Journal of Number Theory 131 (2011) 1743–1769

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Equivariant holomorphic differential operators and finite averages of values of L-functions

Siegfried Böcherer [a], Bernhard Heim [b,*]

[a] Kunzenhof 4B, 79117 Freiburg, Germany

[b] German University of Technology in Oman (GUtech), Department of Applied Information Technology, PO Box 1816, Athaibah PC 130, Corner of Beach Road and Wadi Athaibah Way, Oman

A R T I C L E I N F O

Article history:
Received 14 September 2010
Accepted 8 March 2011
Available online 18 May 2011
Communicated by Dipendra Prasad

Keywords:
Differential operators
Special values L-functions
Eisenstein series

A B S T R A C T

Using pullback formulas for both Siegel–Eisenstein series and Jacobi–Eisenstein series the second author obtained relations between critical values of certain L-functions. To extend these relations to other critical values we use holomorphic differential operators for both types of pullbacks. The differential operators in question are well known in the Siegel case whereas for the Jacobi case they have to be developed from scratch. To compare the two pullbacks, we have furthermore to establish a relation of unexpected nature between the two types of differential operators.

0. Introduction

In [20] the second author obtained an identity for weighted finite averages over values of triple product L-functions (in the largest critical point) on one side and central critical values of L-functions $L(f \otimes Sym^2(g), s)$ on the other side. In particular, the identity relates, for a given cuspidal eigenform f of weight k, the two finite sums

$$\sum_{g \in S^k} \Lambda(f \otimes f \otimes g, s^+) \quad \text{and} \quad \sum_{h \in S^{2k-2}} \Lambda\big(h, s^+\big) \cdot \Lambda\big(h \otimes Sym^2(f), s_c\big).$$

In both cases, the sum should run over normalized Hecke eigenforms of weight k and weight $2k-2$ respectively and s^+ and s_c denote the largest and central critical points of the L-functions in question.

* Corresponding author.
E-mail addresses: boecherer@math.uni-mannheim.de (S. Böcherer), bernhard.heim@gutech.edu.om (B. Heim).

0022-314X/$ – see front matter
doi:10.1016/j.jnt.2011.03.008

We use $\Lambda(\ldots)$ instead of $L(\ldots)$ to indicate that we have divided the L-value by an appropriate period, the summands above are therefore all algebraic.

It is quite mysterious that by this identity central critical values of L-functions get related to critical values in the range of convergence of other L-functions.

This identity was obtained by restricting a degree three Siegel–Eisenstein series in different ways and combining our knowledge about such restrictions, in particular:

- The pullback formulas for triple product L-functions (following Garrett [15]).
- The identification of the first Fourier–Jacobi coefficient of a Siegel–Eisenstein series with a Jacobi–Eisenstein series of index one [2].
- The pullback formula of Arakawa for Jacobi–Eisenstein series [1].
- Ichino's work on the Gross–Prasad conjecture for the Saito–Kurokawa liftings [23], rephrased as an explicit spectral decomposition of restrictions of Jacobi forms [18].

From previous works on restrictions (in particular the doubling method, see e.g. [3]) it is quite natural to expect, that there exist variants of the identity in question, which can be obtained by applying differential operators before doing the restriction. "Variants" means that the weights k and $2k-2$ can be changed and also the largest critical points should be moved.

The construction of appropriate differential operators for our situation is one of the main topic of this paper; interesting new problems arise for the following reasons:

The holomorphic differential operators should be equivariant with respect to the groups stabilizing the subdomains, to which we restrict. Here an amusing new feature comes up: There are no holomorphic differential operators, which fit to all the restrictions which have to be considered at the same time. So what we do is to consider two types of differential operators separately for different restrictions, one of them is well know (the one for a restriction a la Garrett), but the other one (which fits to the work of Arakawa) cannot be found in the literature and has to be constructed; it can be viewed as a Jacobi forms version of the differential operators studied by the first author and Ibukiyama for symplectic groups [3,21]. This is indeed the main technical part of our paper.

At the end, we can glue these differential operators together in the sense that we can write the operator of the Arakawa–Ichino side as a finite linear combination of the ones adopted to the Garrett side to obtain again such an identity for L-values.

We present our main results in two ways, one in terms of periods of Jacobi forms and the other in terms of central critical values. The formulation in terms of periods is more general, its translation into central critical values relies on the spectacular results of Ichino. Indeed we will work as long as possible without using Ichino's formula, arriving at more general identities between values of triple-product L-functions and periods for Jacobi forms of arbitrary squarefree index. These results are formulated more generally than in [20], but they need some results about pullbacks of Jacobi–Eisenstein series going beyond Arakawa (worked out by K. Bringmann and the second author in [19,10]).

The first section is devoted to the definition and properties of the differential operators. In the second and third sections we study pullbacks modified by differential operators whereas in Section 4 we will compare the resulting formulas and in the final section we specialize our identities to the case, where the results of Ichino are available.

0.1. Preliminaries

For matrices A, B of appropriate size we put $A[B] := B^t \cdot A \cdot B$. For basic facts about elliptic modular forms we refer to [27] and for Siegel modular forms to [24]. In particular, we denote by M^k and S^k the spaces of modular and cuspidal modular forms for weight k with respect to the full modular group $SL(2,\mathbb{Z})$. We equip S^k with the usual Petersson scalar product $\langle\,,\rangle$. For normalized Hecke eigenforms $f_i = \sum a_i(n)e^{2\pi iz} \in S^{k_i}$ we define "Satake parameters" $\alpha_i(p), \beta_i(p) \in \mathbb{C}$ by

$$a_i(p) = \alpha_i(p) + \beta_i(p), \qquad \alpha_i(p)\cdot\beta_i(p) = p^{k-1}$$

and we put

$$\begin{aligned} L(f_i,s) &= \prod_p \frac{1}{(1-\alpha_i(p)p^{-s})(1-\beta_i(p)p^{-s})}, \\ L_2(f_i,s) &= D(f_i, s-k_i+1) \\ &= \prod_p \frac{1}{(1-p^{k_i-1-s})(1-\alpha_i(p)^2p^{-s})(1-\beta_i(p)^2p^{-s})}. \end{aligned}$$

Moreover we define the triple product L-function $L(f_1 \otimes f_2 \otimes f_3, s)$ by

$$\prod_p \det\left(1_8 - \begin{pmatrix} \alpha_1(p) & 0 \\ 0 & \beta_1(p) \end{pmatrix} \otimes \begin{pmatrix} \alpha_2(p) & 0 \\ 0 & \beta_2(p) \end{pmatrix} \otimes \begin{pmatrix} \alpha_3(p) & 0 \\ 0 & \beta_3(p) \end{pmatrix} p^{-s}\right)^{-1}$$

and $L(f_1 \otimes Sym^2(f_2), s)$ by

$$\prod_p \det\left(1_6 - \begin{pmatrix} \alpha_1(p) & 0 \\ 0 & \beta_1(p) \end{pmatrix} \otimes \begin{pmatrix} \alpha_2(p)^2 & 0 & 0 \\ 0 & p^{k_2-1} & 0 \\ 0 & 0 & \beta_2(p)^2 \end{pmatrix} p^{-s}\right)^{-1}.$$

We mention the identity

$$L(f_1 \otimes f_2 \otimes f_2, s) = L(f_1, s-k_2+1) \cdot L\big(f_1 \otimes Sym^2(f_2), s\big).$$

Hecke eigenforms for Jacobi groups and associated zeta functions will be explained in Sections 4 and 5.

1. Differential operators on $\mathbb{H}_3$

We give a construction of the differential operators in question in a direct and explicit way appropriate for the purpose of this paper. A more general exposition will be given elsewhere [5].

1.1. *Embeddings of symplectic groups and Jacobi groups in symplectic groups of higher rank*

1.1.1. *Symplectic groups*

The symplectic group $Sp(n,\mathbb{R})$ acts on Siegel's upper half space $\mathbb{H}_n$ in the usual way. Moreover, $Sp(n,\mathbb{R})$ acts by a "slash-operator" $|_k$ on functions $f : \mathbb{H}_n \longrightarrow \mathbb{C}$ by

$$(f|_k M)(Z) := j(M,Z)^{-k} f\big(M\langle Z\rangle\big) \quad \big(M \in Sp(n,\mathbb{R}),\ Z \in \mathbb{H}_n\big).$$

Here $j(M,Z) = \det(CZ+D)$ is the standard automorphy factor and

$$M\langle Z\rangle := (AZ+B)(CZ+D)^{-1} \quad \text{for } M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R}).$$

In the sequel we embed small symplectic groups $Sp(n)$ as subgroups into bigger symplectic groups $Sp(n+m)$ by

$$\uparrow: \begin{cases} Sp(n) \longrightarrow Sp(n+m) \\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \longmapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1_m & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1_m \end{pmatrix} \end{cases}$$

and

$$\downarrow: \begin{cases} Sp(n) \longrightarrow Sp(n+m) \\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \longmapsto \begin{pmatrix} 1_m & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1_m & 0 \\ 0 & c & 0 & d \end{pmatrix} \end{cases}.$$

It should always be clear from the context, what n and m are.

1.1.2. Jacobi groups

We prefer to consider a Jacobi group as a subgroup of an appropriate symplectic group:

$$G_n := \left\{ \begin{pmatrix} a & 0 & b & \mu \\ \lambda' & 1 & \mu' & \kappa \\ c & 0 & d & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} \in Sp(n+1) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(n) \right\}$$

with $(\lambda', \mu') = (\lambda, \mu) \cdot \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

We put $n = n_1 + n_2$ and we consider embeddings of G_{n_1} and G_{n_2} in a big symplectic group $Sp(n+1)$:

$$\iota(n_1, n)^+ : G_{n_1} \longrightarrow Sp(n_1 + n_2 + 1)$$

$$\begin{pmatrix} a & 0 & b & \mu \\ \lambda' & 1 & \mu' & \kappa \\ c & 0 & d & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} a & 0 & 0 & b & 0 & \mu \\ 0 & 1_{n_2} & 0 & 0 & 0 & 0 \\ \lambda' & 0 & 1 & \mu' & 0 & \kappa \\ c & 0 & 0 & d & 0 & -\lambda \\ 0 & 0 & 0 & 0 & 1_{n_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\iota(n_2, n)^- : G_{n_2} \longmapsto Sp(n+1)$$

$$\begin{pmatrix} a & 0 & b & \mu \\ \lambda' & 1 & \mu' & \kappa \\ c & 0 & d & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1_{n_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & b & \mu \\ 0 & \lambda' & 1 & 0 & \mu' & \kappa \\ 0 & 0 & 0 & 1_{n_1} & 0 & 0 \\ 0 & c & 0 & 0 & d & -\lambda \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We look at a corresponding embedding of domains: We decompose an element $\mathcal{Z} \in \mathbb{H}_{n+1}$ as

$$\mathcal{Z} = \begin{pmatrix} \tau_1 & z & z_1 \\ z^t & \tau_2 & z_2 \\ z_1^t & z_2^t & \tau_3 \end{pmatrix}, \quad \tau_1 \in \mathbb{H}_{n_1},\ \tau_2 \in \mathbb{H}_{n_2},\ \tau_3 \in \mathbb{H}_1. \tag{1}$$

Occasionally we use $Z = \left(\begin{smallmatrix} \tau_1 & z \\ z^t & \tau_2 \end{smallmatrix}\right) \in \mathbb{H}_{n_1+n_2}$ to denote the upper left corner of $\mathcal{Z}$.

Both groups $\iota(n_1, n)^+(G_{n_1})(\mathbb{R})$ and $\iota^-(n_2, n)(G_{n_2})(\mathbb{R})$ stabilize the submanifold of $\mathbb{H}_{n+1}$ defined by $z = 0$.

