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Waiting Time Distribution of a Queueing System with Postservice Activity

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Abstract—In this paper, we consider a queueing system with postservice activity. During the time when the server is engaged in the postservice activity (wrap-up time), the waiting customer, if any, cannot receive his or her service. This type of queueing system has been used to model automatic call distribution (ACD) systems. We consider the waiting time distribution of the queueing system. Using the Markovian point process that can be expressed by the so-called Markovian arrival process (MAP), we derive the waiting time distribution in terms of the representing matrices of a particular MAP. Then we apply the Baker-Hausdorff lemma to the matrices and derive the conditional waiting time distribution in closed form by exploiting the specific structure of the matrices. As a byproduct, we give an explicit solution of the number of arrivals for the MAP.

Keywords—Automatic call distribution (ACD), Waiting time distribution, Markovian point process, Baker-Hausdorff lemma, Closed-form solution.

1. INTRODUCTION

Suppose that we have a multiserver queueing system. The queueing system has a single queue with finite capacity. Customers are served according to the first-come first-served (FCFS) way. After completing the service, the customer leaves from the system and the server must finish the additional job (postservice). During the time in which the server is engaged in the postservice activity, the waiting customer, if any, cannot be served.

This type of queueing system is employed to model the automatic call distribution (ACD) systems and has been analyzed extensively so far [1–7]. The ACD systems are used by call centers in, e.g., the travel, banking, insurance companies to handle large volumes of incoming calls on inquiry efficiently. For the literature on the introduction to the queueing models of call centers, see [8].

One of the interesting aspects in the ACD systems is that a server (agent) is required to finish the additional job such as entering or updating data into the customer database after completing the service, while the customer leaves the system, which implies that the telephone line used by the customer is released. During this time, a new incoming call can occupy the released line, and hence, the call is not lost. However, the server cannot begin the service of the new call

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until the server finishes the additional job of the previous call. This extra time spent on the postservice activity is often called the wrap-up time. To our best knowledge, a small number of authors [1,6] focused on the effects of the wrap-up time on the performance measures such as the loss probability and waiting time distribution.

In this paper, we consider the queueing system with postservice activity in the framework of the Markovian modeling approach employed by Jolley and Harris [1]. Our contribution is to obtain an explicit closed-form solution on the waiting distribution. The brief summary of our approach is as follows. First, focusing on the case where customers must wait at their arrival epochs, we model the time between two successive delayed customers by the Markovian point process, or Markovian arrival process (MAP) [9], and construct the representing matrices of a particular MAP. Secondly, we express the waiting time distribution of the queueing system by using the representing matrices of the MAP. Finally, we apply the Baker-Hausdorff lemma [13] to the matrices representing the MAP. The specific structure of the matrices allows us to calculate matrix products in the conditional waiting time distribution in closed form. As a byproduct, we give an explicit solution of the number of arrivals for the MAP. The details of the analysis are described in Section 3.

The paper is organized as follows. In Section 2, we describe the model of the queueing system. Section 3 presents the detailed analysis of the waiting time distribution. Section 4 shows some analytical examples of our closed-form solution. Finally, we summarize our results in Section 5.

2. MODEL DESCRIPTION

Let us consider a queueing system with postservice activity. The system capacity is denoted by K which represents the maximum number of customers (including customers being served) that can be accommodated in the queueing system. The system has c ($\leq K$) identical servers. The service time of a customer is assumed to be exponentially distributed with parameter μ. We also assume that the amount of time that a server spends in the postservice activity is exponentially distributed with parameter ξ. We call the additional working time the wrap-up time. The service and wrap-up time distributions are assumed to be mutually independent. Hence, each server has three states:

(1) the *busy state*, in which the customer is receiving the service from the server;
(2) the *wrap-up state*, in which the server is being engaged in the postservice activity after completing the service; and
(3) the *idle state*, in which the server is idle.

Note that customers cannot receive their service from the servers who are engaged in the post-service activity.

Suppose that customers arrive according to a Poisson process with rate λ. Let $\pi(i,j)$ ($0 \leq i \leq K$, $0 \leq j \leq c$) be the steady-state probability that there are i customers in the system including customers being served and the number of servers in the wrap-up state is j. Furthermore, we introduce the steady-state probability vector $\boldsymbol{\pi}_i$ defined by

$$\boldsymbol{\pi}_i \triangleq (\pi(i,0), \pi(i,1), \dots, \pi(i,c)), \tag{1}$$

for $0 \leq i \leq K$. Then it follows that the steady-state probability can be obtained by solving the system of equations

$$\mathbf{0} = \boldsymbol{\pi}_0 \mathbf{A}_0 + \boldsymbol{\pi}_1 \mathbf{D}_1, \tag{2}$$

$$\mathbf{0} = \boldsymbol{\pi}_{i-1} \mathbf{B}_{i-1} + \boldsymbol{\pi}_i \mathbf{A}_i + \boldsymbol{\pi}_{i+1} \mathbf{D}_{i+1}, \qquad 1 \leq i \leq K-1, \tag{3}$$

