Orbital stability of solitary waves for Kundu equation

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A B S T R A C T

In this paper, we consider the Kundu equation which is not a standard Hamiltonian system. The abstract orbital stability theory proposed by Grillakis et al. (1987, 1990) cannot be applied directly to study orbital stability of solitary waves for this equation. Motivated by the idea of Guo and Wu (1995), we construct three invariants of motion and use detailed spectral analysis to obtain orbital stability of solitary waves for Kundu equation. Since Kundu equation is more complex than the derivative Schrödinger equation, we utilize some techniques to overcome some difficulties in this paper. It should be pointed out that the results obtained in this paper are more general than those obtained by Guo and Wu (1995). We present a sufficient condition under which solitary waves are orbitally stable for $2c_3^2 + s_2 < 0$, while Guo and Wu (1995) only considered the case $2c_3^2 + s_2 > 0$. We obtain the results on orbital stability of solitary waves for the derivative Schrödinger equation given by Colin and Ohta (2006) as a corollary in this paper. Furthermore, we obtain orbital stability of solitary waves for Chen–Lee–Lin equation and Gerdjikov–Ivanov equation, respectively.

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1. Introduction

In this paper, we consider orbital stability of solitary waves for Kundu equation [1] with fifth order nonlinear term

$$iu_t + u_{xx} + c_3|u|^2u + c_5|u|^4u - is_2(|u|^2u)_x - ir(|u|^2)_xu = 0, \quad x \in \mathbb{R},$$

(1a)
where \( c_3, c_5, s_2 \) and \( r \) are real constants. Eq. (1a) was derived by Kundu [1] in the study of integrability and it is an important special case of the generalized complex Ginzburg–Landau equation [2]. Meanwhile, Eq. (1a) and its special cases arise in various physical and mechanical applications, such as plasma physics, nonlinear fluid mechanics, nonlinear optics and quantum physics (see [2–7]). For convenience, we denote the Kundu equation as follows

\[
u_t + i c_3 |\nu|^2 \nu + c_5 |\nu|^4 \nu + \alpha |\nu|^2 \nu_x + \beta \nu_x = 0, \quad x \in \mathbb{R},
\]

where \( \alpha = (2s_2 + r), \beta = (s_2 + r) \).

Obviously, if \( r = 0 \), then Eq. (1) reduces to the derivative Schrödinger equation

\[
u_t + i c_3 |\nu|^2 \nu + s_2 (|\nu|^2 \nu)_x = 0;
\]

if \( c_3 = 0, c_5 = 0, s_2 = -\delta, r = -s_2 \), then Eq. (1) reduces to Chen–Lee-Lin equation [8]

\[
u_t + i \delta |\nu|^2 \nu_x = 0;
\]

if \( c_5 = 2\delta^2, s_2 = 2\delta, r = -2s_2 \), then Eq. (1) reduces to Gerdjikov–Ivanov equation [9]

\[
u_t + u_{xx} + c_3 |\nu|^2 u + 2\delta^2 |\nu|^4 u + 2i \delta u^2 \nu_x = 0.
\]

In [2], by using primary integral method, Saarloos and Hohenberg obtained the exact solitary waves of the form

\[
u(x, t) = e^{-i \omega t} e^{i \psi(x-vt)} a(x-vt) = e^{-i \omega t} e^{i \psi(\xi)} a(\xi), \quad \xi = x-vt,
\]

for Eq. (2). Furthermore, Guo and Wu proved that this kind of solitary waves for Eq. (2) are orbitally stable in [10]. For more results on Eq. (2), we refer the readers to [12–15].

The existence of solutions for initial value problem of Eq. (1) has been proved in [16]. However, to our best knowledge, orbital stability of solitary waves in the form (5) for Eq. (1) have not been considered. We will focus on studying this problem in this paper. For clarity, a solution in the form (5) is called a solitary wave throughout this paper. As mentioned above, Eqs. (2)–(4) are special cases of Eq. (1). In other words, Eq. (1) is more general than Eq. (2), moreover, Eq. (2) does not include Eqs. (3)–(4).