We define Jacobi forms of degree n as holomorphic functions on $\mathbb{H}_{n+1}$ of the form

$$F(Z) = \phi(\tau_1, z)e^{2\pi i m\tau_2}, \quad Z = \begin{pmatrix} \tau_1 & z \\ z^t & \tau_2 \end{pmatrix} \in \mathbb{H}_{n+1},$$

satisfying the transformation law $F|_k M = F$ for all $M \in G_n(\mathbb{Z})$ with the additional requirement of holomorphy in the cusps if $n = 1$.

As an immediate consequence of these definitions we have

Remark. Suppose F is a weight k Jacobi form of degree $n = n_1 + n_2$ of index m with $\mathcal{Z}$ decomposed as in (1)

$$F(\mathcal{Z}) = \phi\left(\begin{pmatrix} \tau_1 & z \\ z^t & \tau_2 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right) \cdot e^{2\pi i m\cdot\tau_3}.$$

Then

$$F_{|z=0} = \phi\left(\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right) \cdot e^{2\pi i m\tau_3}$$

is a Jacobi form of index m for (τ_1, z_1, τ_3) and for (τ_2, z_2, τ_3) of the same weight k.

1.2. Existence of some differential operators

We introduce differential operators on $\mathbb{H}_n$, which are polynomials in the entries $\partial_{i,j}$ of the matrix

$$\partial := \left(\frac{(1 + \delta_{ij})}{2} \frac{\partial}{\partial z_{ij}}\right).$$

The transformation properties of these differential operators, in particular of their minors are described in [13,26]. We will also tacitly use that for any polynomial $\mathcal{P}$ in the entries of ∂ and any complex symmetric matrix $\mathcal{T}$ of size n we have

$$\mathcal{P}(\partial)e^{tr(\mathcal{T}\cdot Z)} = \mathcal{P}(\mathcal{T})e^{tr(\mathcal{T}\cdot Z)} \quad (Z \in \mathbb{H}_n).$$

Most of the statements below will make sense in the general context introduced above, their proofs are however will be completely different and much more complicated. We stick now to differential operators on $\mathbb{H}_3$ where direct and more explicit methods are available.

One of our main aims in this section is the proof of the following

Theorem 1.1 *(Weak version). There exists for each $\nu \geqslant 0$ a holomorphic differential operator D (a polynomial in $\partial_{i,j}$, evaluated in $z = 0$) acting on C^∞-functions defined on $\mathbb{H}_3$ such that for all $M \in G_1(\mathbb{R})$*

$$D\big(F|_k\iota(1,2)^+(M)\big) = D(F)|_{k+\nu}M^+,$$
$$D\big(F|_k\iota(1,2)^-(M)\big) = D(F)|_{k+\nu}M^-.$$

Here M^+ means action of M for the variables (τ_1, z_1, τ_3) and M^- the action for the variables (τ_2, z_2, τ_3).

If F is a Jacobi modular form of degree 2, of index m and of weight k then $D(F)$ defines for $\nu > 0$ a cuspidal Jacobi form of degree one, index m and weight $k+\nu$ for (τ_1, z_1, τ_3) and for (τ_2, z_2, τ_3); moreover, $D(F)$ is symmetric:

$$D(F)(\tau_1, z_1, \tau_2, z_2; \tau_3) = D(F)(\tau_2, z_2, \tau_1, z_1; \tau_3).$$

We obtain the cuspidality property by considering the differential operator together with the action of

$$\iota(1,2)^+\left(\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right) \quad (a \in \mathbb{R},\ a \neq 0)$$

on the constant term in the Fourier expansion or more generally on those holomorphic functions on $\mathbb{H}_3$, which do not depend on (τ_1, z_1) (and similarly for (τ_2, z_2)).

The theorem follows from a statement about differential operators with much stronger properties; instead of the notation ∂_{ij} we use here

$$\partial = \begin{pmatrix} \partial_{\tau_1} & \partial_z & \partial_{z_1} \\ \partial_z & \partial_{\tau_2} & \partial_{z_2} \\ \partial_{z_1} & \partial_{z_2} & \partial_{\tau_3} \end{pmatrix}. \tag{2}$$

Theorem 1.1 *(Strong version). The differential operator*

$$\mathbb{D}_k := (k-1)\partial_{z_1}\partial_{z_2} - (k-1)\partial_z\partial_{\tau_3} + z \cdot \partial^{[3]}$$

acting on C^∞-functions F defined on $\mathbb{H}_3$ satisfies

$$\mathbb{D}_k(F|_k M) = (\mathbb{D}_k F)|_{k+1} M$$

for all $M \in \iota(1,1)^{\pm}(G_1(\mathbb{R}))$. Moreover $\mathbb{D}_k$ satisfies a symmetry relation:

$$\mathbb{D}_k(F|_k V) = (\mathbb{D}_k F)|_{k+1} V$$

where

$$V := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in Sp(3, \mathbb{R}).$$

Proof. The symmetry property can be read off directly from the explicit formula. We put $\partial^{[3]} := \det(\partial)$. This operator has the fundamental transformation property

$$\partial^{[3]}(F|_1 M) = \left(\partial^{[3]} F\right)\Big|_3 M \quad \left(M \in Sp(3, \mathbb{R})\right),$$

see [13, p. 216].

We consider now the differential operator $\mathcal{L}$ defined by

$$F \longmapsto z^{-k+2}\partial^{[3]}\big(F \cdot z^{k-1}\big).$$

The z-coordinate is changed to $\frac{z}{c\tau_1+d}$ or $\frac{z}{c\tau_2+d}$ if we act on $\mathbb{H}_3$ by $\iota(1,1,1)^{\pm}(G_1(\mathbb{R}))$ and $\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$ denotes the "symplectic part" of the element of G_1. Therefore, this operator has the required transformation property. We easily get the identity

$$\mathcal{L} = z \cdot \partial^{[3]} + (k-1)\partial_{z_1}\partial_{z_2} - (k-1)\partial_z\partial_{\tau_3} - \frac{(k-1)(k-2)}{4}z^{-1}\partial_{\tau_3}.$$

The last summand above has itself the requested transformation property and hence it can just be omitted.

Then we get the differential operator D of Theorem 1.1 (weak version) by

$$D := \mathbb{D}^0_{k,\nu} = (\mathbb{D}_{k+\nu-1} \circ \cdots \circ \mathbb{D}_{k+1} \circ \mathbb{D}_k)_{z=0}. \qquad \square$$

Remark. The procedure above does not generalize to higher degree, the analogues of Theorem 1.1 (both versions) for arbitrary degree n are however true, but the proof has to go along the lines of [3].

1.2.1. Basic examples

Changing the weight from k to $k+1$ and $k+2$ (with restriction)

$$\begin{aligned}
\mathbb{D}^0_{k,1} &= (k-1)\{\partial_{z_1}\partial_{z_2} - \partial_z\partial\tau_3\}_{|z=0},\\
\mathbb{D}^0_{k,2} &= \left\{(k\partial_{z_1}\partial_{z_2} - k\partial_z\partial_{\tau_3})\big((k-1)\partial_{z_1}\partial_{z_2} - (k-1)\partial_z\partial_{\tau_3}\big) - \frac{k}{2}\partial_{\tau_3}\partial^{[3]}\right\}_{z=0}\\
&= \left\{k(k-1)\partial^2_{z_1}\partial^2_{z_2} + \big(k-2k^2\big)\partial_z\partial_{z_1}\partial_{z_2}\partial_{\tau_3} + k\left(k-\frac{1}{2}\right)\partial^2_z\partial^2_{\tau_3}\right.\\
&\qquad \left. - \frac{k}{2}\partial_{\tau_1}\partial_{\tau_2}\partial^2_{\tau_3} + \frac{k}{2}\partial^2_{z_2}\partial_{\tau_1}\partial_{\tau_3} + \frac{k}{2}\partial^2_{z_1}\partial_{\tau_2}\partial_{\tau_3}\right\}_{|z=0}.
\end{aligned}$$

Remark. We should point out that the "weight" k in all our considerations about differential operators is allowed to be an arbitrary complex number as long as we use the same branch of $\log j(M,Z)$ on both sides of the transformation laws.

1.3. Explicit formulas for $\mathbb{D}^0_{k,\nu}$

1.3.1. Differential operators for $\mathbb{H} \times \mathbb{H} \hookrightarrow \mathbb{H}_2$

In [3] we introduced differential operators on $\mathbb{H}_{2n}$ with equivariance properties with respect to $Sp(n,\mathbb{R})^{\uparrow} \times Sp(n,\mathbb{R})^{\downarrow} \subset Sp(2n,\mathbb{R})$. For $n=1$ these operators appeared implicitly in [11] and have been studied from a different point of view in [21].

These operators are related to the ones introduced above and they will appear also later on explicitly, so we introduce them here for the special case needed (again in a strong and a weak versions):

We consider functions f defined on $\mathbb{H}_2$ and differential operators

$$\partial_{ij} := \frac{1+\delta_{ij}}{2}\frac{\partial}{\partial z_{ij}} \quad (1 \leqslant i,j \leqslant 2) \text{ for } Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{pmatrix} \in \mathbb{H}_2.$$

Then we put

$$\mathcal{D}_k = \left(-k + \frac{1}{2}\right)\partial_{12} + z_{12} \cdot (\partial_{11}\partial_{22} - \partial_{12}\partial_{12}).$$

For all $M \in SL(2, \mathbb{R})$ we have

$$\mathcal{D}_k\bigl(f|_k M^{\uparrow}\bigr) = (\mathcal{D}_k F)|_{k+1} M^{\uparrow},$$
$$\mathcal{D}_k\bigl(f|_k M^{\downarrow}\bigr) = (\mathcal{D}_k F)|_{k+1} M^{\downarrow}$$

and we put, for $t \geqslant 0$

$$\mathcal{D}^0_{k,t} := (\mathcal{D}_{k+t-1} \circ \cdots \circ \mathcal{D}_k)_{z_{12}=0}.$$

For functions of type $f(Z) = \phi(z_{11}, z_{12}) \cdot e^{2\pi i m z_{22}}$ these operators are quite familiar in the theory of Jacobi forms [11].

There is another construction of such differential operators by means of harmonic polynomials: We define (for even integers d) polynomials in A and B by

$$\sum_{\nu=0}^{\infty} G^{\nu}_d(A, B)X^{\nu} = \frac{1}{(1 - AX + \frac{1}{2}BX^2)^{\frac{d-2}{2}}}.$$

Then the $G^{\nu}_d(A, B)$ are (up to a power of two) Gegenbauer polynomials. By

$$\begin{cases} \mathbb{C}^{2k} \times \mathbb{C}^{2k} \longrightarrow \mathbb{C} \\ (\mathfrak{x}, \mathfrak{y}) \longmapsto G^{\nu}_{2k}\bigl(\{\mathfrak{x}, \mathfrak{y}\}, \{\mathfrak{x}, \mathfrak{x}\} \cdot \{\mathfrak{y}, \mathfrak{y}\}\bigr) \end{cases}$$

we get polynomial functions of $\mathfrak{x}, \mathfrak{y}$, which are harmonic and homogeneous of degree ν in both variables; here $\{\,,\}$ denotes the standard bilinear form on $\mathbb{C}^{2k}$. The general theory of Ibukiyama [21] asserts, that the differential operators, defined (formally) by

$$\mathcal{L}_{k,\nu} := G_{2k}(\partial_{12}, \partial_{11} \cdot \partial_{22})_{z_{12}=0}$$

have the same transformation property as $\mathcal{D}^0_{k,\nu}$; these differential operators were constructed in two completely different manners, but they are proportional (the space of such operators is one-dimensional, see [21] for details). The constant, defined by

$$\mathcal{D}^0_{k,\nu} = c(k, \nu) \cdot \mathcal{L}_{k,\nu}$$

is equal to

$$c(k, \nu) = (-1)^{\nu} 2^{-\nu} \frac{(2k-3+2\nu)!}{(2k-3+\nu)!} \binom{k-2+\nu}{k-2}^{-1}.$$

This is an easy computation (e.g. by comparing the effect of both operators on the function $Z \longmapsto z^{\nu}_{12}$; we omit details).