$$\mathbf{0} = \boldsymbol{\pi}_{K-1} \mathbf{B}_{K-1} + \boldsymbol{\pi}_K \mathbf{A}_K, \tag{4}$$

where $\mathbf{0}$ is the row vector of order $c+1$ whose elements are all zero. The matrix $\mathbf{B}_i$ ($0 \leq i \leq K-1$) represents the (upward) transition from i to $i+1$ customers in the system and is given by $\mathbf{B}_i = \lambda \mathbf{I}$, where $\mathbf{I}$ denotes the identity matrix. The matrix $\mathbf{D}_i$ $(1 \le i \le K)$ represents the (downward) transition from i to $i-1$ customers in the system. If we denote by $\{\mathbf{D}_i\}_{m,n}$ element (m,n) of $\mathbf{D}_i$ for $m, n \in \{0, 1, \dots, c\}$, then it follows that

$$\{\mathbf{D}_i\}_{m,n} = (i \wedge (c-m))\mu\delta_{m,n-1}, \tag{5}$$

where $x \wedge y \equiv \min(x,y)$ and $\delta_{m,n}$ is the Kronecker delta. The matrix $\mathbf{A}_i$ $(0 \le i \le K)$ represents the transition within i customers in the system. The element (m,n) of $\mathbf{A}_i$ for $m, n \in \{0, 1, \dots, c\}$ is given by

$$\{\mathbf{A}_i\}_{m,n} = -a_m^{(i)}\delta_{m,n} + m\xi\delta_{m,n+1}, \tag{6}$$

where

$$a_m^{(i)} = \begin{cases} \lambda + m\xi, & i = 0, \\ \lambda + m\xi + (i \wedge (c-m))\mu, & 1 \le i \le K-1, \\ m\xi + (c-m)\mu, & i = K. \end{cases} \tag{7}$$

Using numerical algorithms [10], the system of equations can be solved numerically with the normalization condition $\sum_{i=0}^{K} \pi_i \mathbf{1}^\top = 1$, where $\mathbf{1}^\top$ is the column vector of order $c+1$ whose elements are all one.

REMARK. Note that the downward transition caused by the customer departure from the system occurs when the server finishes service, not postservice activity. Hence, every nonzero element of the matrix $\mathbf{D}_i$ $(1 \le i \le K)$ should be written in terms of μ, not ξ. If we treat the entire service time as the convolution of exponential distributions with parameters μ and ξ, which corresponds to the generalized Erlang distribution, then the elements of $\mathbf{D}_i$ should be related with ξ. Hence, the steady-state probability of the convolution model is different from our steady-state probability.

3. WAITING TIME ANALYSIS

Suppose that an arriving customer is accepted by the system and finds no waiting customers at his or her arrival time. Furthermore, we assume that i servers are in the wrap-up state and the remaining $c-i$ servers are in the busy state at the arrival time. We consider possible transitions in terms of states of the servers. Recalling that the service and wrap-up times are exponentially distributed with parameters μ and ξ, respectively, we have two cases:

1. The number of servers in the busy state decreases by one with rate $(c-i)\mu$.
2. The number of servers in the wrap-up state decreases by one with rate $i\xi$.

The first case accompanies the departure of a customer being served by a server and puts the server in the wrap-up state, resulting in the increment of the number of servers in the wrap-up state. The waiting customer to be served next, however, cannot immediately receive his or her service after the transition. Hence, the waiting customer remains in the queue and the number of waiting customers does not decrease. The second case corresponds to the situation in which one of the servers in the wrap-up state finishes the postservice activity and immediately continues with the service for the customer waiting in the queue. Thus, the number of waiting customers in the queue decreases by one, the state of the server changes from the wrap-up state to the busy state, and hence the number of servers in the wrap-up state decreases by one.

Observing these transitions, we can construct the matrix of the probability density of the waiting time difference between two successive delayed customers. It is clear that the time between two events at which the number of wrap-up servers decreases by one provides such waiting time difference. In order to express the distribution of the time by using the Markovian arrival process [9], let us consider two matrices $\mathbf{C}$ and $\mathbf{D}$ of order $c+1$. If we set up the matrices $\mathbf{C}$ and $\mathbf{D}$ by

$$\{\mathbf{C}\}_{i,j} = (c-i)\mu\delta_{i,j-1} - [(c-i)\mu + i\xi]\delta_{i,j}, \tag{8}$$

$$\{\mathbf{D}\}_{i,j} = i\xi\delta_{i,j+1}, \tag{9}$$

then we can show that these matrices fulfill the conditions for a MAP. It is well known that element (i,j) of the matrix $e^{\mathbf{C}t}\mathbf{D}$ gives the probability density that an arrival event occurs, and that the state of the MAP is j at time t, given that the process has started from state i [12]. For our queueing system, it can be viewed as the probability density that the waiting time difference between two successive delayed customers is t and j servers are in the wrap-up state, given that there were i servers in the wrap-up state.

