



Origin of constrained maximal CP violation in flavor symmetry



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ABSTRACT

Current data from neutrino oscillation experiments are in good agreement with $\delta = -\frac{\pi}{2}$ and $\theta_{23} = \frac{\pi}{4}$ under the standard parametrization of the mixing matrix. We define the notion of “constrained maximal CP violation” (CMCPV) for predicting these features and study their origin in flavor symmetry. We derive the parametrization-independent solution of CMCPV and give a set of equivalent definitions for it. We further present a theorem on how the CMCPV can be realized. This theorem takes the advantage of residual symmetries in neutrino and charged lepton mass matrices, and states that, up to a few minor exceptions, $(|\delta|, \theta_{23}) = (\frac{\pi}{2}, \frac{\pi}{4})$ is generated when those symmetries are real. The often considered μ - τ reflection symmetry, as well as specific discrete subgroups of $O(3)$, is a special case of our theorem.

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1. Introduction

While a coherent picture in leptonic mixing has emerged, important measurements are still lacking. In particular, the Dirac CP angle δ and the exact value of the atmospheric neutrino mixing angle θ_{23} are of great interest. Whether θ_{23} is maximal or departs sizably from $\frac{\pi}{4}$ has important ramifications for flavor symmetry models [1]. The CP phase also has model building impact, and the question of whether the lepton sector violates CP has conceptual significance in connection to the matter–antimatter asymmetry via leptogenesis [2]. While maximal atmospheric mixing is compatible with data since the observation of atmospheric neutrino oscillations, recently first hints towards a Dirac CP angle $\delta = -\frac{\pi}{2}$ have arisen from the appearance and disappearance measurements of T2K [3] when combined with reactor antineutrino data. Indeed, global fits [4–6] confirm a mild preference for this particular value of CP phase.

With these in mind, it is tempting to study the origin of such values of δ and θ_{23} within theories of flavor symmetry. In particular, the so-called μ - τ reflection symmetry [7–10] was often considered in the literature in this respect. It transforms the neutrino fields as $(\nu_e, \nu_\mu, \nu_\tau) \rightarrow (\nu_e^*, \nu_\tau^*, \nu_\mu^*)$, leading to $|\delta| = \frac{\pi}{2}$ and $\theta_{23} = \frac{\pi}{4}$ in the standard parametrization of the PMNS mixing ma-

trix [11,12]. In our study, we demonstrate that these two features arise as the outcome of “Constrained Maximal CP Violation” (CMCPV), which we will establish in a parametrization-independent way by maximizing the Jarlskog invariant under a minimal constraint.

The framework we will discuss is that a flavor symmetry group G is broken such that the neutrino and charged lepton mass matrices are invariant under certain subgroups of G . We will propose and prove a general theorem revealing that if the residual flavor symmetries are real,¹ then the CMCPV is generated. There are a few minor exceptions to this theorem which we will clarify in Section 3. The μ - τ reflection symmetry is actually a special case of this theorem, which can be shown explicitly after a simple basis transformation. We further deduce some corollaries from the theorem which are practically useful in understanding and building models for the CMCPV. For instance, specific subgroups of $O(3)$ can generate CMCPV, so do the models with certain groups under which all neutrino fields transform as triplets. As an illustration, we will present a simple model to explicitly realize the CMCPV.

This paper is organized as follows. In Section 2, we will establish our definition of CMCPV in a parametrization-independent way, and give a set of equivalent descriptions. Various physical implications (such as leptonic unitarity triangles) from CMCPV are further discussed. In Section 3, we present our theorem for the origin of CMCPV and derive its corollaries which are important for practical model buildings. We will study applications in Section 4,

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¹ Here and henceforth “a symmetry is real” always means that the transformation matrix representing the symmetry is real.

and finally we conclude in Section 5. Some elaborated mathematical proofs are presented in [Appendices A and B](#).

2. Parametrization-independent formulation of constrained maximal CP violation

What we mean by “constrained maximal CP violation” (CMCPV) is not merely $|\delta| = \frac{\pi}{2}$ in the standard parametrization of the PMNS matrix [11,12], but both the $|\delta| = \frac{\pi}{2}$ and $\theta_{23} = \frac{\pi}{4}$. In general, a parametrization-independent definition of the maximal CP violation should be given in terms of Jarlskog invariant J [13], rather than the CP angle δ , because δ is not rephasing invariant. Furthermore, we will clarify shortly that naively maximizing J without constraint is already excluded by experimental data. Hence, introducing the new concept of CMCPV is essential for studying the viable maximal CP violation. For this purpose, we first formulate the CMCPV in a parametrization-independent form.

Definition 1 (CMCPV). Constrained Maximal CP Violation (CMCPV) is defined as the maximum of the absolute value of Jarlskog invariant under the minimal constraint that the absolute values of the elements in the first row of the PMNS matrix are fixed.

Definition 1 is parametrization-independent because it does not invoke any explicit form of the PMNS matrix. Note that this is a constrained maximization problem. The Jarlskog invariant J [13] can be regarded as a function of the PMNS matrix U . We are looking for the maximal values of the function $J(U)$, where U is not an arbitrary unitary matrix but a constrained one. We impose this constraint on U by requiring the absolute values of its elements in the first row, $(|U_{e1}|, |U_{e2}|, |U_{e3}|)$, be fixed to certain given values. (Actually, fixing any two of them in the first row is enough, due to the unitarity condition $|U_{e1}|^2 + |U_{e2}|^2 + |U_{e3}|^2 = 1$.) The reason that we fix absolute values of the elements of U in its first row, rather than any other rows or columns, will become clear shortly (cf. footnote 2).

This constraint is necessary because without it J would reach its maximal value as allowed by unitarity, $|J| = \frac{1}{6\sqrt{3}}$, which is equivalent to all unitarity triangles being equilateral. The corresponding U in this case is just the Wolfenstein mixing matrix [14],

$$U_W = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}, \quad (1)$$

with $\omega = e^{i2\pi/3}$, which has been excluded by oscillation data. In the standard parametrization [12], the Jarlskog invariant is given by

$$J = \frac{1}{8} \sin \delta \cos \theta_{13} \sin 2\theta_{13} \sin 2\theta_{23} \sin 2\theta_{12}. \quad (2)$$

Indeed, if we compute its maximum by $\partial_\theta J = 0$ with $\theta = \{\theta_{12}, \theta_{13}, \theta_{23}, \delta\}$, we obtain the Wolfenstein mixing, $(\theta_{13}, \theta_{12}, \theta_{23}, |\delta|) = (\arctan \frac{1}{\sqrt{2}}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2})$. This includes the desired values of (θ_{23}, δ) , but gives unrealistic $(\theta_{12}, \theta_{13})$. Hence, the Wolfenstein mixing is already excluded by experimental data. To derive acceptable maximization of J , we observe that if we fix θ_{12} and θ_{13} (to their best-fit values for instance) and then maximize J , we still obtain $(|\delta|, \theta_{23}) = (\frac{\pi}{2}, \frac{\pi}{4})$. This is in fact consistent with the above parametrization-independent **Definition 1** of CMCPV, because fixing $(\theta_{12}, \theta_{13})$ corresponds to fixing the absolute values of the elements in the first row of U under the standard parametrization. Note that this is the allowed minimal constraint we could impose on the Jarlskog invariant: fixing any other row or column

of the PMNS matrix and then maximizing J will not lead to experimentally acceptable results.² Hence, the above **Definition 1** gives a minimal parametrization-independent definition of viable maximal CP violation.