Remark. Both constructions have some specific merits: The combinatorics of the $\mathcal{L}_{k,\nu}$ is quite explicit, but when we compute Γ-factors in the doubling method, the $\mathcal{D}^0_{k,\nu}$ are much more convenient.

1.3.2. The operators $\mathbb{D}^0_{k,\nu}$ as "Jacobifications" of the $\mathcal{D}^0_{k,\nu}$

The formula for defining $\mathbb{D}_k$ looks similar to the corresponding one for $\mathcal{D}_{k-\frac{1}{2}}$, more precisely, it follows directly from the definition of the differential operators in question that

$$\mathbb{D}_k(F) = (\partial_{\tau_3} \cdot \mathcal{D}_{k-\frac{1}{2}})(F) \tag{3}$$

for any function F on $\mathbb{H}_2 \times \mathbb{H}$, where on the left hand side of (3) we view F as a function on $\mathbb{H}_3$ not depending on z_1, z_2.

This is not accidental, a systematic study of how to view the $\mathbb{D}_k$ as "jacobified" versions of the $\mathcal{D}_{k-\frac{1}{2}}$ will be given elsewhere [5] in a much more general context.

In this section, we are mainly interested in relating the combinatorics of $\mathbb{D}^0_{k,\nu}$ to the more accessible one of the $\mathcal{D}^0_{k-\frac{1}{2},\nu}$.

We introduce two polynomials $\mathcal{P}_{k,\nu}$ and $\mathrm{Q}_{k,\nu}$ with the entries of symmetric matrices (of size two and three respectively) by

$$\mathcal{D}^0_{k,\nu}\bigl(e^{tr(TZ)}\bigr) = \mathcal{P}_{k,\nu}(T)e^{t_1 z_{11}+z_{22}} \quad (Z \in \mathbb{H}_2),$$
$$\mathbb{D}^0_{k,\nu}\bigl(e^{tr(\mathcal{T}\cdot\mathcal{Z})}\bigr) = \mathrm{Q}_{k,\nu}(\mathcal{T}) \cdot e^{tr(\mathcal{T}\cdot Z)}_{|z=0}.$$

Here $\mathcal{T}$ and T are complex symmetric matrices of size 3 and 2 respectively and t_1, t_2 denote the diagonal entries of T.

As an immediate consequence of (3) we obtain

$$\mathrm{Q}_{k,\nu}\begin{pmatrix} T & 0 \\ 0 & m \end{pmatrix} = \mathcal{P}_{k-\frac{1}{2},\nu}(m \cdot T). \tag{4}$$

Note that the polynomial on the left hand side of the equation above has to be homogeneous of degree ν in the entries of T and at the same time it carries m^ν as a factor, therefore we can write it in the form given on the right side.

Proposition 1.1. *We write the symmetric complex matrix $\mathcal{T}$ as*

$$\mathcal{T} = \begin{pmatrix} T & \mathfrak{r} \\ \mathfrak{r}^t & m \end{pmatrix}.$$

Then

$$\mathrm{Q}_{k,\nu}(\mathcal{T}) = \mathcal{P}_{k-\frac{1}{2}}\bigl(m \cdot T - \mathfrak{r} \cdot \mathfrak{r}^t\bigr).$$

Proof. It is sufficient to consider real symmetric matrices. We decompose

$$\mathcal{T} = \begin{pmatrix} T & \mathfrak{r} \\ \mathfrak{r}^t & m \end{pmatrix} = \begin{pmatrix} T - \frac{1}{m}\mathfrak{r}\mathfrak{r}^t & 0 \\ 0 & m \end{pmatrix}\left[\begin{pmatrix} 1_2 & 0 \\ \frac{1}{m}\mathfrak{r}^t & 1 \end{pmatrix}\right].$$

Then we put

$$M := \begin{pmatrix} 1_2 & 0 & 0 & 0 \\ \mathfrak{r}^t & 1 & 0 & 0 \\ 0 & 0 & 1_2 & -\mathfrak{r} \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \iota(1,2)^+\bigl(G_1(\mathbb{R})\bigr) \cdot \iota(1,2)^-\bigl(G_1(\mathbb{R})\bigr)$$

and write

$$M = \iota(1,2)^{+}(M_1)\cdot\iota(1,2)^{-}(M_2) \quad \text{with } M_i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ r_i & 1 & 0 & 0 \\ 0 & 0 & 1 & -r_i \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then we get, using (4)

$$\begin{aligned} \mathbb{D}^0_{k,\nu} e^{tr(\mathcal{T}\cdot\mathcal{Z})} &= \mathbb{D}^0_{k,\nu}\big(e^{tr\begin{pmatrix} T-\frac{1}{m}\mathfrak{r}\mathfrak{r}^t & 0 \\ 0 & m\end{pmatrix}\cdot\mathcal{Z}} |_k M\big) \\ &= \big(\mathbb{D}^0_{k,\nu} e^{tr\begin{pmatrix} T-\frac{1}{m}\mathfrak{r}\mathfrak{r}^t & 0 \\ 0 & m\end{pmatrix}\cdot\mathcal{Z}}\big)\big|_{k+\nu} M_1^{+}\big|_{k+\nu} M_2^{-} \\ &= \mathcal{P}_{k-\frac{1}{2},\nu}\big(mT-\mathfrak{r}\mathfrak{r}^t\big)\big(e^{(t_1-\frac{1}{m}r_1^2)\tau_1+t_2-\frac{1}{m}r_2^2\tau_2+m\tau_3}\big)\big|_{k+\nu} M_1^{+}\big|_{k+\nu} M_2^{-} \\ &= \mathcal{P}_{k-\frac{1}{2},\nu}\big(mT-\mathfrak{r}\mathfrak{r}^t\big)e^{tr(\mathcal{T}\cdot\mathcal{Z})}\big|_{z=0}. \qquad \square \end{aligned}$$

Corollary 1.1. *For all k and all ν we have*

$$\mathbb{D}^0_{k,\nu} = Q_{k,\nu}(\partial)_{|z=0} = \mathcal{P}_{k-\frac{1}{2},\nu}\left(\partial_{\tau_3}\partial_Z - \begin{pmatrix} \partial^2_{z_1} & \partial_{z_1}\partial_{z_2} \\ \partial_{z_1}\partial_{z_2} & \partial^2_{z_2}\end{pmatrix}\right).$$

The advantage of this is that combinatorial formulas for the polynomials $\mathcal{P}_{k,\nu}$ are known, they can be described in terms of the Gegenbauer polynomials (whose coefficients are explicitly known, see e.g. [11]), in particular, we have

$$\mathbb{D}^0_{k,\nu} = c\left(k-\frac{1}{2},\nu\right)\cdot G^{\nu}_{2k-1}\big((\partial_{\tau_3}\partial_z-\partial_{z_1}\partial_{z_2}),\big(\partial_{\tau_3}\partial_{\tau_1}-\partial^2_{z_1}\big)\big(\partial_{\tau_3}\partial_{\tau_2}-\partial^2_{z_2}\big)\big).$$

We have tacitly used the fact that the Gegenbauer polynomials can also be defined for half-integral weight in any case, the coefficients are polynomials in k and therefore they make sense for any (not necessarily integral weight k).

1.4. Relation to differential operators of Ibukiyama–Zagier

Now we define operators, which map functions on $\mathbb{H}_3$ to functions on $\mathbb{H}\times\mathbb{H}\times\mathbb{H}$: For non-negative integers ν, t_1, t_2 we put

$$\mathbb{D}^{t_1,t_2,0}_{k,\nu} := \mathcal{D}^{0,+}_{k+\nu,t_1}\circ\mathcal{D}^{0,-}_{k+\nu,t_2}\circ\mathbb{D}^0_{k,\nu}.$$

Here the plus (minus respectively) indicates that we view the function as Jacobi form w.r.t. τ_1, z_1, τ_3 (τ_2, z_2, τ_3 respectively).

These operators map Siegel modular forms of weight k and degree 3 to functions on $\mathbb{H}\times\mathbb{H}\times\mathbb{H}$, which are modular forms for τ_1 of weight $k+\nu+t_1$, modular forms for τ_2 of weight $k+\nu+t_2$. In general, these operators do not preserve automorphy for the variable τ_3 (unless $\nu=0$). This is an interesting new phenomenon. We will get a weak substitute for automorphy w.r.t. τ_3 in this section.

1.4.1. Differential operators equivariant for $\mathbb{H}^3 \hookrightarrow \mathbb{H}_3$

We recall from [21,22] that there is indeed for all triples (μ_1, μ_2, μ_3) a differential operator $L_k(\mu_1, \mu_2, \mu_3)$ changing weights from k to $k + \mu_2 + \mu_3, k + \mu_1 + \mu_3, k + \mu_1 + \mu_2$. This operator is unique up to constants. It is explicitly described (and normalized) in [22] in terms of generating functions. In a more abstract (and less explicit) setting it also appears in [16]. For us the description of Ibukiyama–Zagier is quite appropriate:

For formal variables Y_1, Y_2, Y_3 (which we collect as $\mathbf{Y}$) and ∂ as in (2) we put

$$\Delta(\partial, \mathbf{Y}) := 1 - \partial_{23} Y_1 - \partial_{13} Y_2 - \partial_{12} Y_3 + \frac{1}{2}\partial_{23}\partial_1 Y_2 Y_3 + \frac{1}{2}\partial_{13}\partial_2 Y_1 Y_3$$

$$+ \frac{1}{2}\partial_{12}\partial_3 Y_1 Y_2 + \frac{1}{4}\partial_1\partial_2 Y_3^2 + \frac{1}{4}\partial_2\partial_3 Y_1^2 + \frac{1}{4}\partial_1\partial_3 Y_2^2,$$

$$d(\boldsymbol{\partial}) := \frac{1}{2}\partial_1\partial_2\partial_3 - \frac{1}{2}\partial_1\partial_{23}^2 - \frac{1}{2}\partial_2\partial_{13}^2 - \frac{1}{2}\partial_3\partial_{12}^2 + \partial_{13}\partial_{13}\partial_{23}.$$

We also put

$$R(\boldsymbol{\partial}, \mathbf{Y}) := \frac{1}{2}\Big(\Delta(\boldsymbol{\partial}, \mathbf{Y}) + \sqrt{\Delta(\boldsymbol{\partial}, \mathbf{Y})^2 - 4d(\boldsymbol{\partial}) Y_1 Y_2 Y_3}\Big).$$

If we expand

$$G_k(\mathbf{Y}, \boldsymbol{\partial}) := \frac{1}{R(\partial, \mathbf{Y})^{k-2}\sqrt{\Delta(\boldsymbol{\partial}, \mathbf{Y})^2 - 4d(\boldsymbol{\partial}) Y_1 Y_2 Y_3}}$$

$$= \sum_{\mu_1,\mu_2,\mu_3} L_k(\mu_1, \mu_2, \mu_3) Y_1^{\nu_1} Y_2^{\nu_2} Y_3^{\nu_3}$$

as a formal power series, then

$$L_k(\mu_1, \mu_2, \mu_3)^0 := L_k(\nu_1, \nu_2, \nu_3)_{|z=z_1=z_2=0}$$

is a holomorphic differential operator, mapping holomorphic functions on $\mathbb{H}_3$ to holomorphic functions on $\mathbb{H} \times \mathbb{H} \times \mathbb{H}$ and changing the weight from k (on $\mathbb{H}_3$) to $k + \mu_2 + \mu_3, m + \mu_1 + \mu_3, m + \mu_1 + \mu_2$ (on $\mathbb{H} \times \mathbb{H} \times \mathbb{H}$).