We move on now to derive the matrix expression of the waiting time distribution. Let us denote by $\pi^*(i,j)$ the probability that an accepted customer who finds the system in the state (i,j) upon arrival for $0 \le i \le K-1$, $0 \le j \le c$. Then it follows that [11]

$$\pi^*(i,j) = \frac{\pi(i,j)}{1 - \sum\limits_{k=0}^{c} \pi(K,k)}. \tag{10}$$

We can then show the following proposition.

PROPOSITION 3.1. *Let us define* $\mathbb{R}_+ \triangleq [0,+\infty)$ *and introduce* $\Omega_{k,t}$ *as*

$$\Omega_{k,t} = \left\{(t_1,t_2,\dots,t_k) \in \mathbb{R}_+^k;\ 0 < t_1 < t_2 < \cdots < t_k < t\right\}. \tag{11}$$

For a given $t \in \mathbb{R}_+$, *let* $\mathbf{M}_k(t)$ *be the matrix defined by*

$$\mathbf{M}_k(t) \triangleq \int_{\Omega_{k,t}} \left(e^{\mathbf{C}t_1}\mathbf{D}e^{-\mathbf{C}t_1}\right)\left(e^{\mathbf{C}t_2}\mathbf{D}e^{-\mathbf{C}t_2}\right)\cdots\left(e^{\mathbf{C}t_k}\mathbf{D}e^{-\mathbf{C}t_k}\right) dt_1\, dt_2 \cdots dt_k, \tag{12}$$

for $k \ge 1$ *and* $\mathbf{M}_0(t) \triangleq \mathbf{I}$. *Then, the complement of the waiting time distribution* $W^c(x)$ *can be expressed by*

$$W^c(x) = \sum_{k=0}^{K-1} \mathbf{p}_k \sum_{n=0}^{k} \mathbf{M}_n(x) e^{\mathbf{C}x}\mathbf{1}^\top, \tag{13}$$

where $\mathbf{p}_k$ *is the row vector defined by*

$$\mathbf{p}_k \triangleq \begin{cases} (\pi^*(k+c,0), \pi^*(k+c-1,1),\dots,\pi^*(k,c)), & 0 \le k \le K-1-c, \\ (0,0,\dots,\pi^*(K-1,k+c-K+1),\dots,\pi^*(k,c)), & K-1-c < k \le K-1. \end{cases} \tag{14}$$

PROOF. Let $W_{k,l}$ be the conditional waiting time of an arriving customer who finds k waiting customers and l servers are in the wrap-up state at his or her arrival epoch. Denote by $w(t \mid k,l)$ the conditional probability density function defined by

$$w(t \mid k,l) \triangleq \frac{d}{dt} \Pr[W_{k,l} \le t]. \tag{15}$$

Let t_i be the beginning of a service for the i^{th} customer $(i = 1,2,\dots,k)$ from the head of the queue. We then have the constraint $0\ (= t_0) < t_1 < t_2 < \cdots < t_k < t$. The waiting time difference between the i^{th} and $(i-1)^{\text{th}}$ customers is equal to $t_i - t_{i-1}$ for $i = 1,2,\dots,k$. Hence, the conditional probability density function $w(t \mid k,l)$ is equal to the l^{th} element of the column vector given by

$$\left\{\int_{\Omega_{k,t}} e^{\mathbf{C}t_1}\mathbf{D}e^{\mathbf{C}(t_2-t_1)}\mathbf{D}\cdots e^{\mathbf{C}(t_k-t_{k-1})}\mathbf{D}e^{\mathbf{C}(t-t_k)}\mathbf{D}\, dt_1\, dt_2 \cdots dt_k\right\}\mathbf{1}^\top. \tag{16}$$

Since $e^{\mathbf{C}(x-y)} = e^{\mathbf{C}x}e^{-\mathbf{C}y} = e^{-\mathbf{C}y}e^{\mathbf{C}x}$ [13], we can rearrange the above as

$$\left\{\int_{\Omega_{k,t}} \left(e^{\mathbf{C}t_1}\mathbf{D}e^{-\mathbf{C}t_1}\right)\left(e^{\mathbf{C}t_2}\mathbf{D}e^{-\mathbf{C}t_2}\right)\cdots\left(e^{\mathbf{C}t_k}\mathbf{D}e^{-\mathbf{C}t_k}\right)e^{\mathbf{C}t}\mathbf{D}\, dt_1\, dt_2 \cdots dt_k\right\}\mathbf{1}^\top. \tag{17}$$

Hence, $W^c(x)$ can be written by

$$\begin{aligned} W^c(x) &= \sum_{k=0}^{K-1} \sum_{l=0}^{c} \pi^*(k+c-l,l) \int_x^\infty w(t \mid k,l)\, dt \\ &= \sum_{k=0}^{K-1} \mathbf{p}_k \int_x^\infty dt \mathbf{M}_k(t) e^{\mathbf{C}t} \mathbf{D} \mathbf{1}^\top. \end{aligned} \tag{18}$$