It is worth pointing out that it is difficult for us to transform Eq. (1) into a standard Hamiltonian system. Therefore, the results about orbital stability obtained in [17] and [18] cannot be applied directly to prove orbital stability of solitary waves for Eq. (1). Motivated by the approach in [10], we study this problem by constructing three appropriate invariants of motion and using detailed spectral analysis. Due to Eq. (1) being more complex than Eq. (2), we overcome some difficulties by using some techniques. The results obtained in this paper are more general than those in [10]; (a) Using Theorem 3 in this paper, we obtain not only the results shown in [10], but also the results which have not been obtained in [10]. The conclusions in [10] were given in the case of \( 2c_3 + s_2 \nu > 0 \). When \( s_2 < 0, c_3 > 0 \), [10] obtained the orbital stability of solitary waves traveling left. In this paper, we give not only the whole results obtained in [10], but also a sufficient condition under which solitary waves are orbitally stable as \( 2c_3 + s_2 \nu < 0 \). Corollary 1 obtained in this paper can be used to discriminate orbital stability for both solitary waves traveling left and solitary waves traveling right of Eq. (2). (b) Corollary 2 obtained in this paper enables us to derive the same results on orbital stability of solitary waves for the following derivative Schrödinger equation

\[
u_t + u_{xx} + i (|\nu|^2 \nu)_x = 0
\]

as those presented by Colin and Ohta [11]. By using variational method, they proved that solitary waves of Eq. (6) are orbitally stable if the wave speed \( \nu \) satisfies \( \nu^2 < -4\omega \). Since Eq. (2) becomes
Eq. (6) when $s_2 = -1, c_3 = c_5 = 0$, from the results obtained in [10], it is easy to see that solitary waves of Eq. (6) are orbitally stable as $\nu < 0$ and $\nu^2 < -4\omega$. However, it is unable to derive the conclusion that solitary waves of Eq. (6) are orbitally stable as $\nu > 0$ and $\nu^2 < -4\omega$. By Theorem 3 obtained in this paper, solitary waves of Eq. (6) are orbitally stable as $\nu^2 < -4\omega$ and $\nu \neq 0$. (c) By means of the results obtained in this paper, we have Corollary 3 on orbital stability of solitary waves for Chen–Lee–Lin equation, and Corollary 4 on orbital stability of solitary waves for Gerdjikov–Ivanov equation. Eqs. (3)–(4) are special cases of Eq. (1), but Eq. (2) does not include them.

2. Exact solitary waves of Kundu equation

Due to the complexity of transforming Eq. (1) to primary integral, Saarloos and Hohenberg [2] obtained the exact solitary waves in the form (5) for Eq. (2), but they did not give any exact solution to Eq. (1). In this section, we give exact solitary waves of Eq. (1) by using proper transformations and undetermined coefficient method.

Assume that Eq. (1) has solutions of the form (5). Let

$$\hat{a}(\xi) = \hat{a}(x - \nu t) = e^{i\psi(x - \nu t)}a(x - \nu t),$$

and substitute $u(x, t) = e^{-i\omega t}\hat{a}(x - \nu t)$ into Eq. (1), then $\hat{a}(\xi)$ satisfies

$$-\hat{a}_{xx} - g(a^2)\hat{a} + i\beta(a^2\hat{a})_x + i(\alpha - 2\beta)a^2\hat{a}_x - \omega\hat{a} + i\nu\hat{a}_x = 0,$$

where $g(a^2) = c_3a^2 + c_5a^4$. Substituting (7) into Eq. (8) and making the real part and imaginary part of Eq. (8) equal to zero, we have

$$\psi''a + 2\psi'a' - \alpha a^2a' - \beta a^2a' - \nu a' = 0,$$

$$a'' + [g(a^2) + \alpha \psi^2 a^2 - \beta \psi^2 a^2 - (\psi')^2 + \omega + \nu \psi]a = 0.$$