We remark, that these differential operators include the previously defined operators $\mathcal{L}_{k,\nu}$ on $\mathbb{H}_2$ as a special case:

$$\mathcal{L}_{k,\nu} = L_k(0, 0, \nu)^0,$$

if we apply the right hand side to a function on $\mathbb{H}_2$, viewed as a function on $\mathbb{H}_3$ not depending on the variables τ_3, z_1, z_2.

Proposition 1.2. *There are rational numbers $\alpha(w) = \alpha_{k,\nu,t_1,t_2}(w)$ such that*

$$\mathbb{D}_{k,\nu}^{t_1,t_2,0} = \sum_{w=0}^{\nu+Min(t_1,t_2)} \alpha(w) \cdot \partial_{\tau_3}^w \cdot L_k(\nu + t_2 - w, \nu + t_1 - w, w)^0. \tag{5}$$

The $\alpha(w)$ are uniquely determined by the property above.

Actually, this is a statement about harmonic polynomials: Define a polynomial function $\mathcal{Q}:\mathbb{C}^{2k}\times\mathbb{C}^{2k}\times\mathbb{C}^{2k}\longrightarrow\mathbb{C}$ by

$$\mathcal{Q}(\mathfrak{x},\mathfrak{y},\mathfrak{z}):=Q_{k,\nu}^{t_1,t_2}\left(\begin{pmatrix}\mathfrak{x}^t\cdot\mathfrak{x} & \mathfrak{x}^t\cdot\mathfrak{y} & \mathfrak{x}^t\cdot\mathfrak{z}\\ \mathfrak{y}^t\cdot\mathfrak{x} & \mathfrak{y}^t\cdot\mathfrak{y} & \mathfrak{y}^t\cdot\mathfrak{z}\\ \mathfrak{z}^t\cdot\mathfrak{x} & \mathfrak{z}^t\cdot\mathfrak{y} & \mathfrak{z}^t\cdot\mathfrak{z}\end{pmatrix}\right).$$

This polynomial is homogeneous of degree

$$\begin{aligned}\nu+t_1 &\quad \text{in } \mathfrak{x} \text{ and harmonic in } \mathfrak{x},\\ \nu+t_2 &\quad \text{in } \mathfrak{y} \text{ and harmonic in } \mathfrak{y},\\ 2\nu+t_1+t_2 &\quad \text{in } \mathfrak{z} \text{ and not harmonic in } \mathfrak{z} \text{ (in general)}.\end{aligned}$$

The harmonicity is a consequence of the transformation properties of the differential operators defining the polynomials, see e.g. [2,21]. It follows from the representation theory of the orthogonal group $O(2k,\mathbb{C})$ (see e.g. [17, p. 255]) that such a polynomial can be written as

$$\mathcal{Q}=\sum_{w=0}^{\nu+[\frac{t_1+t_2}{2}]}\left(\mathfrak{z}^t\cdot\mathfrak{z}\right)^w\cdot\mathcal{L}_w(\mathfrak{x},\mathfrak{y},\mathfrak{z}),$$

where $\mathcal{L}_w$ is now harmonic in all three sets of variables (and homogeneous of degrees $\nu+t_1$, $\nu+t_2$, $2\nu+t_1+t_2-2w$). Furthermore, the expression above is unique.

It is an extra feature that the polynomial $\mathcal{Q}$ is invariant under the simultaneous action of the orthogonal group $O(2k,\mathbb{C})$ on the three variables. This carries over to the polynomials $\mathcal{L}_w$. Therefore these polynomials are again polynomials in the entries of the Gram matrix associated with $\mathfrak{x},\mathfrak{y},\mathfrak{z}$.

It is known that such invariant harmonic polynomials $\mathcal{L}_w$ can only exist if the sum of the two smaller degrees is bigger than the largest degree ("balanced case"). This comes down to the condition

$$w\leqslant\nu+Min\{t_1,t_2\}.$$

If this condition is satisfied, there is indeed such a polynomial, it is unique up to constants. After rephrasing everything in terms of differential operators, we get the proposition.

1.4.2. On the coefficients $\alpha(w)$

Using polyindices $\boldsymbol{\mu}=(\mu_1,\mu_2,\mu_3)$, $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\lambda_3)$, $\boldsymbol{\kappa}=(\kappa_1,\kappa_2,\kappa_3)$ we may expand

$$L_k(\boldsymbol{\mu})^0=\sum_{\boldsymbol{\lambda},\boldsymbol{\kappa}}R^{\boldsymbol{\mu}}(\boldsymbol{\lambda},\boldsymbol{\kappa})(\partial_{\tau_1}\partial_{\tau_2}\partial_{\tau_3})^{\boldsymbol{\lambda}}(\partial_{z_2}\partial_{z_1}\partial_z)^{\boldsymbol{\kappa}}.$$

The coefficients $R^{\boldsymbol{\mu}}(\boldsymbol{\lambda},\boldsymbol{\kappa})$ are analyzed in detail in [22].

The only property we need here is that $R^{\boldsymbol{\mu}}(0,\boldsymbol{\kappa})\neq 0$ only if $\boldsymbol{\mu}=\boldsymbol{\kappa}$; in that case

$$R^{\boldsymbol{\mu}}(0,\boldsymbol{\mu})=\frac{(k-2+\mu_2+\mu_3)!(k-2+\mu_1+\mu_3)!(k-2+\mu_1+\mu_2)!}{\mu_1!\mu_2!\mu_3!(k-2+\mu_1)!(k-2+\mu_2)!(k-2+\mu_3)!}.$$

Now we fix a number w' satisfying $0\leqslant w'\leqslant\nu+Min(t_1,t_2)$. We compare the coefficient of $\partial_{\tau_3}^{w'}\partial_{z_1}^{\nu+t_1-w'}\partial_{z_2}^{\nu+t_2-w'}\partial_z^{w'}$ on both sides of (5). For a fixed w' this gives a linear equation for the $\alpha(w)$:

$$\sum_{w+\lambda_3=w'} \alpha(w) R^{\nu+t_2-w,\nu+t_1-w,w}\big(0,0,\lambda_3,\nu+t_2-w',\nu+t_1-w',w'\big)=(*),$$

where $(*)$ denotes the corresponding coefficient on the left hand side of (5). The matrix of the corresponding homogeneous system is then a lower triangular matrix with diagonal elements equal to

$$R^{\nu+t_2-w',\nu+t-w',w'}\big(0,0,0;\nu+t_2-w',\nu+t-w',w'\big)\neq 0.$$

This procedure allows us to compute the rational numbers $\alpha(w)$ in each particular case (of course we must compute some additional coefficients on the left and right hand sides arising either from the generating series of [22] or from Gegenbauer polynomials). It would be nice to have a simple closed formula for the coefficients $\alpha(w)$; it would in particular be desirable to show them to be different from zero.

We compute a few cases below.

1.4.3. Basic examples

We start with the formulas

$$\begin{aligned}
L_k(0,0,1)^0 &= (k-1)\partial_z,\\
L_k(1,1,0)^0 &= -\frac{k-1}{2}\partial_{\tau_3}\partial_z + k(k-1)\partial_{z_1}\partial_{z_2},\\
L_k(0,0,2)^0 &= -\frac{1}{4}(k-1)\partial_{\tau_1}\partial_{\tau_2} + \frac{k(k-1)}{2}\partial_z^2,\\
L_k(1,1,1)^0 &= k^3\partial_z\partial_{z_1}\partial_{z_2} - \frac{k^2}{2}\big(\partial_{\tau_1}\partial_{z_2}^2 + \partial_{\tau_2}\partial_{z_1}^2 + \partial_{\tau_3}\partial_z^2\big) + \frac{k}{2}\partial_{\tau_1}\partial_{\tau_2}\partial_{\tau_3},\\
L_k(2,2,0)^0 &= \frac{k(k-1)}{8}\partial_z^2\partial_{\tau_3}^2 + \frac{k(k-1)}{16}\partial_{\tau_1}\partial_{\tau_2}\partial_{\tau_3}^2\\
&\quad - \frac{(k+1)k(k-1)}{8}\partial_{\tau_2}\partial_{\tau_3}\partial_{z_1}^2 - \frac{(k+1)k(k-1)}{8}\partial_{\tau_1}\partial_{\tau_3}\partial_{z_2}^2\\
&\quad - \frac{(k+1)k(k-1)}{2}\partial_{\tau_3}\partial_z\partial_{z_1}\partial_{z_2} + \frac{(k+2)(k+1)k(k-1)}{4}\partial_{z_1}^2\partial_{z_2}^2.
\end{aligned}$$

All cases (except the forth one) can easily be obtained from the generating series of Ibukiyama–Zagier by the observation that it simplifies considerably for those $L_k(\mu_1,\mu_2,\mu_3)$, for which at least one of the μ_i is zero: In that case everything comes down to the consideration of a series of type

$$\frac{1}{(1-X)^{k-1}} = \sum \binom{k-2+j}{k-2} X^j.$$

The case $L_k(1,1,1)$ was obtained in a different way: We determine the polynomials in the entries of Gram matrix associated to $(\mathfrak{x},\mathfrak{y},\mathfrak{z})\in\mathbb{C}^{(2k,3)}$ with the right degrees and normalize them properly.

We treat 3 examples following the pattern of the previous section.

Example 1. $\nu=1$, $t_1=t_2=0$

$$\mathbb{D}_{k,1}^0 = \alpha(0)L_k(1,1,0)^0 + \alpha(1)\partial_{\tau_3}L_k(0,0,1)^0.$$

This gives the linear equations

$$\begin{aligned} k(k-1)\alpha(0) \qquad\qquad &= k-1, \\ -\frac{k-1}{2}\alpha(0)+(k-1)\alpha(1) &= -(k-1) \end{aligned}$$

and from this

$$\mathbb{D}^0_{k,1} = \frac{1}{k}L_k(1,1,0)^0 - \frac{2k-1}{2k}\partial_{\tau_3}L_k(0,0,1)^0.$$

Example 2. $\nu = t_1 = t_2 = 1$

$$\begin{aligned} \mathbb{D}^{1,1,0}_{k,1} &= \left\{\left(-k-\frac{1}{2}\right)^2(k-1)\left(\partial^2_{z_1}\partial^2_{z_2} - \partial_z\partial_{z_1}\partial_{z_2}\partial_{\tau_3}\right)\right\}_{z=0} \\ &= \alpha(0)L_k(2,2,0)^0 + \alpha(1)\partial_{\tau_3}L_k(1,1,1)^0 + \alpha(2)\partial^2_{\tau_3}L_k(0,0,2)^0. \end{aligned}$$

The linear equations are

$$\begin{aligned} \frac{(k+2)(k+1)k(k-1)}{4}\alpha(0) \qquad\qquad\qquad &= \left(k+\frac{1}{2}\right)^2(k-1), \\ -\frac{(k+1)k(k-1)}{2}\alpha(0)+k^3\alpha(1) \qquad\qquad &= -\left(k+\frac{1}{2}\right)^2(k-1), \\ \frac{k(k-1)}{8}\alpha(0)-\frac{k^2}{2}\alpha(1)+\frac{k(k-1)}{2}\alpha(2) &= 0 \end{aligned}$$

and from this

$$\begin{aligned} \mathbb{D}^{1,1,0}_{k,1} = {}& \frac{4(k+\frac{1}{2})^2}{(k+2)(k+1)k}L_k(2,2,0)^0 - \frac{(k+\frac{1}{2})^2(k-1)}{k^2(k+2)}\partial_{\tau_3}L_k(1,1,1)^0 \\ & - \frac{(k+\frac{1}{2})^2}{k(k+1)}\partial^2_{\tau_3}L_k(0,0,2)^0. \end{aligned}$$