It is worth noting that the Jarlskog invariant can be further cast into a manifestly parametrization-independent form [15],

$$J^2 = |U_{\alpha i}|^2 |U_{\alpha j}|^2 |U_{\beta i}|^2 |U_{\beta j}|^2 - \frac{1}{4} (1 - |U_{\alpha i}|^2 - |U_{\alpha j}|^2 - |U_{\beta i}|^2 - |U_{\beta j}|^2 + |U_{\alpha i}|^2 |U_{\beta j}|^2 + |U_{\alpha j}|^2 |U_{\beta i}|^2)^2 \quad (3a)$$

$$= |U_{e1}|^2 |U_{e3}|^2 |U_{\mu 1}|^2 |U_{\mu 3}|^2 - \frac{1}{4} (1 - |U_{e1}|^2 - |U_{e3}|^2 - |U_{\mu 1}|^2 - |U_{\mu 3}|^2 + |U_{e1}|^2 |U_{\mu 3}|^2 + |U_{e3}|^2 |U_{\mu 1}|^2)^2. \quad (3b)$$

The above Eq. (3a) is a general expression with $(\alpha, \beta) = e, \mu, \tau$ and $(i, j) = 1, 2, 3$, where $\alpha \neq \beta$ and $i \neq j$. Eq. (3b) corresponds to the case of $(\alpha, \beta) = (e, \mu)$ and $(i, j) = (2, 3)$. It is clear that the Jarlskog invariant can be fully determined by any 4 independent matrix elements $\{|U_{\alpha i}|, |U_{\alpha j}|, |U_{\beta i}|, |U_{\beta j}|\}$ with $\alpha \neq \beta$ and $i \neq j$, as shown in Eq. (3a). According to our above **Definition 1** of CMCPV and using Eq. (3b), we can maximize Jarlskog invariant by imposing the extremal conditions with the two elements $(|U_{e1}|, |U_{e3}|)$ of first row fixed, $\partial J^2 / \partial |U_{\mu 1}|^2 = 0$ and $\partial J^2 / \partial |U_{\mu 3}|^2 = 0$. From these two equations, we can derive the solution of CMCPV,

$$|U_{\mu j}|^2 = |U_{\tau j}|^2 = \frac{1}{2} (1 - |U_{ej}|^2) \quad (j = 1, 2, 3), \quad (4)$$

which is the maximum of J under the minimal constraint of **Definition 1**, and is manifestly parametrization-independent. The detail of this derivation is presented in [Appendix A](#).

When adopting the standard parametrization [12] of PMNS mixing matrix U , we can use Eq. (4) to immediately deduce the explicit realization of CMCPV, $(|\delta|, \theta_{23}) = (\frac{\pi}{2}, \frac{\pi}{4})$, which is shown in [Appendix A](#). Furthermore, we find that once we realize $(|\delta|, \theta_{23}) = (\frac{\pi}{2}, \frac{\pi}{4})$ under the standard parametrization [12], the PMNS matrix exhibits an interesting feature, which we explain as follows. The standard parametrization of the PMNS matrix is expressed as [12],

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - c_{23}s_{12}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \quad (5)$$

where we have used the common notations $(s_{ij}, c_{ij}) = (\sin \theta_{ij}, \cos \theta_{ij})$. Under the rephasing

$$U' \equiv \text{diag}(1, e^{-i\delta}, e^{-i\delta}) U \text{diag}(1, 1, e^{i\delta}), \quad (6)$$

we obtain

$$U' = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\ -c_{12}s_{23}s_{13} - s_{12}c_{23}e^{-i\delta} & -s_{12}s_{23}s_{13} + c_{12}c_{23}e^{-i\delta} & s_{23}c_{13} \\ -c_{12}c_{23}s_{13} + s_{12}s_{23}e^{-i\delta} & -s_{12}c_{23}s_{13} - c_{12}s_{23}e^{-i\delta} & c_{23}c_{13} \end{pmatrix}. \quad (7)$$

For $(\delta, \theta_{23}) = (\pm \frac{\pi}{2}, \frac{\pi}{4})$, we find that U' becomes

$$U'_m = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}c_{12}c_{13} & \sqrt{2}s_{12}c_{13} & \sqrt{2}s_{13} \\ -c_{12}s_{13} \pm i s_{12} & -s_{12}s_{13} \mp i c_{12} & c_{13} \\ -c_{12}s_{13} \mp i s_{12} & -s_{12}s_{13} \pm i c_{12} & c_{13} \end{pmatrix}. \quad (8)$$

² To be explicit, we have directly verified that fixing the second row or the third row of the PMNS matrix will result in $|U_{ej}| = |U_{\tau j}|$, or $|U_{ej}| = |U_{\mu j}|$ ($j = 1, 2, 3$), respectively. Fixing the first, second or third columns will lead to $|U_{\ell 2}| = |U_{\ell 3}|$, $|U_{\ell 1}| = |U_{\ell 3}|$, or $|U_{\ell 1}| = |U_{\ell 2}|$ ($\ell = e, \mu, \tau$), respectively. All these cases are already excluded by the current neutrino oscillation data.

Eq. (8) explicitly reveals that the first row is real, while the second row and the third row are complex conjugates of each other. We call this feature the “row conjugation equality” (RCE). It is easy to see that the reverse is also true: holding RCE will result in $(|\delta|, \theta_{23}) = (\frac{\pi}{2}, \frac{\pi}{4})$. Even though we have demonstrated RCE by using the standard parametrization (5) [12], we stress that the RCE form should be independent of parametrizations (up to trivial rephasing). We can use any other parametrizations [16,17] and maximize Jarlskog invariant under the same constraint as in Definition 1. Then, we find that the mixing matrix always exhibits the RCE form, up to a trivial rephasing. In fact, we see that any specific RCE form does obey $|U_{\mu j}| = |U_{\tau j}|$ as in our parametrization-independent general solution (4) of the CMCPV.

For later usage, let us introduce the following lemma on the RCE.

Lemma (*O(3) invariance of RCE*). *If a unitary matrix V has the form of RCE, then after a right-handed real transformation $V \rightarrow V' = VR$, the matrix V' should still have the form of RCE, where $R \in O(3)$ is an arbitrary orthogonal matrix.*

The proof of this statement is delegated to Appendix B.1.