Note that $\mathbf{p}_k$ is the row vector whose l^{th} element gives the steady-state probability that an arriving customer finds k waiting customers and l servers are in the wrap-up state at arrival epoch. Since each eigenvalue of $\mathbf{C}$ is strictly negative, $\mathbf{C}$ is invertible and $\lim_{t\to\infty} \exp(\mathbf{C}t) = O$ [10], where O is the matrix whose elements are all zero. Recalling the relation $(\mathbf{C}+\mathbf{D})\mathbf{1}^\top = \mathbf{0}^\top$ [9], where $\mathbf{0}^\top$ is the transpose of vector $\mathbf{0}$, we can show that

$$\begin{aligned} \int_x^\infty dt\, \mathbf{M}_k(t) e^{\mathbf{C}t} \mathbf{D} \mathbf{1}^\top &= \mathbf{M}_k(t) e^{\mathbf{C}t} \mathbf{C}^{-1} \mathbf{D} \mathbf{1}^\top \Big|_x^\infty - \int_x^\infty dt \frac{d\mathbf{M}_k(t)}{dt} e^{\mathbf{C}t} \mathbf{C}^{-1} \mathbf{D} \mathbf{1}^\top \\ &= \mathbf{M}_k(x) e^{\mathbf{C}x} \mathbf{1}^\top + \int_x^\infty dt\, \mathbf{M}_{k-1}(t) e^{\mathbf{C}t} \mathbf{D} \mathbf{1}^\top \\ &\vdots \\ &= \sum_{n=0}^{k} \mathbf{M}_n(x) e^{\mathbf{C}x} \mathbf{1}^\top. \end{aligned} \tag{19}$$

∎

In order to evaluate the waiting time distribution, we need to calculate $\mathbf{M}_k(x)$ which is in the form of the multiple integral of the matrix products in general. However, it is difficult to evaluate the multiple integral of $e^{\mathbf{C}t}\mathbf{D}e^{-\mathbf{C}t}$ analytically and even numerically except for the special case such as when $\mathbf{C}$ and $\mathbf{D}$ commute. In what follows, we further analyze the waiting time distribution and make it in as simple a way as possible for analytically and numerically easy evaluation. To this end, we need some preliminaries of algebraic properties on $\mathbf{C}$ and $\mathbf{D}$ for the queueing system with postservice activity.

DEFINITION 3.1. *For two square matrices $\mathbf{X}$ and $\mathbf{Y}$, we define the commutator $[\mathbf{X},\mathbf{Y}]$ by*

$$[\mathbf{X},\mathbf{Y}] \triangleq \mathbf{XY} - \mathbf{YX}. \tag{20}$$

We can directly calculate the commutator on $\mathbf{C}$ and $\mathbf{D}$ matrices given by equations (8) and (9). We summarize the results as the following lemma.

LEMMA 3.1. *The elements (i,j) of the commutators $[\mathbf{C},\mathbf{D}]$ and $[\mathbf{C},[\mathbf{C},\mathbf{D}]]$ are calculated as*

$$\{[\mathbf{C},\mathbf{D}]\}_{i,j} = (c-2i)\mu\xi\delta_{i,j} + i(\mu-\xi)\xi\delta_{i,j+1}, \tag{21}$$

and

$$\{[\mathbf{C},[\mathbf{C},\mathbf{D}]]\}_{i,j} = -2(c-i)\mu^2\xi\delta_{i,j-1} + (c-2i)(\mu-\xi)\mu\xi\delta_{i,j} + i(\mu-\xi)^2\xi\delta_{i,j+1}. \tag{22}$$

Furthermore, the commutator $[\mathbf{C},[\mathbf{C},[\mathbf{C},\mathbf{D}]]]$ is calculated as

$$[\mathbf{C},[\mathbf{C},[\mathbf{C},\mathbf{D}]]] = (\mu-\xi)^2[\mathbf{C},\mathbf{D}]. \tag{23}$$

Hence, we have

$$\overbrace{[\mathbf{C},[\mathbf{C},\cdots,[\mathbf{C}}^{n},\mathbf{D}]\cdots]] = \begin{cases} (\mu-\xi)^{2k}[\mathbf{C},\mathbf{D}], & n = 2k+1, \\ (\mu-\xi)^{2k}[\mathbf{C},[\mathbf{C},\mathbf{D}]], & n = 2k+2, \end{cases} \tag{24}$$

for $k \ge 0$.

PROOF. We can check this lemma by direct calculation. Details are described in the Appendix. ∎

Before proceeding, we need the following key lemma for our analysis.

LEMMA 3.2. *(See [13].) For a given $t \in \mathbb{R}$ and square matrices $\mathbf{X}$ and $\mathbf{Y}$, the matrix $e^{\mathbf{X}t}\mathbf{Y}e^{-\mathbf{X}t}$ can be expanded as*

$$e^{\mathbf{X}t}\mathbf{Y}e^{-\mathbf{X}t} = \mathbf{Y} + \sum_{n=1}^{\infty} \frac{t^n}{n!} \overbrace{[\mathbf{X}, [\mathbf{X}, \cdots, [\mathbf{X}}^{n}, \mathbf{Y}] \cdots]]. \tag{25}$$

Combining Lemma 3.1 and the Baker-Hausdorff lemma immediately leads us to the next lemma.