Example 3. $\nu = 2$, $t_1 = t_2 = 0$

$$\begin{aligned} \mathbb{D}^0_{k,2} &= \left\{k(k-1)\partial^2_{z_1}\partial^2_{z_2} + \left(k-2k^2\right)\partial_z\partial_{z_1}\partial_{z_2}\partial_{\tau_3} + k\left(k-\frac{1}{2}\right)\partial^2_z\partial^2_{\tau_3}\right. \\ &\qquad \left. - \frac{k}{2}\partial_{\tau_1}\partial_{\tau_2}\partial^2_{\tau_3} + \frac{k}{2}\partial^2_{z_2}\partial_{\tau_1}\partial_{\tau_3} + \frac{k}{2}\partial^2_{z_1}\partial_{\tau_2}\partial_{\tau_3}\right\}_{|z=0} \\ &= \alpha(0)L_k(2,2,0)^0 + \alpha(1)\partial_{\tau_3}L_k(1,1,1)^0 + \alpha(2)\partial^2_{\tau_3}L_k(0,0,2)^0. \end{aligned}$$

The linear equations are

$$\begin{aligned} \frac{(k+2)(k+1)k(k-1)}{4}\alpha(0) \qquad\qquad\qquad &= k(k-1), \\ -\frac{(k+1)k(k-1)}{2}\alpha(0)+k^3\alpha(1) \qquad\qquad &= k-2k^2, \\ \frac{k(k-1)}{8}\alpha(0)-\frac{k^2}{2}\alpha(1)+\frac{k(k-1)}{2}\alpha(2) &= k\left(k-\frac{1}{2}\right) \end{aligned}$$

and from this

$$\mathbb{D}^0_{k,2} = \frac{4}{(k+2)(k+1)} L_k(2,2,0)^0 + \frac{-1-2k}{(k+2)k}\partial_{\tau_3} L_k(1,1,1)^0 + \frac{2k^3+3k^2-3k-2}{(k+2)(k+1)(k-1)}\partial^2_{\tau_3} L_k(0,0,2)^0.$$

2. Pullback formulas with differential operators I: the Arakawa side

2.1. Jacobi– and Siegel–Eisenstein series

We first define the degree n Siegel–Eisenstein series:

$$E^n_k(Z) := \sum_{M\in Sp(n,\mathbb{Z})_\infty\backslash Sp(n,\mathbb{Z})} j(M,Z)^{-k} \quad (k>n+1,\ Z\in\mathbb{H}_n).$$

Also we recall the definition of the degree n Jacobi–Eisenstein series: For

$$\mathcal{Z} = \begin{pmatrix} * & * \\ * & \tau_3 \end{pmatrix} \in \mathbb{H}_{n+1} \quad (\tau_3\in\mathbb{H})$$

we start from the function (with $m\in\mathbb{N}$)

$$\mathbf{e}_m : \begin{cases} \mathbb{H}_3 \longrightarrow \mathbb{C} \\ \mathcal{Z} \longmapsto e^{2\pi i m\tau_3} \end{cases}$$

and we put (for $k>n+2$)

$$E^n_{k,m}(\mathcal{Z}) := \sum_{M\in(G_n)_\infty\backslash G_n(\mathbb{Z})} \mathbf{e}_m|_k M.$$

More explicitly, the summation runs over

$$\bigl\{M=[\lambda,0,0]\cdot R^{\uparrow} \bigm| R\in Sp(n,\mathbb{Z})_\infty\backslash Sp(n,\mathbb{Z}),\ \lambda\in\mathbb{Z}^n\bigr\},$$

where as usual we use $[\lambda,\mu,\kappa]$ as notation for an element of the Heisenberg part of G_n:

$$[\lambda,\mu,\kappa] := \begin{pmatrix} 1_n & 0 & 0 & \mu' \\ \lambda & \mu & 0 & \kappa \\ 0 & 0 & 1_n & -\lambda^t \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \bigl(\lambda,\mu\in\mathbb{R}^n,\ \kappa\in\mathbb{R}\bigr).$$

These two Eisenstein are connected via the Fourier–Jacobi expansion of E^{n+1}_k:

$$E^{n+1}_k(\mathcal{Z}) = \sum_{m=0}^{\infty} e^n_{k,m}(\tau_1,z)e^{2\pi i m\tau_3}.$$

If m is squarefree the relation is quite simple, see [2]:

$$e^n_{k,m}(\tau,Z)e^{2\pi i m\tau_3} = A_k\cdot\sigma_{k-1}(m)E^n_{k,m}(\mathcal{Z})$$

with

$$A_k := \frac{2}{\zeta(1-k)}$$

and $\sigma_{k-1}(m) = \sum_{d|m} d^{k-1}$.

2.2. The pullback formula of Arakawa

We need a general pullback formula for $\mathbb{D}^0_{k,\nu} E^2_{k,m}$. The case of plain restriction (i.e. $\nu = 0$) and index $m = 1$ was worked out by Arakawa [1] and in a refined version by the second author [19,10] for squarefree index m.

A key tool in Arakawa's pullback formula is a double coset decomposition due to Garrett [15]: A complete set of representatives for

$$Sp(2,\mathbb{Z})_\infty \backslash Sp(2,\mathbb{Z})/SL(2,\mathbb{Z})^{\uparrow} \times SL(2,\mathbb{Z})^{\downarrow}$$

is given by

$$\{\gamma_d \mid d \geqslant 0\}$$

with

$$\gamma_d = \begin{pmatrix} \begin{matrix} 1_2 \end{matrix} & 0_2 \\ \begin{matrix} 0 & d \\ d & 0 \end{matrix} & 1_2 \end{pmatrix}.$$

Note that the $\gamma_d^{\uparrow} \in Sp(3,\mathbb{Z})$ are not compatible with the differential operators $\mathbb{D}^0_{k,\nu}$ in the sense that the theorem of the previous section does not apply to the $\gamma_d^{\uparrow}$.

We first consider the action of $\mathbb{D}_k$ on $\mathbf{e}_m|_k[\lambda, 0_2, 0]|_k \gamma_d^{\uparrow}$.

To study this, we remark that $[\lambda, 0_2, 0]$ and $\gamma_d^{\uparrow}$ commute and therefore we just have to study $\mathbb{D}_k(\mathbf{e}_m|\gamma_d^{\uparrow})$.

The key result is

Proposition 2.1. *For all k, d we have*

$$\mathbb{D}_k\big(\mathbf{e}_m|_k\gamma_d^{\uparrow}\big) = \left(k - \frac{1}{2}\right)(k-1)md(-2\pi i)\mathbf{e}_m|_{k+1}\gamma_d^{\uparrow}.$$

Corollary 2.1. *For all $\nu > 0$ we get*

$$\mathbb{D}_{k,\nu}\big(\mathbf{e}_m|_k\gamma_d^{\uparrow}\big) = 2^{-2\nu}(-2\pi i)^{\nu}\frac{\Gamma(2k+2\nu-2)}{\Gamma(2k-2)}(md)^{\nu}\mathbf{e}_m|_{k+\nu}\gamma_d^{\uparrow}.$$

It is clear that we could prove this proposition by direct computation. However, we prefer to follow Lemma 10 of [3], which can (hopefully) be generalized to higher degree cases.

Using the matrix $\mathbb{J}(x) := \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&-1\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&1&0&0&x \end{pmatrix}$ and $\mathbb{J} := \mathbb{J}(0)$ we put (for $d > 0$)

$$
\begin{aligned}
M_d &:= \mathbb{J} \cdot \gamma_d^{\uparrow} \cdot \begin{pmatrix} 0 & -d^{-1} \\ d & 0 \end{pmatrix}^{+} \\
&= \begin{pmatrix} d & \overset{*}{d} & 0 & 0 & \overset{*}{0} & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}
$$

Using the decomposition (1) for $\mathcal{Z} \in \mathbb{H}_3$ we study the auxiliary function

$$
h_{k,M_d} := 1|_k M_d = d^{-k}\big(\tau_1\tau_3 + 2z\tau_3 + \tau_2\tau_3 - z_1^2 - z_2^2 - 2z_1z_2\big)^{-k}.
$$

We compute $\mathbb{D}_k h_k$ (with $h_k := h_{k,M_1}$), using the elementary formulas

$$
\begin{aligned}
\partial_{z_2} h_k &= -\frac{k}{2} h_{k+1} \cdot (-2z_2 - 2z_1), \\
\partial_{z_1}\partial_{z_2} h_k &= k(k+1)h_{k+2}(z_1 + z_2)^2 + \frac{k}{2} h_{k+1}, \\
\partial_z h_k &= -k h_{k+1}\tau_3, \\
\partial_{\tau_3}\partial_z h_k &= -k h_{k+1} + k(k+1)h_{k+2}(\tau_1 + 2z + \tau_2)\tau_3, \\
\partial^{[3]} h_{k,M_1} &= 0.
\end{aligned}
$$

The last equation follows from studying the linear map $Z \longmapsto C_M Z + D_M$ with $\det(C_M) = 0$.

In summary, this gives

$$
\begin{aligned}
\mathbb{D}_k h_k &= (k-1)\left\{\frac{k}{2} + k\right\} h_{k+1} \\
&\quad + (k-1)\big\{k(k+1)(z_1 + z_2)^2 - k(k+1)(\tau_1 + 2z + \tau_2)\tau_3\big\} h_{k+2} \\
&= -k\left(k - \frac{1}{2}\right)(k-1)h_{k+1}.
\end{aligned}
$$

Then we get

$$
\mathbb{D}_k h_{k,M_d} = -k\left(k - \frac{1}{2}\right)(k-1) \cdot d \cdot h_{k+1,M_d}
$$

and for $N_d := \mathbb{J} \cdot \gamma_d^{\uparrow}$

$$
\mathbb{D}_k(1|_k N_d) = -k\left(k - \frac{1}{2}\right)(k-1)d \cdot (1|_{k+1} N_d).
$$

Here we may also include the case $d = 0$, because the formula above is analytic in d – this is in accordance with the fact that these operators produce cusp forms (after restriction).

To get from this a result about $\mathbb{D}_k(\mathbf{e}_m|_k\gamma^\uparrow)$, we start from the identity

$$\frac{(-2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty} m^{k-1}\mathbf{e}_m(\tau_3)=\sum_{l\in\mathbb{Z}}(\tau_3+l)^{-k}$$

and we apply to both sides first the operator $|_k\gamma_d^\uparrow$ and then the differential operator $\mathbb{D}_k$.

The left hand side then becomes

$$\frac{(-2\pi i)^k}{(k-1)!}\sum_{m=1} m^{k-1}\mathbb{D}_k(\mathbf{e}_m|_k\gamma_d).$$

Concerning the right side, we recall that $(\tau_3+l)^{-k}=j(\mathbb{J}(l),\mathcal{Z})^{-k}$ and $\mathbb{J}(l)=\mathbb{J}\cdot\mathbb{T}(l)$ for an appropriate translation $\mathbb{T}(l)$, hence we can view $\mathbb{D}_k((\tau_3+l)^{-k}|\gamma_d^\uparrow)$ as

$$\begin{aligned}
\mathbb{D}_k\big(1|_k\mathbb{J}(l)\gamma_d^\uparrow\big)&=\mathbb{D}_k\big(1|_k\mathbb{J}\gamma_d^\uparrow\mathbb{T}(l)\big)\\
&=\mathbb{D}_k\big(1|_k\mathbb{J}\gamma_d^\uparrow\big)\big|_k\mathbb{T}(l)\\
&=\mathbb{D}_k(1|_kN_d)|\mathbb{T}(l)\\
&=-k\left(k-\frac{1}{2}\right)(k-1)d\cdot 1|_{k+1}N_d|_k\mathbb{T}(l)\\
&=-k\left(k-\frac{1}{2}\right)(k-1)d(\tau_3+l)^{-k-1}|_{k+1}\gamma_d^\uparrow.
\end{aligned}$$

Comparison of both sides gives

$$\mathbb{D}_k\big(\mathbf{e}_m|_k\gamma_d^\uparrow\big)=\left(k-\frac{1}{2}\right)(k-1)dm(-2\pi i)\mathbf{e}_m|_{k+1}\gamma_d.$$

This proves the proposition.