Another interesting feature of RCE concerns leptonic unitarity triangles (LUTs) of the PMNS matrix U , in connection to its column orthogonality,

$$U_{ei}^* U_{ej} + U_{\mu i}^* U_{\mu j} + U_{\tau i}^* U_{\tau j} = 0, \quad (9)$$

where the column indices $i \neq j$. We call these unitarity triangles the *column triangles*. It is evident that if U has the form of RCE, then all column triangles should be *isosceles triangles* because under the RCE the two sides $U_{\mu i}^* U_{\mu j}$ and $U_{\tau i}^* U_{\tau j}$ have equal length, $|U_{\mu i}^* U_{\mu j}| = |U_{\tau i}^* U_{\tau j}|$.

Unitarity triangles are intrinsically connected to CP violation because all these triangles have the same area, which equals half of the absolute value of Jarlskog invariant J . The LUTs are less studied than the unitarity triangles in quark sector since measuring the LUTs and thus the leptonic CP violation is much harder. Nevertheless, the LUTs can be directly measured in principle via oscillation experiments [18]. Furthermore, the LUTs can provide a geometrical formulation of the CMCPV. Since we define CMCPV as J reaching its maximum under certain constraints, it also means that the area of the LUT reaches its maximum under those constraints. How do these constraints appear in our current geometrical picture? The constraint in our Definition 1 is that the first row of U is fixed, namely, $|U_{e1}^* U_{e2}|$, $|U_{e2}^* U_{e3}|$ and $|U_{e3}^* U_{e1}|$ are fixed, which means that the e -sides of the column triangles are fixed. Hence, Definition 1 is equivalent to saying that each column triangle reaches its maximal area with its e -side fixed. This provides a *geometrical formulation* of the CMCPV.

Note that for a triangle with its e -side fixed and its perimeter (the sum of the lengths of its three sides) bounded from above, its area reaches the maximum if and only if it is an isosceles triangle. This is clear from geometrical intuition. In Ref. [19], we proved that a unitarity triangle must always have its perimeter equal or less than 1. This is a necessary and sufficient condition for a triangle to be unitarity triangle, and requires the perimeter of each unitarity triangle to be bounded from above, which ensures the area of each unitarity triangle to have a maximum. With these, we give a geometrical formulation of the CMCPV: it corresponds to the maximal area of the LUT by fixing its e -side, and such LUT is an isosceles triangle.

Finally, we summarize the analysis of this section into the following theorem.

Theorem 1 (*Equivalent definitions of CMCPV*). *For the PMNS mixing matrix U , the following statements are equivalent:*

- it has the CMCPV (cf. Definition 1);
- for any parametrization of U , the general condition (4) holds;
- in the standard parametrization, $(|\delta|, \theta_{23}) = (\frac{\pi}{2}, \frac{\pi}{4})$ holds;
- it has the form of RCE (up to rephasing);
- each column triangle reaches the maximal area with its e -side fixed;
- each column triangle is an isosceles triangle.

After setting up the above preliminaries, we are ready to study the origin of CMCPV in flavor symmetry in the next section.

3. Origin of constrained maximal CP violation

In this section, we will trace CMCPV to the “residual symmetries”, i.e., the subgroups of the original flavor symmetry group that remain intact after the full group is broken.

Consider that the flavor symmetry group G is broken down to two residual symmetries G_ν and G_ℓ for neutrinos and charged leptons, respectively. They are defined as follows,

$$G \rightarrow \begin{cases} G_\nu : \{S | S^T M_\nu S = M_\nu\}; \\ G_\ell : \{T | T^\dagger M_\ell M_\ell^\dagger T = M_\ell M_\ell^\dagger\}; \end{cases} \quad (10)$$

where M_ν is the Majorana mass matrix of neutrinos, and $M_\ell M_\ell^\dagger$ is the effective mass matrix of left-handed charged leptons. Thus, the mixing matrices U_ν and U_ℓ (which diagonalize M_ν and $M_\ell M_\ell^\dagger$, respectively) are directly determined by S and T [20],

$$\begin{aligned} U_\nu^\dagger S U_\nu &= D_S, \\ U_\ell^\dagger T U_\ell &= D_T. \end{aligned} \quad (11)$$

Here the matrices D_S and D_T are diagonal matrices. Since Eq. (11) demonstrates a direct connection between flavor symmetry and lepton mixings, it also attracted extensive studies [21–24] via the approach of symmetry and group theory, without resorting to explicit mass matrices or a fundamental Lagrangian. With this *general mass-independent approach*, we will analyze the origin of CMCPV.

Roughly speaking, our theorem states that the CMCPV can be realized if the residual symmetries are real. In rigorous manner, we formulate this theorem in the following form.

Theorem 2 (*Origin of CMCPV*). *If the residual flavor symmetries in the lepton sector (including charged leptons and neutrinos) are real and fully determine the mixing pattern, then the CMCPV always holds, up to a few minor exceptions:*

- one of the three mixing angles in the PMNS matrix is zero;
- neutrinos are not Majorana fermions;
- the residual symmetry for charged leptons is a Klein group, i.e., $G_\ell = Z_2 \otimes Z_2$.

It is clear that exception (i) is already excluded by current oscillation data, and exception (ii) is not a concern for most neutrino theories. Exception (iii) is less trivial, but can be easily evaded in model-buildings. Besides, in Theorem 2, for the residual symmetries being real, we mean that there always exists a basis under which these symmetries become real.

To illustrate this theorem explicitly, we first consider a simple (unrealistic) example. A rotation of 120° around the axis $(1, 1, 1)^T$ can be represented by

$$R = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \quad (12)$$

Suppose that this R corresponds to the residual symmetry group G_ℓ . Thus, we have

$$U_W^\dagger R U_W = \text{diag}(1, \omega, \omega^2), \quad (13)$$

where U_W is the Wolfenstein mixing matrix defined in Eq. (1). According to the relation (11), we have $U_\ell = U_W$, which shows that U_ℓ^\dagger exhibits RCE. If we further assume that the neutrino mass matrix is diagonal (and thus $U_\nu = I$), then the PMNS matrix $U = U_\ell^\dagger U_\nu$ gives, $(|\delta|, \theta_{23}) = (\frac{\pi}{2}, \frac{\pi}{4})$, because Theorem 1 states that the RCE always leads to CMCPV.

Two remarks are in order for this example. One is that both residual symmetries are real. This is explicit for G_ℓ . The form of G_ν is, $G_\nu \supset \{\text{diag}(1, 1, -1), \text{diag}(1, -1, 1)\}$, since neutrino mass matrix is taken to be diagonal. The other point is that U_ℓ^\dagger exhibits RCE.