LEMMA 3.3. *Let $\mathbf{C}$ and $\mathbf{D}$ be matrices given by equations (8) and (9), respectively. If $\mu \neq \xi$, then we have*

$$e^{\mathbf{C}t}\mathbf{D}e^{-\mathbf{C}t} = \mathbf{D} + f(t)[\mathbf{C}, \mathbf{D}] + g(t)[\mathbf{C}, [\mathbf{C}, \mathbf{D}]], \tag{26}$$

where $f(t)$ and $g(t)$ are scalar functions defined by

$$f(t) \triangleq \frac{\sinh((\mu - \xi)t)}{\mu - \xi}, \tag{27}$$

$$g(t) \triangleq \frac{\cosh((\mu - \xi)t) - 1}{(\mu - \xi)^2}. \tag{28}$$

If $\mu = \xi$, then $e^{\mathbf{C}t}\mathbf{D}e^{-\mathbf{C}t}$ can be further simplified as

$$e^{\mathbf{C}t}\mathbf{D}e^{-\mathbf{C}t} = \mathbf{D} + t[\mathbf{C}, \mathbf{D}] + \frac{t^2}{2}[\mathbf{C}, [\mathbf{C}, \mathbf{D}]]. \tag{29}$$

PROOF. Applying the Baker-Hausdorff lemma with the combination of Lemma 3.1, we have for $\mu \neq \xi$

$$\begin{aligned} e^{\mathbf{C}t}\mathbf{D}e^{-\mathbf{C}t} &= \mathbf{D} + t[\mathbf{C}, \mathbf{D}] + \frac{t^2}{2!}[\mathbf{C}, [\mathbf{C}, \mathbf{D}]] + \cdots + \frac{t^n}{n!} \overbrace{[\mathbf{C}, [\mathbf{C}, \cdots, [\mathbf{C}}^{n}, \mathbf{D}] \cdots]] + \cdots \\ &= \mathbf{D} + \left(t + \frac{t^3}{3!}(\mu - \xi)^2 + \frac{t^5}{5!}(\mu - \xi)^4 + \cdots \right) [\mathbf{C}, \mathbf{D}] \\ &\quad + \left(\frac{t^2}{2!} + \frac{t^4}{4!}(\mu - \xi)^2 + \frac{t^6}{6!}(\mu - \xi)^4 + \cdots \right) [\mathbf{C}, [\mathbf{C}, \mathbf{D}]] \\ &= \mathbf{D} + \left(\frac{e^{(\mu-\xi)t} - e^{-(\mu-\xi)t}}{2(\mu - \xi)} \right) [\mathbf{C}, \mathbf{D}] \\ &\quad + \left(\frac{e^{(\mu-\xi)t} + e^{-(\mu-\xi)t}}{2(\mu - \xi)^2} - \frac{1}{(\mu - \xi)^2} \right) [\mathbf{C}, [\mathbf{C}, \mathbf{D}]]. \end{aligned} \tag{30}$$

Recalling the definitions of $\sinh(x)$ and $\cosh(x)$, we obtain the first statement. In case of $\mu = \xi$, we can easily derive the result by taking the limit of $\mu \to \xi$. ∎

REMARK. Lemma 3.3 indicates that $e^{\mathbf{C}t}\mathbf{D}e^{-\mathbf{C}t}$ has the form of sum of the three matrices $\mathbf{D}$, $[\mathbf{C}, \mathbf{D}]$, and $[\mathbf{C}, [\mathbf{C}, \mathbf{D}]]$. It also indicates that we can factorize the t-dependence of the matrix $e^{\mathbf{C}t}\mathbf{D}e^{-\mathbf{C}t}$. Hence, $\mathbf{M}_n(x)$ can be written in terms of the products of the three matrices in closed form. Moreover, each product has the coefficient given by the multiple integral of the scalar functions $f(t)$ and $g(t)$ which are explicitly given.

The remaining task to calculate the waiting time distribution is to evaluate the matrix exponential $\exp[\mathbf{C}x]$. Because $\mathbf{C}$ is the triangular form, we can easily obtain its eigenvalues from the characteristic equation. Note that $\mathbf{C}$ has $c + 1$ distinct eigenvalues $-c\mu, -(c-1)\mu - \xi, \ldots, -\mu - (c-1)\xi, -c\xi$ when $\mu \neq \xi$. Hence, if we can find the right and left eigenvectors corresponding each eigenvalue explicitly, $\exp[\mathbf{C}x]$ can be easily calculated by using the spectral representation of $\mathbf{C}$. Let us denote the eigenvalue of $\mathbf{C}$ by $\lambda_m = -(c - m)\mu - m\xi$, and let $\mathbf{u}_m$ and $\mathbf{v}_m$ be the corresponding the right and left eigenvectors, $m \in \{0, 1, \dots, c\}$. Then, they are given by solving the system of equations

$$\mathbf{C}\mathbf{u}_m = \lambda_m \mathbf{u}_m, \; \mathbf{v}_m \mathbf{C} = \lambda_m \mathbf{v}_m. \tag{31}$$