The proposition and its corollary allow us immediately to generalize Arakawa's pullback formula to the case of $\mathbb{D}^0_{k,\nu}E^2_{k,m}$. The reason is that the differential operator commutes with all substitutions entering into the definition of the Jacobi–Eisenstein series (when written in terms of representatives arising from Garrett's double coset decomposition) except the $\gamma_d^\uparrow$; for these substitutions, we use the corollary above.

Furthermore, we recall that (at least if the index m is squarefree) the space $J^{cusp}_{k,m}$ of Jacobi cusp forms of index m and weight k has an (orthogonal) basis $\{\Phi_i\}$ consisting of eigenforms of the Jacobi–Hecke operators given by the double cosets

$$G_1(\mathbb{Z})\cdot diag\big(1,n^{-1},1,n\big)\cdot G_1(\mathbb{Z})\quad (n\in\mathbb{N}).$$

The corresponding eigenvalues will be denoted by $\lambda(\Phi_i,n)$. We associate to these eigenforms Φ_i the zeta functions

$$Z^J(s,\Phi_i)=\sum\frac{\lambda(\Phi_i,n)}{n^s},$$

which were studied in [1,19].

With these facts at hand, the generalization of Arakawa's pullback formula is just a formal manipulation following the original computation [1,10,19] line by line; our version here is the analogue of [10, Proposition 4.1].

Theorem 2.1. *Let m be a squarefree positive integer and ν a non-negative integer. We denote by $\{\Phi_i\}$ an orthogonal Hecke eigenbasis of the space of Jacobi cusp forms of index m and weight $k+\nu$. Then*

$$\left(\mathbb{D}^0_{k,\nu} E^2_{k,m}\right)(\tau_1, z_2, \tau_2, z, \tau_3) = \delta_{\nu,0} E^1_{k,m}(\tau_1, z_1) \times E^1_{k,m}(\tau_2, z_2) e^{2\pi i m\tau_3} + \beta(k,\nu) \sum_i \frac{Z^J(k,\Phi_i)}{\|\Phi_i\|^2} \Phi_i(\tau_1, z_1) \times \Phi(\tau_2, z_2) e^{2\pi i m\tau_3}.$$

Here we have changed the notion of Jacobi forms: They appear now as functions on $\mathbb{H} \times \mathbb{C}$. Furthermore $\| \ \|$ denotes the Petersson norm of the Jacobi form Φ (as explained e.g. in [1, Section 2]) and

$$\beta(k,\nu) = 2^{-2\nu}(-2\pi i)^\nu \frac{\Gamma(2k+2\nu-2)}{\Gamma(2k-2)} m^\nu \frac{(-1)^{\frac{k+\nu}{2}} \pi 2^{1-k-\nu}}{m(k+\nu-\frac{3}{2})}.$$

Comment. The first part of the constant $\beta(k,\nu)$ comes from the corollary above, the second part comes from [10, Proposition 4.1] with k replaced by $k+\nu$. Furthermore, we should point out that factor d^ν on the right side of the corollary is responsible for changing the critical point of the zeta function $Z^J(s,\Phi_i)$ from $k+\nu$ to k.

Remark. For a reinterpretation of the zeta function $Z^J(s,\Phi)$ in terms of L-functions for elliptic modular forms we refer the reader to Section 5.

2.3. *The final restriction to $\mathbb{H} \times \mathbb{H}$*

We may now apply the operators $\mathcal{D}^0_{k+\nu,t}$ to the Jacobi forms of the theorem: Formally we get the spectral decomposition

$$\mathcal{D}^0_{k+\nu,t}\left(\Phi_i \cdot e^{2\pi i m\tau_3}\right) = \sum_{\{f\}} I^{(t)}(\Phi_i, f) \cdot f \cdot e^{2\pi i m\tau_3}$$

where the f run over an orthogonal basis of $S^{k+\nu+t}$, e.g. a basis of normalized Hecke eigenforms. By abuse of notation, we write

$$\mathcal{D}^0_{k+\nu,t}(\Phi_i) := \mathcal{D}^0_{k+\nu,t}\left(\Phi_i \cdot e^{2\pi i m\tau_3}\right) e^{-2\pi i m\tau_3}.$$

Due to reasons to be explained later, we call

$$I^{(t)}(\Phi_i, f) := \frac{\langle \mathcal{D}^0_{k+\nu,t}(\Phi_i), f\rangle}{\langle f, f\rangle}$$

an *Ichino period* for the Jacobi form Φ_i.

Theorem 2.1 *(Final general pullback formula for the Arakawa–Ichino side (first version)). For integers $k > 4$, $m > 0$, m squarefree, $\nu, t_1, t_2 \geqslant 0$ we have*

$$\mathbb{D}_{k,\nu}^{t_1,t_2,0} E_{k,m}^2 = \delta_{\nu,0} X + \beta(k,\nu) \sum_i \frac{Z^J(k,\Phi_i)}{\|\Phi_i\|^2} \sum_{\{f\}} \sum_{\{g\}} I^{(t_1)}(\Phi_i, f) \cdot I^{(t_2)}(\Phi_i, g) f \otimes g \cdot e^{2\pi i m \tau_3},$$

where the f run over an orthogonal basis of $S^{k+\nu+t_1}$ and the g over an orthogonal basis of $S^{k+\nu+t_2}$. The contribution X only occurs for $\nu = 0$; it equals

$$X = \left(\delta_{t,0} E_k + \frac{\kappa(k,t_1)}{A_k \sigma_{k-1}(m) \zeta(k) \zeta(2k-2)} \times \sum_{\{f\}} a_f(m) \frac{D(f,k-1)}{\langle f, f\rangle} f \right) \otimes \left(\delta_{k,t_2} E_k + \frac{\kappa(k,t_2)}{A_k \sigma_{k-1}(m) \zeta(k) \zeta(2k-2)} \sum_{\{g\}} a_g(m) \frac{D(g,k-1)}{\langle g, g\rangle} g \right) e^{2\pi i m \tau_3}.$$

As for the contribution X we recall that

$$\left\langle \mathcal{D}_{k,t}^0 \left(E_k^2\right) \begin{pmatrix} * & 0 \\ 0 & \tau' \end{pmatrix}, f \right\rangle = \kappa(k,t) \frac{D(f,k-1)}{\zeta(k)\zeta(2k-2)} \overline{f\left(-\bar{\tau}'\right)}$$

for a Hecke eigenform in S^{k+t}. This is a version of the doubling method with differential operators as in [4]. The constant is a natural product of three factors

$$\kappa(k,t) = \frac{(-1)^{\frac{k+t}{2}} 2^{3-k-t} \pi}{k+t-1} \times \prod_{i=0}^{t-1} (-k-i) \times \prod_{i=0}^{t-1} \left(-k-i+\frac{1}{2}\right).$$

Again the first factor comes from the analogous formula for weight $(k+t)$ without differential operator, the second factor comes from the iteration of the formula $\partial_{12}(z_{11}+2z_{12}+z_{22})^{-k} = (-k)(z_{11}+2z_{12}+z_{22})^{-k-1}$, which arises naturally in the doubling method (see [3,4]), and the third factor comes from the normalization of the differential operator $\mathcal{D}_{k,t}$.

We can reformulate this for the Jacobi–Eisenstein series as

$$\left\langle \mathcal{D}_{k,t}^0 \left(E_{k,m}^1\right), f \right\rangle = \frac{\kappa(k,t)}{A_k \sigma_{k-1}(m)} \frac{D(f,k-1)}{\zeta(k)\zeta(2k-2)} \overline{a_f(m)}.$$

The formula above is the final general Arakawa–Ichino side of our identity. The summands will be rewritten in Section 5.

3. Pullback formulas with differential operators II: the Garrett side

In Section 1.4 we introduced differential operators $L_k(\mu_1, \mu_2, \mu_3)$, which were defined by means of a generating series. We have used these differential operators in the Garrett integral representation for triple product L-functions already in previous works [7,6]. In [7,6] we only considered the integrals against cusp forms, here however we need the exact pullback formula including also possible contributions from Eisenstein series:

We have to study $L_k(\mu_1, \mu_2, \mu_3)(E_k^3)$ in detail. We start with a few remarks concerning the non-cuspidal part:

- $L_k(\mu_1, \mu_2, \mu_3)(F)$ is cuspidal in all three variables unless two of the μ_i are zero; this is true for any degree 3 Siegel modular form F of weight k by a reasoning similar to the one given in the proof of the theorem in Section 1.

- For the remaining case (say e.g. $\mu_1 = \mu_2 = 0$) we recall that $L_k(0,0,\mu)^0 = G_{2k}^{\mu}(\partial_z, \partial_{\tau_1}\partial_{\tau_2})_{|z=0}$, so $L_k(0,0,\mu)^0$ acts on F like $\mathcal{L}_{k,\mu}$ acts on $F\left(\left(\begin{smallmatrix} Z & 0 \\ 0 & \tau_3 \end{smallmatrix}\right)\right)$, when viewed as a function of $Z = \left(\begin{smallmatrix} \tau_1 & z \\ z & \tau_2 \end{smallmatrix}\right)$.
- Over any field K there are 5 orbits of

$$Sp(3,K)_\infty \backslash Sp(3,K)/SL(2,K)^3,$$

given by

$$\mathcal{O}_0 := 1_6, \qquad \mathcal{O}_1 := \begin{pmatrix} & 1_3 & & 0_3 \\ 0 & 0 & 0 & \\ 0 & 0 & 1 & 1_3 \\ 0 & 1 & 0 & \end{pmatrix}, \qquad \mathcal{O}_2 := \begin{pmatrix} & 1_3 & & 0_3 \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1_3 \\ 1 & 0 & 0 & \end{pmatrix},$$

$$\mathcal{O}_3 := \begin{pmatrix} & 1_3 & & 0_3 \\ 0 & 1 & 0 & \\ 1 & 0 & 0 & 1_3 \\ 0 & 0 & 0 & \end{pmatrix}, \qquad \mathcal{O}_{main} = \begin{pmatrix} & 1_3 & & 0_3 \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & 1_3 \\ 1 & 1 & 1 & \end{pmatrix}.$$

Applying the differential operators $L_k(\mu_1,\mu_2,\mu_3)$ to the degree three Siegel–Eisenstein series, we first notice, which orbits (and their contributions to the pullbacks) get killed. If at least two of the μ_i are non-zero, then only the "main orbit" $\mathcal{O}_{main}$ remains. If only μ_i is different from zero, then the two orbits $\mathcal{O}_i$ and $\mathcal{O}_{main}$ survive.