In general, the validity of Theorem 2 (excluding its exceptions) implies the following important points:

- 1). real G_ν leads to real U_ν , which will be explicitly proven in Appendix B.2;
- 2). real G_ℓ leads to RCE in U_ℓ^\dagger , which will be explicitly proven in Appendix B.3;
- 3). if U_ℓ^\dagger has RCE and U_ν is real, then the PMNS matrix $U = U_\ell^\dagger U_\nu$ has RCE.

The three points above are combined to prove Theorem 2. The last point is just based on the lemma given above Eq. (9), and the first two points can be understood by the following reasoning. (We delegate the mathematical proofs to Appendix B.) Since both G_ℓ and G_ν contain only real transformations, they can be geometrically regarded as rotations in 3-dimensional Euclidean space. (Here a trivial minus sign between the determinants of SO(3) and O(3) does not matter.) For such a rotation represented by a matrix R , the rotation axis is one of its eigenvectors with the corresponding eigenvalue equal to 1. The remaining two eigenvectors must be complex conjugate to each other, which is a general property of SO(3) matrices. (The explicit forms of the two eigenvectors are given in Appendix B.3.) Hence, if $R \in G_\ell$, then the eigenvectors of R are the columns of U_ℓ , which implies two columns in U_ℓ are conjugate to each other, and thus U_ℓ^\dagger has RCE.

There are differences in the neutrino sector, because we consider the neutrinos as Majorana particles here. Hence, the residual symmetry has to be constructed with Z_2 's, i.e., $G_\nu = Z_2 \otimes Z_2$, which geometrically correspond to two rotations of 180° . These are special rotations in the sense that only such rotations may commute with rotations around different axes. For rotations of 180° , the eigenvalues are $(1, -1, -1)$, cf. Eq. (B.7). Due to a partial degeneracy of the eigenvalues, the neutrino mass matrix M_ν should be determined by two Z_2 -rotations with orthogonal axes. Each axis determines one column of U_ν , so U_ν only contains real column vectors. This in turn implies that G_ℓ cannot be $Z_2 \otimes Z_2$, which is the exceptional case (iii) pointed out in Theorem 2: if $G_\ell = Z_2 \otimes Z_2$, then U_ℓ will be real and no CP violation exists. Now, it is also easy to understand why Theorem 2 requires neutrinos to be Majorana fermions, since the symmetries $Z_2 \otimes Z_2$ are needed for G_ν .

Theorem 2 further leads to a series of corollaries which we will discuss as follows.

Corollary A ($O(3)$ subgroups). *If an $O(3)$ subgroup G contains sufficient residual symmetries that can fully determine a mixing matrix, then it*

leads to the CMCPV after avoiding the three exceptions listed in Theorem 2.

This is manifest because the constraint which requires the residual symmetries to be subgroups of O(3) makes G_ℓ and G_ν automatically real. According to Theorem 2, this leads to CMCPV. Examples of such residual symmetries include popular groups like A_4 , S_4 , and A_5 , corresponding to tetrahedral, octahedral, and icosahedral symmetries, respectively.

We should comment on the phrase “sufficient residual symmetries” in Corollary A. As is well-known, the maximal residual symmetries in the charged lepton and neutrino sectors are $U(1) \otimes U(1) \otimes U(1)$ and $Z_2 \otimes Z_2 \otimes Z_2$, respectively [20]. But when seeking flavor groups to unify the residual symmetries, it is not necessary to cover those large groups. For charged leptons, the minimal choice is to take a Z_3 subgroup from those $U(1)$'s, which is in fact sufficient to determine the mixing U_ℓ . For the neutrino sector, the minimal sufficient residual symmetry should be $Z_2 \otimes Z_2$. So, this smaller set of residual symmetries should be included in the O(3) subgroup for Corollary A.

However, in some models, especially those based on A_4 , sometimes the flavor group does not contain sufficient residual symmetries, and the so-called accidental symmetries are present to fully determine the mixings. Those accidental symmetries depend on the detailed dynamics of the model (instead of the flavor group), so Corollary A does not apply. But, if the accidental symmetry is a real symmetry, then Theorem 2 applies and there is still CMCPV.

Corollary B (Real M_ν). *If G_ℓ is real and M_ν is real or M_ν can be written as $M_\nu = z_1 \mathcal{I} + z_2 \tilde{M}_\nu$, where \tilde{M}_ν and \mathcal{I} are real and identity matrices, respectively, and (z_1, z_2) are complex numbers, then there is CMCPV, after evading the three exceptions listed in Theorem 2.*

We first consider the case that M_ν is real. Then, as a real symmetric matrix, M_ν can be diagonalized by a real orthogonal matrix, which implies U_ν and G_ν are real. Hence, according to Theorem 2, we have CMCPV. Multiplying M_ν by an overall complex phase will not change U_ν . Thus, if $M_\nu = z_2 \tilde{M}_\nu$ with real \tilde{M}_ν , then this means that M_ν is essentially real, up to an overall complex phase factor. Hence, this case also leads to CMCPV. Next, consider $M_\nu = z_1 \mathcal{I} + \tilde{M}_\nu$ with real \tilde{M}_ν . This means that M_ν is real up to subtracting a constant from all diagonal elements. In this case, for $S (\in G_\nu)$ satisfying $S^T \tilde{M}_\nu S = \tilde{M}_\nu$, we have $S^T M_\nu S = S^T (z_1 \mathcal{I} + \tilde{M}_\nu) S = z_1 \mathcal{I} + \tilde{M}_\nu$, which shows that M_ν is also invariant under S . Hence, M_ν has invariance under real G_ν and there is CMCPV. Finally, combining the two cases above, we have thus proven Corollary B for the general form $M_\nu = z_1 \mathcal{I} + z_2 \tilde{M}_\nu$.

The form $M_\nu = z_1 \mathcal{I} + z_2 \tilde{M}_\nu$ has important applications in model buildings for CMCPV. Typically, for building flavor symmetry models, at least one flavon ϕ is introduced to couple with neutrinos ν and forms a Yukawa term $\nu\nu\phi$, which contributes to neutrino masses if the vacuum expectation values (VEV) $\langle\phi\rangle \neq 0$, where ϕ is a scalar field acting as a multiplet under the flavor symmetry. Neutrinos are commonly considered as flavor triplets in many models, so a $\nu\nu$ term or $\nu\nu\xi$ term will usually show up, where ξ is a flavor singlet. These terms will contribute to M_ν as a diagonal mass term $z_1 \mathcal{I}$, where z_1 is complex because the coefficients (Yukawa couplings) of these terms are complex in general. The $\nu\nu\phi$ term will contribute in a form of $z_2 \tilde{M}_\nu$ if $\langle\phi\rangle$ is real (up to an overall complex phase) and the Clebsch–Gordan (CG) coefficients for the term are real.