For each eigenvalue, in principle, we can express $\mathbf{u}_m$ and $\mathbf{v}_m$ in terms of μ and ξ. However, it is often difficult to obtain the left and right eigenvectors in closed form even if the eigenvalues are analytically obtained. In our case, we can analytically evaluate them by virtue of the specific structure of $\mathbf{C}$. In fact, we can check the following lemma (proof is given in the Appendix), which presents a formal, explicit solution in terms of $\mathbf{U}$ and $\mathbf{V}$ composed of the right and left eigenvectors $\mathbf{u}_m$ and $\mathbf{v}_m$ ($m \in \{0, 1, \dots, c\}$).

LEMMA 3.4. *Let $\mathbf{U}$ be a $(c+1) \times (c+1)$ matrix whose first column is $\mathbf{u}_0$, second column is $\mathbf{u}_1$, and so on. Similarly, let $\mathbf{V}$ be a $(c+1) \times (c+1)$ matrix whose first row is $\mathbf{v}_0$, second row is $\mathbf{v}_1$, and so on. We can then show that these matrices are explicitly given by*

$$\mathbf{U} = (\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_c), \tag{32}$$

$$\mathbf{V} = (\mathbf{v}'_0, \mathbf{v}'_1, \dots, \mathbf{v}'_c), \tag{33}$$

where $\mathbf{u}_m$ and $\mathbf{v}'_m$ ($m \in \{0, 1, \dots, c\}$) are both column vectors which are formally expressed by differentiating the vectors

$$\mathbf{u}_c = \begin{pmatrix} z^c \\ z^{c-1} \\ \vdots \\ 1 \end{pmatrix}, \qquad \mathbf{v}'_c = \begin{pmatrix} w^c \\ w^{c-1} \\ \vdots \\ 1 \end{pmatrix}, \qquad z = \frac{\mu}{\mu - \xi}, \qquad z + w = 0, \tag{34}$$

element by element as

$$\mathbf{u}_{c-m} = \frac{1}{m!}\frac{d^m}{dz^m}\mathbf{u}_c, \quad \mathbf{v}'_{c-m} = \frac{1}{m!}\frac{d^m}{dw^m}\mathbf{v}'_c, \qquad \text{for } m = 0, 1, \dots, c. \tag{35}$$

It can also be shown by direct calculation that the relation $\mathbf{UV} = \mathbf{I}$ holds. By using the spectral representation $\mathbf{C} = \mathbf{U\Lambda V}$ with $\mathbf{\Lambda} = \text{diag}\{-c\mu, -(c-1)\mu - \xi, \dots, -\mu - (c-1)\xi, -c\xi\}$, $\exp[\mathbf{C}x]$ can be explicitly written in terms of μ, ξ, c. When $\mu = \xi$, the characteristic equation has the multiple roots of order $c+1$, i.e., all eigenvalues of $\mathbf{C}$ are $-c\mu$. In this case, we can also have an explicit solution of $\exp[\mathbf{C}x]$ by just taking the limit of $\xi \to \mu$ in the spectral representation of $\mathbf{C}$.

In summary, the conditional waiting time distribution can be expressed in terms of several matrix products which are explicitly given by the system parameters μ, ξ, and c. Because $\mathbf{M}_n(x)e^{\mathbf{C}x}$ is indeed related to the number of arrival events in $(0, x]$ for the Markovian arrival process with representing matrices $\mathbf{C}$ and $\mathbf{D}$, our closed-form solution gives an analytically exact expression of the probability matrix $\mathbf{P}(n, x)$ for the counting process of the MAP [9]. The successful derivation depends on the fact that the matrices $\mathbf{C}$ and $\mathbf{D}$ have a specific structure. In practice, it is desirable to provide an algorithm for computing the waiting time distribution, rather than deriving an explicit formula. We, however, believe that a closed-form solution is interesting to derive, at least, from the mathematical point of view. The effectiveness of the closed-form solution is not pursued in this paper but is left for another study.

4. EXAMPLE

We show examples of the closed-form solution. For simplicity, we calculate the queueing system with $c = 2$ servers. In this case, the matrix $\mathbf{C}$ has the spectral representation given by

$$\mathbf{C} = \begin{pmatrix} 1 & 2z & z^2 \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2\mu & 0 & 0 \\ 0 & -(\mu+\xi) & 0 \\ 0 & 0 & -2\xi \end{pmatrix} \begin{pmatrix} 1 & 2w & w^2 \\ 0 & 1 & w \\ 0 & 0 & 1 \end{pmatrix}. \tag{36}$$