With the remarks from above and using the results from [15,14,7,6,20] we get

Proposition 3.1. *Let k be an even integer, $k > 4$ and let μ_1, μ_2, μ_3 be any non-negative integers. Then*

$$\begin{aligned} L_k(\mu_1,\mu_2,\mu_3)\big(E_k^3\big) &= \delta_{0,\mu_1}\delta_{0,\mu_2}\delta_{0,\mu_3} E_k \otimes E_k \otimes E_k \\ &+ \delta_{0,\mu_1}\delta_{0,\mu_2} \frac{\kappa(k,\mu_3)}{c(k,\mu_3)} \sum_{g \in S^{k+\mu_3}} \frac{D(g,k-1)}{\langle g,g\rangle} g \otimes g \otimes E_k \\ &+ \delta_{0,\mu_1}\delta_{0,\mu_3} \frac{\kappa(k,\mu_2)}{c(k,\mu_2)} \sum_{g \in S^{k+\mu_2}} \frac{D(k-1,g)}{\langle g,g\rangle} g \otimes E_k \otimes g \\ &+ \delta_{0,\mu_2}\delta_{0,\mu_3} \frac{\kappa(k,\mu_1)}{c(k,\mu_1)} \sum_{g \in S^{k+\mu_1}} \frac{D(k-1,g)}{\langle g,g\rangle} E_k \otimes g \otimes g \\ &+ \frac{\gamma(k,\mu_1,\mu_2,\mu_3)}{\zeta(k)\zeta(2k-2)} \sum_{f,g,h} \frac{L(f \otimes g \otimes h, 2k+\mu_1+\mu_2+\mu_3-2)}{\langle f,f\rangle\langle g,g\rangle\langle h,h\rangle} f \otimes g \otimes h. \end{aligned}$$

The last sum goes over a normalized basis of Hecke eigenforms for $S^{k+\mu_2+\mu_3}$, $S^{k+\mu_1+\mu_3}$ and $S^{k+\mu_1+\mu_2}$ and

$$\begin{aligned} \gamma(k,\mu_1,\mu_2,\mu_3) &= (i)^{-k+\mu_1+\mu_2+\mu_3} 2^{-5k+8-4\mu_1-4\mu_2-4\mu_3} \pi^{3-2k-\mu_1-\mu_2-\mu_3} \\ &\times \frac{\Gamma(2k+\mu_1+\mu_2+\mu_3-2)}{\Gamma(2k-2)\Gamma(k)} \\ &\times \frac{\Gamma(\mu_1+\mu_2+k-1)\Gamma(\mu_1+\mu_3+k-1)\Gamma(\mu_2+\mu_3+k-1)}{\mu_1!\mu_2!\mu_3!}. \end{aligned}$$

We extract this archimedian factor from the computation in [7, (2.41)]. The simple form of this factor comes out only after some tedious calculations. The reason is that the archimedian integral is not computed directly for the differential operator $L_k(\mu_1, \mu_2, \mu_3)$, but for a composition of (not necessarily holomorphic) differential operators (Maaß type operators together with the operators from [8]) whose "holomorphic component" is then up to a factor equal to $L_k(\mu_1, \mu_2, \mu_3)$. This procedure avoids a lot of combinatorial problems.

Remark. The critical point which appears in the proposition above is the largest critical point for the triple L-function only if at least one of he μ_i is zero. Note also that we do not get the central critical value (because $k = 2$ is not allowed!).

4. Comparison: first version via periods

We follow the basic strategy of [20]: We fix an even number $k > 4$, non-negative integers ν, t_1, t_2 and a squarefree positive integer m and also we choose normalized Hecke eigenforms $f \in S^{k+\nu+t_1}$ and $g \in S^{k+\nu+t_2}$. Starting from the Siegel–Eisenstein series E_k^3 of degree 3 we first pick out the m-th Fourier–Jacobi coefficient. As mentioned at the beginning of Section 2, this m-th Fourier–Jacobi coefficient equals

$$A_k \sigma_{k-1}(m) \cdot E_{k,m}^2,$$

i.e. it equals, up to a factor, the Jacobi–Eisenstein series of index m defined earlier.

We apply the differential operator $\mathbb{D}_{k,\nu}^{t_1,t_2,0}$ to $E_{k,m}^2$ and pick out the coefficient of $f \otimes g \cdot e^{2\pi i m\tau_3}$. According to Theorem 2.2 this coefficient is equal to

$$\delta_{\nu,0}\left(\frac{\kappa(k,t_1)\kappa(k,t_2)a_f(m)a_g(m)}{A_k\sigma_{k-1}(m)\zeta(k)^2\zeta(2k-2)^2}\frac{D(f,k-1)}{\langle f,f\rangle}\frac{D(g,k-1)}{\langle g,g\rangle}\right) + \beta(k,\nu)\sum_i \frac{Z^J(k,\Phi_i)}{\|\Phi_i\|^2} I^{(t_1)}(\Phi_i,f)I^{(t_2)}(\Phi_i,g).$$

On the other hand,

$$\mathbb{D}_{k,\nu}^{t_1,t_2,0} = \sum_{w=0}^{\nu+Min(t_1,t_2)} \alpha_{k,\nu,t_1,t_2}(w)\partial_{\tau_3}^w \cdot L_k(\nu+t_2-w, \nu+t_1-w, w)^0.$$

We may as well apply the operators $L_k(\ldots)^0$ to E_k^3, use Garrett's pullback formula (Proposition 3.1) and then pick out the m-th Fourier coefficient with respect to τ_3. If f and g are different, we get

$$\sum_{k,\nu,t_1,t_2} \alpha_{k,\nu,t_1,t_2}(w)(2\pi i m)^w \sum_{h\in S^{k+2\nu+t_1+t_2-2w}} \gamma(k,\nu+t_2-w,\nu+t_1-w,w) \times a_h(m)\frac{L(f\otimes g\otimes h, 2k+2\nu+t_1+t_2-w-2)}{\langle f,f\rangle\langle g,g\rangle\langle h,h\rangle}.$$

The case $f = g$ is slightly more complicated, because an additional term may arise if $t_1 = t_2 = t$ and $w = \nu + t$. The additional term is

$$\delta_{t_1,t_2}\alpha_{k,\nu,t,t}(\nu+t)(2\pi i m)^{\nu+t}\frac{\kappa(k,\nu+t)}{c(k,\nu+t)}\frac{A_k\sigma_{k-1}(m)}{\zeta(k)\zeta(2k-2)}\frac{D(g,k-1)}{\langle g,g\rangle},$$

where the last factor in front of the L-function comes from the Fourier expansion of the Eisenstein series:

$$E_k = 1 + A_k \sum \sigma_{k-1}(n) e^{2\pi i n z}.$$

Summarizing this, we obtain

Main Theorem 4.1. *Let k be an even integer with $k > 4$ and let ν, t_1, t_2 be arbitrary non-negative integers such that $k+\nu+t_1$ and $k+\nu+t_2$ are even; furthermore let m be a squarefree positive integer. Let $f \in S^{k+\nu+t_1}$ and $g \in S^{k+\nu+t_2}$ be normalized Hecke eigenforms and denote by $(\Phi_i)_{i\in I}$ an orthogonal Hecke eigenbasis of $J^{cusp}_{k+\nu,m}$. Then we have the identity*

$$\begin{aligned}
&\delta_{\nu,0} \frac{\kappa(k,t_1)\kappa(k,t_2)a_f(m)a_g(m)}{A_k\sigma_{k-1}(m)\zeta(k)^2\zeta(2k-2)^2} \frac{D(f,k-1)}{\langle f,f\rangle} \frac{D(g,k-1)}{\langle g,g\rangle} \\
&\quad + \beta(k,\nu) A_k \sigma_{k-1}(m) \sum_i \frac{Z^J(k,\Phi_i)}{\|\Phi_i\|^2} I^{(t_1)}(\Phi_i,f) I^{(t_2)}(\Phi_i,g) \\
&= \delta_{t_1,t_2}\delta_{f,g}\alpha_{k,\nu,t,t}(\nu+t)(2\pi i m)^{\nu+t} \frac{\kappa(k,\nu+t)}{c(k,\nu+t)} A_k\sigma_{k-1}(m) \frac{D(g,k-1)}{\zeta(k)\zeta(2k-2)\langle g,g\rangle} \\
&\quad + \sum_{w=0}^{\nu+Min(t_1,t_2)} \alpha_{k,\nu,t_1,t_2}(w)(2\pi i m)^w \frac{\gamma(k,\nu+t_2-w,\nu+t_1-w,w)}{\zeta(k)\zeta(2k-2)} \\
&\quad \times \sum_{h\in S^{k+2\nu+t_1+t_2-2w}} a_h(m) \frac{L(f\otimes g\otimes h, 2k+2\nu+t_1+t_2-w-2)}{\langle f,f\rangle\langle g,g\rangle\langle h,h\rangle}.
\end{aligned}$$

Here h runs over a normalized Hecke eigenbasis of $S^{k+2\nu+t_1+t_2-2w}$ and we write t for t_i if $t_1 = t_2$.

This is our most general identity; its significance depends on understanding (by reinterpretation) the left hand side, in particular the periods $I^{(t)}(\Phi,g)$; one can however as well consider such periods as objects of independent interest.

Remark. To have an identity with algebraic summands, we should divide both sides of the identity above by $(2\pi i)^{2\nu+t_1+t_2}$. The reason is that we should have normalized our differential operators from the beginning in such a way that the application to Fourier series with algebraic coefficients gives again Fourier series with algebraic coefficients.

5. Comparison: final version via central values of *L*-functions

It is desirable to rewrite the Arakawa–Ichino side of our main identity in terms of elliptic modular forms (rather than Jacobi forms). In principle, this should be possible in general, definite results however are available only for $m = 1$, $t_1 = t_2 = 0$. We will briefly indicate where the problems are for the general case, but then we will concentrate on the cases where definite results are available.

5.1. How to rewrite the $Z^J(k,\Phi_i)$

Here a general solution is available, which we briefly sketch: By choosing the basis $\{\Phi_i\}$ of $J^{cusp}_{k+\nu,m}$ properly, it is possible to associate to each Φ_i an elliptic cusp form φ_i in the space $\mathcal{M}^{new,-}_{2k+2\nu-2}(m')$, which was introduced by Skoruppa and Zagier [29]; here m' is a suitable divisor of m. Then one can express $Z^J(s,\Phi_i)$ in terms of $L(\varphi_i,s)$, we refer to [19,10] for details.

Now we stick to the case $m = 1$: here the correspondence between the $\Phi_i \in J^{cusp}_{k+\nu,1}$ and the $\varphi_i \in S_{2k+2\nu-2}$ is already described in [11]. Moreover in this case [1,19]

$$Z^J(s, \Phi_i) = \frac{L(\varphi_i, s+k+\nu-3)}{\zeta(2s-2)}.$$

5.2. *How to rewrite the Ichino periods*

To give an appropriate reformulation of the Ichino periods $I^{(t)}(\Phi, f)$ in general, we would need a version of Ichino's result involving differential operators; such a result is likely to be true (see [9] for the case of Yoshida lifts); we would also need a version, which allows us to treat Jacobi forms of index $m > 1$; it is not clear what in that case the analogue of Ichino's result should look like.

The case $m = 1$, $t_1 = t_2 = 0$ was already considered in [20]; it uses Ichino's result in a crucial way:

$$I(\Phi_i, f)^2 = 2^{-k-\nu} \frac{\|\tilde{\varphi}_i\|^2}{\|\varphi_i\|^2 \|f\|^4} \hat{L}\big(\varphi_i \otimes Sym^2(f), 2k+2\nu-2\big),$$

where $\tilde{\varphi}_i$ is the modular form of half-integral weight corresponding to Φ_i and the hat indicates the completion of the L-function. We should also mention here the identity of Petersson's products [25]

$$\|\Phi_i\|^2 = 2^{2k+2\nu-3}\|\tilde{\varphi}\|^2.$$

5.3. *The main identity using L-functions*

Using the results from above, we can now reformulate our Main theorem 4.1 entirely in terms of elliptic modular forms and L-functions. We do this for $f = g$, otherwise we would need square roots of the central critical values; also we have (at the moment) to restrict ourselves to $t_1 = t_2 = 0$.

