)}{a_{11}(y)} + \int_0^\infty R_2(y, \xi) e^{-i\gamma} d\xi,
\]

\[
F_2(y) = \Theta_2(y) + P_2(y) + Q_2(y) + \Psi_2(y),
\]

\[
P_2 = \frac{p_0 y(y)}{2a_{11}(y)}, \quad Q_2 = \frac{\imath s q_0 y(y)}{2|s| a_{11}(y)},
\]

\[
\Theta_2(y) = \frac{1}{a_{11}(y)} \left[ -\gamma \Theta(y) + \frac{|s|}{2} \int_0^\infty (\xi \Theta(\eta) - \xi \Theta(\eta)) \Theta(\eta) e^{-i\gamma} d\eta \right],
\]

\[
\Psi_2 = -\frac{1}{2|s| a_{11}(y)} \left[ \int_0^\infty \left( \beta_1(\eta) \left( \frac{d\Theta(\eta)}{d\eta} - \imath s \Theta(\eta) \right) - \frac{d}{d\eta} \left( \Theta(\eta) \frac{d\gamma}{d\eta} \right) \right) e^{-i\gamma} d\eta + \int_0^\infty \gamma(\eta) d\eta \right.
\]

\[
\left. \times \int_0^\infty \left( \xi \Theta(\xi) - \frac{d\Theta(\xi)}{d\xi} \right) e^{-i\gamma} d\xi d\eta + \chi(y) \Theta(0) \right],
\]

\[
\gamma(y, \eta) = e^{-i\gamma} \left( 2\beta_1(0) - \frac{1}{C_{\gamma y}(0)} \right) - \int_0^\infty e^{-i\gamma} \gamma(y, \eta) d\eta.
\]

\[
R_2(y, \xi) = \sum_{n=1}^\infty K_{n+1}(y, \xi) K_1(t, \xi) dt, \quad n = 1, 2, \ldots
\]

\[
K_{n+1} = \int_0^\infty K_1(t, \xi) K_n(t, \xi) dt,
\]

\[
\gamma(y, \eta) = -\frac{1}{2} \left( \frac{d^2\beta_1(\eta)}{d\eta^2} + s^2\beta_2(\eta) \right) e^{-|y|\eta},
\]

\[
+ \frac{d}{d\eta} \left( 2\beta_1(\eta) - \frac{1}{C_{\gamma y}(\eta)} \right) e^{-|y|\eta} \text{sgn}(y - \eta),
\]

\[
\sigma_y(y) = \left( \frac{\sqrt{2}q_0 + \sqrt{2} \gamma}{|s|} - \rho_0 \right) e^{-|y|\eta} - 1 - \frac{1}{2s^2} \int_0^\infty \left( \sigma(y) e^{-|y|\eta} \right) dy + |s| \int_0^\infty \sigma(y) e^{-|y|\eta} d\xi,
\]
Appendix B. Elasticity and thermoelasticity solution in the case of plane

For the plane $\Omega_1$, the solution to the problems (1) and (8) appear as:

$$
\sigma_y(y) = \frac{1}{2|s|} \int_{-\infty}^{\infty} \left( \frac{d\bar{Y}(\xi)}{d\xi} - ix\bar{X}(\xi) \right) e^{-|y|/\eta} d\xi + \frac{|s|}{2}
$$

$$
\tau_{xy}(y) = \frac{i}{s} \left( \frac{d\sigma_y(y)}{dy} + \bar{Y}(y) \right),
$$

$$
\sigma_y(y) = \Theta_1(y) + \Psi_1(y) + \int_{-\infty}^{\infty} R_1(y, \xi) (\Theta_1(\xi) + \Psi_1(\xi)) d\xi,
$$

where

$$
\Theta_1(y) = \frac{1}{a_{11}(y)} \left[ -\chi_1(y) \bar{T}(y) + \frac{|s|}{2} \int_{-\infty}^{\infty} (\chi_1(\eta) - \chi_2(\eta)) \bar{T}(\eta) e^{-|y|/\eta} d\eta \right],
$$

$$
\Psi_1(y) = \frac{1}{2|s|a_{11}(y)} \left[ \int_{-\infty}^{\infty} \gamma_1(y, \eta) \int_{-\infty}^{\infty} \left( \frac{d\bar{Y}(\xi)}{d\xi} - ix\bar{X}(\xi) \right) e^{-|\eta|/\eta} d\xi d\eta \right.
- \int_{-\infty}^{\infty} \left[ \beta_1(\eta) \left( \frac{d\bar{Y}(\eta)}{d\eta} - ix\bar{X}(\eta) - \frac{d}{d\eta} \bar{Y}(\eta) \right) \right] e^{-|\eta|/\eta} d\eta \right].
$$

$$
R_1(y, \xi) = \sum_{n=0}^{\infty} K_{n+1}(y, \xi),
K_{n+1}(y, \xi) = \frac{|s|}{2a_{11}(y)} \int_{-\infty}^{\infty} \gamma_1(y, \eta) e^{-|\eta|/\eta} d\eta,
K_{n+1} = \int_{-\infty}^{\infty} K_1(y, t) K_{n}(t, \xi) dt,
$$

$$
\gamma_1(y, \eta) = \frac{1}{2} \left[ \left( \frac{d\beta_1(\eta)}{d\eta} + s^2 \beta_1(\eta) \right) e^{-|y|/\eta} \right.
+ \frac{d}{d\eta} \left[ 2\beta_1(\eta) - \frac{1}{a_{11}(\eta)} e^{-|y|/\eta} \text{sgn}(y - \eta) \right].
$$

Appendix C. Supplementary data

Supplementary data associated with this article can be found, in the online version, at doi:10.1016/j.ijsolstr.2009.07.007.