Hence we have $\exp[\mathbf{C}x]$ explicitly as

$$\exp[\mathbf{C}x] = \begin{pmatrix} 1 & 2z & z^2 \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-2\mu x} & 0 & 0 \\ 0 & e^{-(\mu+\xi)x} & 0 \\ 0 & 0 & e^{-2\xi x} \end{pmatrix} \begin{pmatrix} 1 & 2w & w^2 \\ 0 & 1 & w \\ 0 & 0 & 1 \end{pmatrix}. \tag{37}$$

The closed-form solution of $\mathbf{M}_n(x)$ can be written in terms of the three matrices $\mathbf{D}$, $\mathbf{E} = [\mathbf{C}, \mathbf{D}]$, and $\mathbf{F} = [\mathbf{C}, [\mathbf{C}, \mathbf{D}]]$ for general c. The coefficient of the *diagonal* terms such as $\mathbf{D}^n$ for the general $\mathbf{M}_n(x)$ can be easily obtained as follows:

$$\begin{aligned} \mathbf{D}^n &: \frac{x^n}{n!}, \\ \mathbf{E}^n &: \left(\frac{\cosh((\mu-\xi)x) - 1}{(\mu-\xi)^2}\right)^n \frac{1}{n!}, \\ \mathbf{F}^n &: \left(\frac{\sinh((\mu-\xi)x) - (\mu-\xi)x}{(\mu-\xi)^3}\right)^n \frac{1}{n!}. \end{aligned}$$

The explicit expression of the *cross* terms is relatively involved. However, we can derive them by evaluating the multiple integral of the scalar functions $f(\cdot)$ and $g(\cdot)$. Here, we show the first two matrices $\mathbf{M}_1(x)$ and $\mathbf{M}_2(x)$ as examples. Denoting by $a = \mu - \xi$, and using shorthand notations $\operatorname{sh}(\cdot) \equiv \sinh(\cdot)$ and $\operatorname{ch}(\cdot) \equiv \cosh(\cdot)$, it can be shown by direct calculation that

$$\mathbf{M}_1(x) = \mathbf{D}x + \mathbf{E}\frac{\operatorname{ch}(ax) - 1}{a^2} + \mathbf{F}\frac{\operatorname{sh}(ax) - ax}{a^3}, \tag{38}$$

$$\begin{aligned} \mathbf{M}_2(x) = {} & \mathbf{D}^2\frac{x^2}{2} + \mathbf{DE}\frac{ax\operatorname{ch}(ax) - \operatorname{sh}(ax)}{a^3} \\ & + \mathbf{DF}\frac{2ax\operatorname{sh}(ax) - (ax)^2 - 2\operatorname{ch}(ax) + 2}{2a^4} \\ & + \mathbf{ED}\frac{\operatorname{sh}(ax) - ax}{a^3} + \mathbf{E}^2\frac{(\operatorname{ch}(ax) - 1)^2}{2a^4} \\ & + \mathbf{EF}\frac{\operatorname{sh}(ax)\operatorname{ch}(ax) + 3ax - 4\operatorname{sh}(ax)}{2a^5} \\ & + \mathbf{FD}\frac{2\operatorname{ch}(ax) - (ax)^2 - 2}{2a^4} \\ & + \mathbf{FE}\frac{\operatorname{sh}(ax)\operatorname{ch}(ax) - ax - 2ax\operatorname{ch}(ax) + 2\operatorname{sh}(ax)}{2a^5} \\ & + \mathbf{F}^2\frac{(\operatorname{sh}(ax) - ax)^2}{2a^6}. \end{aligned} \tag{39}$$

In case of $\mu = \xi$, we can obtain the coefficients by taking the limit $a \to 0$.

5. CONCLUSION

In this paper, we have considered a queueing system with postservice activity. Using the Markovian point process, or Markovian arrival process (MAP), we have derived the waiting time distribution in terms of the representing matrices of a particular MAP. We have applied the Baker-Hausdorff lemma to the matrices. Exploiting the specific structure of the matrices, we have obtained the conditional waiting time distribution in closed form. As a byproduct, we have given an analytically exact expression of the probability matrix $\mathbf{P}(n, x)$ for the counting process of the MAP. Some analytic examples have also been shown.

APPENDIX

Proof of Lemma 3.1.

Here we give the detailed calculation of the commutator of $\mathbf{C}$ and $\mathbf{D}$ to prove Lemma 3.1. In the following, we use the Einstein summation convention that double (dummy) indices are summed over automatically, viz., $a_i b_i$ means $\sum_i a_i b_i$. Then, it follows that

$$\begin{aligned}\{\mathbf{C}\}_{i,k}\{\mathbf{D}\}_{k,j} &= \{(c-i)\mu\delta_{i,k-1} - [(c-i)\mu + i\xi]\delta_{i,k}\}k\xi\delta_{k,j+1} \\ &= (c-i)(i+1)\mu\xi\delta_{i,j} - [(c-i)\mu + i\xi]i\xi\delta_{i,j+1}, \\ \{\mathbf{D}\}_{i,k}\{\mathbf{C}\}_{k,j} &= i\xi\delta_{i,k+1}\{(c-k)\mu\delta_{k,j-1} - [(c-k)\mu + k\xi]\delta_{k,j}\} \\ &= i(c-i+1)\mu\xi\delta_{i,j} - i[(c-i+1)\mu + (i-1)\xi]\xi\delta_{i,j+1}.\end{aligned}$$