Theorem 5.1. *Let f be a normalized Hecke eigenform of weight $k+\nu$ with $k > 4$. Then*

$$\begin{aligned}
&\delta_{\nu,0} \frac{\kappa(k,0)^2}{A_k \zeta(k)^2 \zeta(2k-2)^2} \left(\frac{D(f, k-1)}{\langle f, f\rangle}\right)^2 \\
&\quad + 2^{-3k-3\nu+3} A_k \beta(k,\nu) \sum_{\varphi \in S^{2k+2\nu-2}} \frac{L(\varphi, 2k+\nu-3)\hat{L}(\varphi \otimes Sym^2(f), 2k+2\nu-2)}{\zeta(2k-2)\|\varphi\|^2\|f\|^4} \\
&= \alpha_{k,\nu,0,0}(\nu)(2\pi i)^\nu \frac{\kappa(k,\nu)}{c(k,\nu)} A_k \frac{D(f,k-1)}{\zeta(k)\zeta(2k-2)\langle f, f\rangle} \\
&\quad + \sum_{w=0}^{\nu} \alpha_{k,\nu,0,0}(w)(2\pi i)^w \frac{\gamma(k, \nu-w, \nu-w, w)}{\zeta(k)\zeta(2k-2)} \\
&\quad \times \sum_{h \in S^{k+2\nu-2w}} \frac{L(f \otimes f \otimes h, 2k+2\nu-w-2)}{\langle f, f\rangle^2 \langle h, h\rangle}.
\end{aligned}$$

Remarks.

A) As in the case $\nu = 0$ discussed in [20] our identity connects central critical values to critical values in the range of convergence. It should be possible to extract information about the central critical values from this identity.

B) It would be desirable to include the case $k = 4$ here (by Hecke summation).

C) The (excluded) case $k=2$ is also remarkable: Then for $\nu=0$ the contributions on both sides are the same (say, for a congruence subgroup $\Gamma_0(p)$). For a completely different approach to such averages for weight 2 we refer to the work of Feigon and Whitehouse [12].

6. A degenerate case

6.1. The case ν odd

There are no Jacobi forms of index one and odd weight on $\mathbb{H}\times\mathbb{C}$, see [11, Theorem 2.2]; therefore we get for all odd positive integers ν

$$\mathbb{D}^0_{k,\nu} E^2_{k,1} = 0.$$

Nevertheless we may apply our comparison procedure using differential operators $\mathbb{D}^{t_1,t_2,0}_{k,\nu}$ for ν, t_1, t_2 all odd. The left hand side of our Main theorem 4.1 then degenerates and we get

Corollary 6.1. *Let f, g, k, ν, t_1, t_2 be as in our main theorem but with the additional condition that ν, t_1, t_2 are all odd and the index m is one; then*

$$\begin{aligned}
&\delta_{t_1,t_2}\delta_{f,g}\alpha_{k,\nu,t,t}(\nu+t)(2\pi i)^{\nu+t}\frac{\kappa(k,\nu+t)}{c(k,\nu+t)}A_k\frac{D(g,k-1)}{\zeta(k)\zeta(2k-2)\langle g,g\rangle}\\
&\quad+\sum_{w=0}^{\nu+Min(t_1,t_2)}\alpha_{k,\nu,t_1,t_2}(w)(2\pi i)^w\frac{\gamma(k,\nu+t_2-w,\nu+t_1-w,w)}{\zeta(k)\zeta(2k-2)}\\
&\quad\times\sum_{h\in S^{k+2\nu+t_1+t_2-2w}}a_h(m)\frac{L(f\otimes g\otimes h,2k+2\nu+t_1+t_2-w-2)}{\langle f,f\rangle\langle g,g\rangle\langle h,h\rangle}=0.
\end{aligned}$$

6.2. An example

It may be of interest to consider simple examples of our identities, where only a few terms appear.

The simplest possible case of the corollary above is $k=10$, $\nu=t_1=t_2=1$. With $f=g=\Delta\in S^{12}$, only $w=1$ gives a contribution, again with $h=\Delta$ as the only contributing h. Then the corollary gives

$$\begin{aligned}
&\alpha_{10,1,1,1}(2)\bigl(2\pi(-1)^{1/2}\bigr)A_{10}\frac{\kappa(10,2)}{c(10,2)}\pi^{29}\frac{D(\Delta,9)}{\pi^{29}\|\Delta\|^2}\\
&\quad=(-1)\alpha_{10,1,1,1}(1)\gamma(10,1,1,1)\pi^{51}\frac{L(\Delta\otimes\Delta\otimes\Delta,21)}{\pi^{51}\|\Delta\|^6}
\end{aligned}\qquad(*)$$

where we have divided the L-values by appropriate powers of π to obtain algebraic values.

We confirm the identity $(*)$ numerically, use the explicit values:

$$\begin{aligned}
\alpha_{10,1,1,1}(2)&=-\frac{(\frac12+k)^2}{k(k+1)}=-\frac{3^2\cdot 7^2}{2^3\cdot 5\cdot 11},\\
\alpha_{10,1,1,1}(1)&=-\frac{(\frac12+k)^2(k-1)}{k^2(k+2)}=-\frac{3^3\cdot 7^2}{2^6\cdot 5^2},\\
A_{10}&=-2^3\cdot 3\cdot 11,\\
\kappa(10,2)&=\frac{3\cdot 5\cdot 7\cdot 19}{2^{10}}\cdot\pi,
\end{aligned}$$

$$
\begin{aligned}
c(10,2) &= \frac{7}{3}, \\
\frac{D(\Delta,9)}{\pi^{29}\|\Delta\|^2} &= \frac{2\cdot 4^{20}}{245\cdot 20!} = \frac{2^{23}}{3^8\cdot 5^5\cdot 7^4\cdot 11\cdot 13\cdot 17\cdot 19}, \\
\gamma(10,1,1,1) &= (-1)^{(3-k)/2}\pi^{-2k}\frac{(k!)^3(2k)!}{2^{5k+4}(k-1)!(2k-3)!} \\
&= (-1)^{-7/2}\pi^{-20}\frac{3^{10}\cdot 5^6\cdot 7^2\cdot 19}{2^{34}}, \\
\frac{L(\Delta\otimes\Delta\otimes\Delta,21)}{\pi^{51}\|\Delta\|^6} &= \frac{2^{54}}{3^{16}\cdot 5^9\cdot 7^6\cdot 11\cdot 13\cdot 17\cdot 19}.
\end{aligned}
$$

Here the values of $D(\Delta,9)$ and $L(\Delta\otimes\Delta\otimes\Delta,21)$ are taken from [30] and [28] respectively.

With these values we can compute both sides of $(*)$ independently and obtain on both sides the same value

$$i\cdot\pi^{31}\times\frac{2^{14}}{3^3\cdot 5^5\cdot 7^2\cdot 11\cdot 13\cdot 17}.$$

More examples will be considered in subsequent work.

Acknowledgments

During the preparation of this paper, the authors were supported by several institutions: The first author thanks the Tata Institute for Fundamental Research (Mumbai), in particular Dipendra Prasad for kind hospitality and excellent working conditions during his visit in October/November 2009, where a first draft of this paper was started; he also thanks A. Panchichkin and the Institute Fourier (Grenoble) for two fruitful periods of stay in 2010, supported by CNRS. The second author thanks the Max-Planck-Institut für Mathematik in Bonn for an invitation in July/August 2010, where the final version was completed. During Mai 2010 both authors stayed at the GUtech (German University of Technology in Oman) in Muscat. They are very thankful for the hospitality and the supportive working conditions.

References

[1] T. Arakawa, Jacobi Eisenstein series and a basis problem for Jacobi forms, Comment. Math. Univ. St. Pauli 43 (1994) 181–216.

[2] S. Böcherer, Über die Fourier-Jacobi-Entwicklung Siegelscher Eisensteinreihen, Math. Z. 183 (1983) 21–46.

[3] S. Böcherer, Über die Fourier-Jacobi-Entwicklung Siegelscher Eisensteinreihen II, Math. Z. 189 (1985) 81–110.

[4] S. Böcherer, Siegel Modular Forms and Theta Series, Proc. Sympos. Pure Math., vol. 49, Amer. Math. Soc., 1989.

[5] S. Böcherer, Jacobification of some holomorphic differential operators of the symplectic group $Sp(2n,\mathbb{R})$ and applications, in preparation.

[6] S. Böcherer, A. Panchichkin, p-Adic interpolation of triple product L-functions: analytic aspects, in: Automorphic Forms and L-functions II, in: Contemp. Math., vol. 489, 2009, pp. 1–39.

[7] S. Böcherer, R. Schulze-Pillot, The central critical value of the triple product L-function, in: Number Theory 1993–1994, Cambridge University Press, 1996, pp. 1–46, a revised version is available from the web page of the second author: http://www.math.uni-sb.de/ag/schulze.

[8] S. Böcherer, T. Satoh, T. Yamazaki, On the pullback of a differential operator and its application to vector-valued Eisenstein series, Comment. Math. Univ. St. Pauli 41 (1992) 1–22.

[9] S. Böcherer, M. Furusawa, R. Schulze-Pillot, On the global Gross–Prasad conjecture for Yoshida liftings, in: Contributions to Automorphic Forms, Geometry, and Number Theory, Johns Hopkins University Press, 2004, pp. 105–130.

[10] K. Bringmann, B. Heim, Ramanujan type congruences and Jacobi forms, Int. Math. Res. Not. IMRN 2007 (2007).

[11] M. Eichler, D.B. Zagier, The Theory of Jacobi Forms, Birkhäuser, 1985.

[12] B. Feigon, D. Whitehouse, Exact averages of central values of triple product L-functions, arXiv:1005.0646, 2010.

[13] E. Freitag, Siegelsche Modulfunktionen, Springer, 1983.

[14] P.B. Garrett, Pullbacks of Eisenstein series; applications, in: Automorphic Forms in Several Complex Variables, Katata, 1983, Birkhäuser, 1984.

[15] P.B. Garrett, Decomposition of Eisenstein series: Triple product L-functions, Ann. of Math. 125 (1987) 209–235.
[16] P.B. Garrett, M. Harris, Special values of triple product L-functions, Amer. J. Math. 115 (1993) 159–238.
[17] R. Goodman, N. Wallach, Representations and Invariants of the Classical Groups, Cambridge University Press, 1998.
[18] B. Heim, Über Poincare Reihen und Restriktionsabbildungen, Abh. Math. Semin. Univ. Hambg. 68 (1998) 79–89.
[19] B. Heim, L-functions for Jacobi forms and the basis problem, Manuscripta Math. 106 (2001) 489–503.
[20] B. Heim, A trace formula of special values of automorphic L-functions, MPIM-Bonn, preprint, 2009.
[21] T. Ibukiyama, On differential operators on automorphic forms and invariant pluri-harmonic polynomials, Comment. Math. Univ. St. Pauli 48 (1999) 103–118.
[22] T. Ibukiyama, D. Zagier, Higher spherical polynomials, in preparation.
[23] I. Ichino, Pullbacks of Saito–Kurokawa lifts, Invent. Math. 162 (2005) 551–647.
[24] H. Klingen, Introductory Lectures on Siegel Modular Forms, Cambridge University Press, 1990.
[25] W. Kohnen, N. Skoruppa, A certain Dirichlet series attached to Siegel modular forms of degree two, Invent. Math. 95 (1989) 449–476.
[26] H. Maaß, Siegel Modular Forms and Dirichlet Series, Lecture Notes in Math., vol. 216, Springer, 1971.
[27] T. Miyake, Modular Forms, Springer, 1989.
[28] S. Mizumoto, Special values of triple product L-functions and nearly holomorphic Eisenstein series, Abh. Math. Semin. Univ. Hambg. 70 (2000) 191–210.
[29] N. Skoruppa, D. Zagier, Jacobi forms and a certain space of modular forms, Invent. Math. 94 (1988) 113–146.
[30] D. Zagier, Modular Forms Whose Coefficients Involve Zeta-Functions of Quadratic Fields, Lecture Notes in Math., vol. 627, Springer, 1977.