Note that here we only consider the case with all 3 generations of neutrinos unified into a triplet of the flavor group. Otherwise,

the $\nu\nu$ term would not be diagonal.³ The flavon ϕ can be in any non-trivial representation. We should point out that both real $\langle\phi\rangle$ and real CG coefficients are very common in many groups. For instance, in the 3-dimensional representation of A_4 , the CG coefficients for $3 \otimes 3 \otimes 3 \rightarrow 1$ are real in both the real basis (used for example in [25]) and the complex basis (used for example in [26]). This corollary does not apply to groups with inherent complex CG coefficients, like T' [27] or $\Delta(27)$ [28]. As for real $\langle\phi\rangle$, if ϕ is a real scalar field by definition, then $\langle\phi\rangle$ is real. If ϕ has to be a complex field, then as known from minimization of scalar potentials in many models, it is still common to have VEV alignment according to $\langle\phi\rangle \propto (1, 1, 1)$, $(1, 0, 0)$, etc., which is real. If $\langle\phi\rangle$ is however complex, then in general CMCPV does not follow. For those “real $\nu\nu\phi$ models”, where only the $\nu\nu\phi$ term makes a non-trivial contribution (not proportional to the unit matrix) to M_ν , we have the following corollary.

Corollary C (Real $\nu\nu\phi$ models). *Real $\nu\nu\phi$ models always lead to CMCPV, if the three minor exceptions listed in Theorem 2 do not happen.*

For a demonstration of the above general discussion, we will build a simple “real $\nu\nu\phi$ model” in the following Section 4.3.

4. Applications

In this section, we apply our theorems and corollaries to various situations and understand why CMCPV is realized in certain cases. We will illustrate how to achieve the CMCPV in model buildings. There are extensive recent literature [29] studying specific models of $(|\delta|, \theta_{23}) = (\frac{\pi}{2}, \frac{\pi}{4})$.

4.1. μ - τ reflection symmetry

The μ - τ reflection symmetry was studied before [7–10], which is sometimes also called the generalized μ - τ symmetry. This symmetry is defined in the flavor eigenbasis (with M_ℓ diagonal),

$$\nu_e \rightarrow \nu_e^*, \quad \nu_\mu \rightarrow \nu_\tau^*, \quad \nu_\tau \rightarrow \nu_\mu^*. \quad (14)$$

Imposing this symmetry leads to the following form of the neutrino and lepton mass matrices,

$$\tilde{M}_\nu = \begin{pmatrix} r_1 & z_1 & z_1^* \\ \cdot & z_2 & r_2 \\ \cdot & \cdot & z_2^* \end{pmatrix}, \quad \tilde{M}_\ell^2 = \begin{pmatrix} m_e^2 & & \\ & m_\mu^2 & \\ & & m_\tau^2 \end{pmatrix}, \quad (15)$$

where \tilde{M}_ν is symmetric and $\tilde{M}_\ell^2 \equiv \tilde{M}_\ell \tilde{M}_\ell^\dagger$ is diagonal. Note that the elements $r_{1,2}$ are real, but $z_{1,2}$ are complex in general. The operation (14) will transform $\nu^T \tilde{M}_\nu \nu$ to its Hermitian conjugate. We can directly check that the Lagrangian term, $\mathcal{L} \supset \nu^T \tilde{M}_\nu \nu + \text{h.c.}$, is invariant under the transformation (14) if and only if \tilde{M}_ν takes the form of Eq. (15).

Let us make a transformation,

$$M_\nu = U_\ell \tilde{M}_\nu U_\ell^T, \quad M_\ell^2 = U_\ell \tilde{M}_\ell^2 U_\ell^\dagger, \quad (16)$$

with

$$U_\ell = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}. \quad (17)$$

Thus, we derive

$$M_\nu = \begin{pmatrix} r_1 & \sqrt{2}z_{11} & \sqrt{2}z_{12} \\ \cdot & r_2 + z_{21} & z_{22} \\ \cdot & \cdot & r_2 - z_{21} \end{pmatrix},$$

$$M_\ell^2 = \begin{pmatrix} a & 0 & 0 \\ 0 & b_+ & ib_- \\ 0 & -ib_- & b_+ \end{pmatrix}, \quad (18)$$

where we have defined notations, $z_j \equiv z_{j1} + iz_{j2}$ ($j = 1, 2$), and $a \equiv m_e^2$, $b_\pm \equiv \frac{1}{2}(m_\mu^2 \pm m_\tau^2)$. The quantities (z_{j1}, z_{j2}) and (a, b_\pm) are all real. Note that M_ν is a real matrix, and the charged lepton sector has an $\text{SO}(2)$ residual symmetry,

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}. \quad (19)$$

This is because $RM_\ell^2 R^\dagger = M_\ell^2$ holds for $\theta \in [0, 2\pi)$. Since M_ν and G_ℓ are all real, this will lead to CMCPV according to our Corollary B. Note that real M_ν implies that G_ν is real.

The μ - τ reflection symmetry is certainly not the only possibility to generate CMCPV. From Eq. (18), we see that M_ν is real, while Corollary B shows that M_ν can have a more general form including complex numbers. Hence, the μ - τ reflection symmetry is just a special case of real residual symmetries, although this is not manifest before the transformation of basis in Eq. (16).

4.2. CMCPV from geometrical symmetry breaking

As another example illustrating our theorem, we revisit a model from Ref. [23], which predicted $|\delta| = \frac{\pi}{2}$ and $\theta_{23} = \frac{\pi}{4}$ (as well as $\theta_{13} \simeq \frac{\pi}{4} - \theta_{12}$). We will show that this model fulfills the criteria for CMCPV.

This model identifies a Z_4 rotation around the x -axis as G_ℓ , and the product reflections $Z_2 \otimes Z_2$ as G_ν , where one Z_2 reflects $y \rightarrow -y$ and the other Z_2 transforms $(x, z) \rightarrow -(z, x)$. These rotations are subgroups of the octahedral symmetry O_h , and can be shown by simple geometrical picture. This group setting generates the bimaximal mixing, $\theta_{12} = \theta_{23} = \frac{\pi}{4}$ and $\theta_{13} = 0$. The necessary deviation from this leading order scheme was generated by slightly tilting the axis of Z_4 rotation by a small angle (defined as $\sqrt{2}\epsilon$) that turns out to be related to nonzero θ_{13} . We also verified that this geometrical symmetry breaking can arise from certain flavon models. For example, we may set up a concrete realization, where a flavor triplet ϕ is responsible for mass-generation of the charged leptons and the Yukawa terms involving ϕ are $\text{SO}(3)$ symmetric in the 3-dimensional flavor space. With these, the geometrical breaking is connected to the VEV misalignment of flavons. After the axis tilt, the residual symmetry of charged lepton mass matrix is represented by [23],

$$R_\ell = \begin{pmatrix} 1 & -2\epsilon & 0 \\ 0 & 0 & -1 \\ 2\epsilon & 1 & 0 \end{pmatrix} + \mathcal{O}(\epsilon^2). \quad (20)$$

The neutrino mass matrix is still invariant under the original reflections $Z_2 \otimes Z_2$, as represented by [23],

$$R_{\nu 1} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad R_{\nu 2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (21)$$

³ For certain groups, such as $\Delta(3n^2)$ or $\Sigma(3n^3)$ with $n \geq 3$, two triplets cannot form a singlet. Those are certain subgroups of $\text{SU}(3)$, to be precise, subgroups with faithful irreducible 3-dimensional representation whose determinant equals 1, that have complex representations and are not subgroups of $\text{SO}(3)$. In this case, the $\nu\nu$ term is absent, which means that in the general form $M_\nu = z_1 I + z_2 \tilde{M}_\nu$, only $z_2 \tilde{M}_\nu$ exists. So the problem becomes simpler.