Consequently, the commutator $[\mathbf{C},\mathbf{D}]$ has element (i,j) given by

$$\begin{aligned}\{[\mathbf{C},\mathbf{D}]\}_{i,j} &= \{\mathbf{C}\}_{i,k}\{\mathbf{D}\}_{k,j} - \{\mathbf{D}\}_{i,k}\{\mathbf{C}\}_{k,j} \\ &= (c-2i)\mu\xi\delta_{i,j} + i(\mu-\xi)\xi\delta_{i,j+1}.\end{aligned} \tag{40}$$

In the same way, we can confirm by direct calculation that

$$\begin{aligned}\{\mathbf{C}\}_{i,k}\{[\mathbf{C},\mathbf{D}]\}_{k,j} &= (c-i)[c-2(i+1)]\mu^2\xi\delta_{i,j-1} + (c-i)(i+1)(\mu-\xi)\mu\xi\delta_{i,j} \\ &\quad - [(c-i)\mu + i\xi](c-2i)\mu\xi\delta_{i,j} - [(c-i)\mu + i\xi]i(\mu-\xi)\xi\delta_{i,j+1}, \\ \{[\mathbf{C},\mathbf{D}]\}_{i,k}\{\mathbf{C}\}_{k,j} &= (c-2i)(c-i)\mu^2\xi\delta_{i,j-1} - (c-2i)[(c-i)\mu + i\xi]\mu\xi\delta_{i,j} \\ &\quad + i[c-(i-1)](\mu-\xi)\mu\xi\delta_{i,j} \\ &\quad - i[(c-(i-1))\mu + (i-1)\xi](\mu-\xi)\xi\delta_{i,j+1}.\end{aligned}$$

Hence, we can express element (i,j) of the commutator $[\mathbf{C},[\mathbf{C},\mathbf{D}]]$ as

$$\{[\mathbf{C},[\mathbf{C},\mathbf{D}]]\}_{i,j} = -2(c-i)\mu^2\xi\delta_{i,j-1} + (c-2i)(\mu-\xi)\mu\xi\delta_{i,j} + i(\mu-\xi)^2\xi\delta_{i,j+1}. \tag{41}$$

Furthermore, we can calculate element (i,j) of the commutator $[\mathbf{C},[\mathbf{C},[\mathbf{C},\mathbf{D}]]]$ by direct calculation, which results in

$$\{[\mathbf{C},[\mathbf{C},[\mathbf{C},\mathbf{D}]]]\}_{i,j} = (\mu-\xi)^2\{[\mathbf{C},\mathbf{D}]\}_{i,j}. \tag{42}$$

Proof of Lemma 3.4.

Next, we give the explicit left and right eigenvectors of $\mathbf{C}$ given in Lemma 3.4. Define the right eigenvector $\mathbf{x}^{(i)}$ for its eigenvalue $\lambda_i = -(c-i)\mu - i\xi$ for $i \in \{0,1,\ldots,c\}$. Denoting the m^{th} element of the right eigenvector $\mathbf{x}^{(i)}$ by $x_m^{(i)}$, $\mathbf{C}\mathbf{x}^{(i)} = \lambda_i\mathbf{x}^{(i)}$ gives

$$-[(c-m)\mu + m\xi]x_m^{(i)} + (c-m)\mu x_{m+1}^{(i)} = -[(c-i)\mu + i\xi]x_m^{(i)}$$

for $m \in \{0,1,\ldots,c\}$ with $x_{c+1}^{(c)} \equiv 0$. Setting $m = i$, we have $x_{i+1}^{(i)} = 0$ and we can choose $x_i^{(i)} = 1$. Since $\mu \neq 0$, $\xi \neq 0$, and $\mu \neq \xi$, we have $x_{i+1}^{(i)} = x_{i+2}^{(i)} = \cdots = x_c^{(i)} = 0$. For $m \in \{0,1,\ldots,i-1\}$, we have the relation

$$x_m^{(i)} = \frac{(c-m)\mu}{(m-i)(\mu-\xi)}x_{m+1}^{(i)} = \frac{c-m}{i-m}zx_{m+1}^{(i)}.$$

Substituting 1 for $x_i^{(i)}$, we obtain

$$x_{i-k}^{(i)} = \binom{c-i+k}{k}z^k$$

for $k \in \{0,1,\ldots,i\}$. Differentiating $x_{i-k}^{(i)}$ with respect to z and dividing by $(c-i+1)$, we obtain

$$\frac{1}{c-i+1}\frac{d}{dz}x_{i-k}^{(i)} = \binom{c-(i-1)+k-1}{k-1}z^{k-1}$$

for $i \in \{1,2,\ldots,c\}$. The right-hand side is exactly the element of the $(i-1)^{\text{th}}$ right eigenvector. In case of the left eigenvector, we can similarly obtain the explicit expression.

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