Since all the residual symmetries are real, [Theorem 2](#) applies and the model should realize CMCPV. This is indeed the case, as found in Ref. [\[23\]](#).

4.3. A real $\nu\nu\phi$ model

As stated in [Corollary C](#), the real $\nu\nu\phi$ models should always produce CMCPV. In the following, we will build such a model as an explicit illustration.

We use $A_4 \otimes Z_2$ as flavor symmetry group and introduce two scalar fields ϕ^ℓ and ϕ^ν , in addition to the SM Higgs doublet H . The relevant particle content of this model is summarized in [Table 1](#). The Lagrangian for the lepton–neutrino sector contains,

$$\mathcal{L} \supset y_e(L\phi^\ell)He^c + y_\mu(L\phi^\ell)'H\mu^c + y_\tau(L\phi^\ell)''H\tau^c + y_{\nu 1}(LL)HH + y_{\nu 2}(\phi^\nu LL)HH + \text{h.c.}, \quad (22)$$

where L stands for the left-handed lepton doublet of $SU(2)_L$ and H is the Higgs doublet. Since ϕ^ℓ and ϕ^ν do not have any charge other than the Z_2 assignment, they can be real fields. Consider that they acquire the following alignment of VEVs,

$$\langle \phi^\nu \rangle \propto (1, \epsilon_2, \epsilon_3), \quad \langle \phi^\ell \rangle \propto (1, 1, 1), \quad (23)$$

where similar to [\[30\]](#), we have introduced a small perturbation on the usual VEV alignment in $\langle \phi^\nu \rangle$, with $\epsilon_2, \epsilon_3 \ll 1$. Similar VEV alignment was already considered in the literature [\[30\]](#), but its further elaboration is irrelevant to the current illustration purpose of realizing CMCPV on the ground of residual symmetries [\[20\]](#); it is also fully beyond the main goal of this short Letter. For $\epsilon_2 = \epsilon_3 = 0$, one would obtain the tri-bimaximal mixing; and the small (ϵ_2, ϵ_3) should produce the necessary corrections [\[30\]](#). Note that for $\epsilon_2 = \epsilon_3 = 0$, although $\langle \phi^\nu \rangle$ is real as required by our definition of real $\nu\nu\phi$ models, one of the mixing angles, θ_{13} , is zero, which just matches exception (i) of our [Theorem 2](#). Hence, CMCPV does not follow in this case. For $\epsilon_{2,3} \neq 0$, the charged leptons and neutrinos acquire masses as follows,

$$M_\ell \propto \begin{pmatrix} a & b & c \\ a & b\omega^2 & c\omega \\ a & b\omega & c\omega^2 \end{pmatrix}, \quad M_\nu \propto \begin{pmatrix} d & \epsilon_3 & \epsilon_2 \\ \epsilon_3 & d & 1 \\ \epsilon_2 & 1 & d \end{pmatrix}, \quad (24)$$

where the mass parameters (a, b, c, d) are complex in general. This type of lepton and neutrino mass matrices are often studied in the literature [\[31\]](#). For instance, diagonalizing the lepton mass matrix $M_\ell^\dagger M_\ell$ gives the mass-eigenvalues $(m_e, m_\mu, m_\tau) \propto (|a|, |b|, |c|)$. [This also shows that using the observed mass values (m_e, m_μ, m_τ) does not fully fix the parameters (a, b, c) themselves; while inputting the model parameters (a, b, c) can fully accommodate the observed lepton masses.] The focus of our paper is on the origin of leptonic mixings from flavor symmetry. Thus, by diagonalizing M_ℓ and M_ν , we derive the following lepton and neutrino mixing matrices,

$$U_\nu \simeq \begin{pmatrix} \frac{\epsilon_2 + \epsilon_3}{\sqrt{2}} & 1 & \frac{-\epsilon_2 + \epsilon_3}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\epsilon_2 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\epsilon_3 & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad (25)$$

and

$$U_\ell = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}. \quad (26)$$

From $U = U_\ell^\dagger U_\nu$, it is straightforward to extract the PMNS parameters in the standard parametrization,

Table 1
Particle content of the $A_4 \otimes Z_2$ model.

Groups	L	(e^c, μ^c, τ^c)	ϕ^ℓ	ϕ^ν	H
A_4	3	$(1, 1'', 1')$	3	3	1
Z_2	–1	1	–1	1	1
$SU(2)_L$	2	1	1	1	2

$$\theta_{23} = \frac{\pi}{4}, \quad |\delta| = \frac{\pi}{2}, \quad (27a)$$

$$\sin \theta_{13} \simeq \frac{\epsilon_3 - \epsilon_2}{\sqrt{6}}, \quad \tan \theta_{12} \simeq \frac{\sqrt{2}(1 - \epsilon_2 - \epsilon_3)}{2 + \epsilon_2 + \epsilon_3}. \quad (27b)$$

These results show that, apart from model-specific deviations of $(\theta_{13}, \theta_{12})$ from their tri-bimaximal values, we have realized the CMCPV, as expected from [Corollary C](#). From Eq. [\(27b\)](#), we can determine the perturbative parameters (ϵ_2, ϵ_3) in terms of $(\sin \theta_{13}, \tan \theta_{12})$ via

$$\begin{aligned} \epsilon_2 &= -\sqrt{\frac{3}{2}} \sin \theta_{13} + \frac{1 - \sqrt{2} \tan \theta_{12}}{2 + \sqrt{2} \tan \theta_{12}}, \\ \epsilon_3 &= \sqrt{\frac{3}{2}} \sin \theta_{13} + \frac{1 - \sqrt{2} \tan \theta_{12}}{2 + \sqrt{2} \tan \theta_{12}}. \end{aligned} \quad (28)$$

Taking $\theta_{13} \simeq 9^\circ$ and $\theta_{12} \simeq 34^\circ$, we derive $(\epsilon_2, \epsilon_3) \simeq (-0.18, 0.21)$.

In general, we can extend the real $\nu\nu\phi$ models to type-I neutrino seesaw. In this case, we may introduce three right-handed neutrinos ν_R in the 3-dimensional representation of A_4 . Thus, the neutrino Dirac mass matrix will be proportional to unit matrix, $m_D \propto \mathcal{I}$, while the heavy Majorana mass matrix M_R shares similar structure with M_ν in Eq. [\(24\)](#). Hence, we find that the seesaw mass matrix of light neutrinos $M_\nu \propto M_R^{-1}$.

5. Conclusions

In this work, we stressed that a general parametrization-independent definition of the maximal CP violation should be constructed in terms of Jarlskog invariant J , rather than the CP angle δ (which is rephasing non-invariant). We pointed out that naively maximizing J without constraint is already excluded by oscillation data. We further demonstrated the crucial importance of introducing the new concept of constrained maximal CP violation (CMCPV) for studying the viable maximal CP violation. For this purpose, we constructed CMCPV in [Definition 1](#), and formulated it by a set of equivalent ways, as summarized in [Theorem 1](#) (Section 2). We derived the parametrization-independent realization of the CMCPV via solution [\(4\)](#), which was proven to be the maximum of Jarlskog invariant under a minimal constraint on the PMNS matrix U (Section 2 and [Appendix A](#)). We found that the CMCPV just corresponds to $(|\delta|, \theta_{23}) = (\frac{\pi}{2}, \frac{\pi}{4})$ in the standard parametrization of the PMNS matrix [\(5\)](#). In Section 3 and [Appendix B](#), we proved [Theorem 2](#), stating that if the residual symmetries of neutrinos and charged leptons are real, then the CMCPV should be realized, up to a few minor exceptions. It was shown that the conditions for CMCPV are actually quite common, and we presented several sample models in Section 4, demonstrating that in particular the often considered μ – τ reflection symmetry is a special case of our theorem. We also note that the current formulation cannot be naively applied to the quark sector. The reason is that our [Theorem 1](#) proves RCE to be essential for the CMCPV, but RCE cannot hold for the CKM mixing matrix due to experimental data. Namely, any two rows (or columns) in the CKM matrix cannot be conjugate to each other (up to rephasing).

If indeed the values of $\delta \simeq -\frac{\pi}{2}$ and $\theta_{23} \simeq \frac{\pi}{4}$ continue to be favored by neutrino data, our general theorems and corollaries of

CMCPV should be important, and provide strong guidelines for the model buildings with flavor symmetry.

Note added

While we were finalizing the present paper, Ref. [32] appeared on arXiv, which has some partial overlap. It was also pointed out there that μ - τ reflection symmetry can be generated by discrete residual subgroups of $O(3)$. In Section 4.1, we explicitly showed that with a proper basis transformation the μ - τ reflection symmetry is actually a real symmetry. Our general theorems are independent and complementary to [32], and we presented a set of equivalent formulations for the CMCPV as well as its parametrization-independent realization.

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Appendix A. Parametrization-independent solution of CMCPV

In this appendix, we derive the general solution of CMCPV by using the manifestly parametrization-independent formula of Jarlskog invariant (3).

Following our Definition 1 for CMCPV, we can use Eq. (3b) to derive the extremal conditions of Jarlskog invariant respect to $|U_{\mu 1}|$ and $|U_{\mu 3}|$ by fixing $|U_{e 1}|$ and $|U_{e 3}|$. Thus, we have

$$\frac{\partial J^2}{\partial z} = xyw - \frac{1}{2}(1-y)[(1-y)z + (1-x)w - (1-x-y)] = 0, \quad (\text{A.1a})$$

$$\frac{\partial J^2}{\partial w} = xyz - \frac{1}{2}(1-x)[(1-y)z + (1-x)w - (1-x-y)] = 0, \quad (\text{A.1b})$$

where for convenience we have used the notations, $(x, y, z, w) \equiv (|U_{e 1}|^2, |U_{e 3}|^2, |U_{\mu 1}|^2, |U_{\mu 3}|^2)$. From the extremal conditions (A.1a)–(A.1b), we deduce the solutions,

$$z = \frac{1}{2}(1-x), \quad w = \frac{1}{2}(1-y). \quad (\text{A.2})$$

Hence, we have

$$|U_{\mu j}|^2 = \frac{1}{2}(1 - |U_{ej}|^2) \quad (j = 1, 2, 3), \quad (\text{A.3})$$

where we have used the unitarity condition for the second row, $\sum_{j=1}^3 |U_{\mu j}|^2 = 1$. With Eq. (A.3) and making use of the unitarity conditions for each column of the mixing matrix U , we further deduce

$$|U_{\tau j}|^2 = \frac{1}{2}(1 - |U_{ej}|^2) \quad (j = 1, 2, 3). \quad (\text{A.4})$$

Finally, comparing Eqs. (A.3) and (A.4), we arrive at

$$|U_{\mu j}|^2 = |U_{\tau j}|^2 = \frac{1}{2}(1 - |U_{ej}|^2) \quad (j = 1, 2, 3). \quad (\text{A.5})$$

This just reproduces Eq. (4), which we presented in the text.

Next, we prove that the above extremal solution (A.2) or (A.3) indeed corresponds to a maximum of Jarlskog invariant. For this purpose, we compute the second derivatives of the squared Jarlskog invariant respect to (z, w) ,

$$\begin{aligned} (J^2)''_{zz} &= -\frac{1}{2}(1-y)^2, & (J^2)''_{ww} &= -\frac{1}{2}(1-x)^2, \\ (J^2)''_{zw} &= (J^2)''_{wz} = -\frac{1}{2}(1-x-y-xy). \end{aligned} \quad (\text{A.6})$$

Then, we inspect the eigenvalues of the 2×2 matrix $\{(J^2)''\}$, whose elements are given by Eq. (A.6). The eigenvalues $\{\lambda_1, \lambda_2\}$ satisfy the following quadratic eigenvalue equation,

$$\lambda^2 - B\lambda + C = 0, \quad (\text{A.7a})$$

$$B = (J^2)''_{zz} + (J^2)''_{ww} = -\frac{1}{2}[(1-x)^2 + (1-y)^2] < 0, \quad (\text{A.7b})$$

$$C = (J^2)''_{zz}(J^2)''_{ww} - [(J^2)''_{zw}]^2 = xy(1-x-y) > 0, \quad (\text{A.7c})$$

where we have, $1-x-y = 1 - |U_{e 1}|^2 - |U_{e 3}|^2 = |U_{e 2}|^2 > 0$, due to the unitarity condition on the first row. Thus, we have the two eigenvalues obey $\lambda_1 + \lambda_2 = B < 0$ and $\lambda_1 \lambda_2 = C > 0$. This means that the two eigenvalues of $\{(J^2)''\}$ are both negative, $\lambda_1, \lambda_2 < 0$. Hence, we conclude that the extremal solution (A.2) or (A.3) is indeed the maximum of the Jarlskog invariant (under the constraint on the first row of U), and provides the parametrization-independent realization of the CMCPV as given in our Definition 1.

Finally, using the parametrization-independent solution (A.5) or (4) of CMCPV, we can readily derive the explicit realization of CMCPV under the standard parametrization (5). From the first equality of Eq. (A.5), we have two independent conditions $|U_{\mu 1}| = |U_{\tau 1}|$ and $|U_{\mu 3}| = |U_{\tau 3}|$, which take the following forms under the standard parametrization (5),

$$|s_{12}c_{23} + c_{12}s_{23}s_{13}e^{i\delta}| = |s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta}|, \quad (\text{A.8a})$$

$$|s_{23}c_{13}| = |c_{23}c_{13}|. \quad (\text{A.8b})$$

The condition (A.8b) leads to $s_{23} = c_{23}$ and thus $\theta_{23} = \frac{\pi}{4}$. Given this, we can rewrite (A.8a) as

$$|s_{12} + c_{12}s_{13}e^{i\delta}| = |s_{12} - c_{12}s_{13}e^{i\delta}|. \quad (\text{A.9})$$

Since $c_{12}s_{13} \neq 0$, this must require $\cos \delta = 0$, i.e., $|\delta| = \frac{\pi}{2}$. Hence, the explicit realization of our CMCPV under the standard parametrization (5) just gives $(|\delta|, \theta_{23}) = (\frac{\pi}{2}, \frac{\pi}{4})$, which we mentioned in the text above Eq. (5).

Appendix B. Proofs

In this appendix, we present proofs that are needed to establish the lemma given after Eq. (8) and the main Theorem 2 given in Section 3.

B.1. RCE is invariant under right-handed real transformations

For a unitary matrix V with the form of “row conjugation equality” (RCE) and a real orthogonal matrix R , we need to prove that $V' = VR$ still has RCE. The proof is straightforward. Defining the elements of these matrices,

$$V = (u_{ij}), \quad R = (r_{ij}), \quad V' = (u'_{ij}), \quad (\text{B.1})$$

we have

$$u'_{ij} = \sum_k u_{ik} r_{kj}. \quad (\text{B.2})$$

Note that the matrix elements (u_{1k}) and (r_{kj}) ($k, j = 1, 2, 3$) are real numbers from the start. The RCE feature of matrix V gives, $(u_{2j})^* = u_{3j}$ for $j = 1, 2, 3$. This implies

$$u'_{1j} = \sum_k u_{1k} r_{kj} = \text{real numbers}, \quad (\text{B.3})$$

and

$$(u'_{2j})^* = \sum_k u_{2k}^* r_{kj} = \sum_k u_{3k} r_{kj} = u'_{3j}. \quad (\text{B.4})$$

We have thus proven explicitly that RCE is invariant under right-handed real transformations.

B.2. Real G_ν leads to real U_ν

Consider Majorana neutrinos with residual symmetry $G_\nu = Z_2 \otimes Z_2$. In the following, we will prove that a real G_ν leads to real U_ν , and vice versa.

Let us set S to be a 3×3 unitary matrix, which is real and is a Z_2 transformation (i.e., $S^2 = \mathcal{I}$). As S is real, it follows that $S^\dagger = S^T$, and the unitarity condition $SS^\dagger = \mathcal{I}$ implies that the real matrix S is orthogonal, $SS^T = \mathcal{I}$. Without losing generality, we set $S \in \text{SO}(3)$. So, S is a rotation in 3-dimensional Euclidean space. Furthermore, since $S^2 = \mathcal{I}$, it must be a 180° -rotation.

For $G_\nu = Z_2 \otimes Z_2$, we may use S_1 and S_2 to represent the transformations of the two Z_2 's, respectively. Thus, $[S_1, S_2] = 0$ should hold, which implies that their rotation axes must be orthogonal. Hence, geometrically G_ν contains two 180° -rotations with orthogonal axes. These two axes can be represented by two normalized real vectors v_1 and v_2 with

$$\begin{aligned} S_1 v_1 &= v_1, & S_1 v_2 &= -v_2, \\ S_2 v_1 &= -v_1, & S_2 v_2 &= v_2, \end{aligned} \quad (\text{B.5})$$

where v_1 and v_2 are column vectors, of the 3×1 matrix form. Taking $v_3 = v_1 \times v_2$ and $U_\nu = (v_1, v_2, v_3)$, we see that U_ν is a real matrix and can diagonalize S_1 and S_2 simultaneously in the way given by Eq. (11).

Therefore, if $G_\nu \supset \{S_1, S_2\}$ contains only real matrices, then U_ν is real. The converse proposition that a real U_ν leads to real G_ν is also true, and can be readily proven.

B.3. Real G_ℓ leads to complex U_ℓ^\dagger with RCE

We need to prove that any $\text{SO}(3)$ matrix R can be diagonalized by $U_R^\dagger R U_R$, where the unitary matrix U_R contains one real column and two other columns which are complex conjugate to each other. This can be explicitly proven as below.

The most general rotation in 3d Euclidean space which rotates the space around an axis $\mathbf{n} = (n_1, n_2, n_3)^T$ by an angle ϕ is [24],

$$\begin{aligned} R(\mathbf{n}, \phi) &= \begin{pmatrix} n_1^2 + c(n_2^2 + n_3^2) & (1-c)n_1 n_2 + sn_3 & -sn_2 + (1-c)n_1 n_3 \\ (1-c)n_1 n_2 - sn_3 & c + n_2^2 - cn_2^2 & sn_1 + (1-c)n_2 n_3 \\ sn_2 + (1-c)n_1 n_3 & -sn_1 + (1-c)n_2 n_3 & c + n_3^2 - cn_3^2 \end{pmatrix}, \end{aligned} \quad (\text{B.6})$$

where $\mathbf{n} \cdot \mathbf{n} = 1$ and $(s, c) = (\sin \phi, \cos \phi)$. We can directly verify that this matrix is diagonalized as

$$U_R^\dagger R U_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c + is & 0 \\ 0 & 0 & c - is \end{pmatrix}, \quad (\text{B.7})$$

where

$$U_R = \begin{pmatrix} n_1 & -\frac{\sqrt{1-n_1^2}}{\sqrt{2}} & -\frac{\sqrt{1-n_1^2}}{\sqrt{2}} \\ n_2 & \frac{n_1 n_2 - i n_3}{\sqrt{2(1-n_1^2)}} & \frac{n_1 n_2 + i n_3}{\sqrt{2(1-n_1^2)}} \\ n_3 & \frac{n_1 n_3 + i n_2}{\sqrt{2(1-n_1^2)}} & \frac{n_1 n_3 - i n_2}{\sqrt{2(1-n_1^2)}} \end{pmatrix}. \quad (\text{B.8})$$

We see explicitly that the first column of U_R is real, and the second and third columns are conjugate to each other, i.e., U_R^\dagger has RCE. Hence, if $R \in G_\ell$, then $U_\ell^\dagger = U_R^\dagger$ has RCE.

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