Let

$$\psi'(\xi) = E + Da^2(\xi).$$

Substituting (11) into Eq. (9) and equating the coefficients of these terms $a, a', a''$ to zero, we get $E = \frac{\nu}{2}, D = \frac{(\alpha + \beta)}{4}$. Therefore, when

$$\psi'(\xi) = \frac{\nu}{2} + \frac{(\alpha + \beta)}{4} a^2(\xi).$$

Eq. (9) is identical to zero. Combining (12) and (10), we know that $a(\xi)$ satisfies the following equation

$$a'' - d_1a - 2d_2a^3 - 3d_4a^5 = 0,$$

where $d_1 = -\omega - \frac{\nu^2}{4}, d_2 = -\frac{c_3}{2} - \frac{(\alpha - \beta)\nu}{4}, d_4 = -\frac{1}{4}(c_3 + \frac{\alpha + \beta)(3\alpha - 5\beta)}{16}$.

In order to solve Eq. (13), we make the following transformation

$$a(\xi) = \sqrt{\varphi(\xi)}.$$

Then, $\varphi(\xi)$ satisfies

$$2\varphi\varphi'' - \varphi'^2 - 4d_1\varphi^2 - 8d_2\varphi^3 - 12d_4\varphi^4 = 0.$$
Now, we assume that Eq. (15) has solutions in the following form

\[ \varphi(\xi) = \frac{A e^{C(\xi + \xi_0)}}{(1 + e^{C(\xi + \xi_0)})^2 + B e^{C(\xi + \xi_0)}} = \frac{A \sech^2 \frac{C}{2} (\xi + \xi_0)}{4 + B \sech^2 \frac{C}{2} (\xi + \xi_0)}, \]

(16)

where \( A, B, C \) are undetermined constants, and \( \xi_0 \) is an arbitrary constant.

By substituting (16) into Eq. (15), we have

\[
\begin{align*}
&\begin{cases}
C^2 - 4d_1 = 0, \\
-4d_2A - 4d_1(2 + B) - C^2(2 + B) = 0, \\
-5C - 2d_1(2 + (2 + B)^2) - 4d_2A(2 + B) - 6d_4A^2 = 0.
\end{cases}
\end{align*}
\]

(17)

Further, we obtain

\[
A = \pm \frac{4d_1}{\sqrt{d_2^2 - 4d_1d_4}}, \quad B = -2 \pm \frac{-2d_2}{\sqrt{d_2^2 - 4d_1d_4}}, \quad C = \pm \sqrt{4d_1}, \quad d_1 > 0.
\]

(18)

From (18) and (16), we get two solutions for Eq. (15) as follows

\[
\begin{align*}
\varphi_1(\xi) &= \frac{2d_1}{\sqrt{d_2^2 - 4d_1d_4}} \sech^2 \sqrt{d_1}(\xi + \xi_0) \\
&\quad \times \frac{\sqrt{d_1}(\xi + \xi_0)}{2 - (1 + \frac{d_2}{\sqrt{d_2^2 - 4d_1d_4}}) \sech^2 \sqrt{d_1}(\xi + \xi_0)},
\end{align*}
\]

(19)

\[
\varphi_2(\xi) = \frac{-2d_1}{\sqrt{d_2^2 - 4d_1d_4}} \sech^2 \sqrt{d_1}(\xi + \xi_0) \times \frac{\sqrt{d_1}(\xi + \xi_0)}{2 + (1 - \frac{d_2}{\sqrt{d_2^2 - 4d_1d_4}}) \sech^2 \sqrt{d_1}(\xi + \xi_0)}.
\]

(20)

Under the condition \( d_1 > 0 \), it is easy to verify that

1. if \( d_4 < 0 \) or if \( d_4 \geq 0, d_2 < 0 \) and \( d_2^2 - 4d_1d_4 > 0 \), then for any \( \xi \in \mathbb{R} \), \( \varphi_1(\xi) > 0 \);
2. if \( d_4 \geq 0, d_2 > 0 \) and \( d_2^2 - 4d_1d_4 > 0 \), then \( \varphi_1(\xi) \) is an unbounded function;
3. if \( d_4 < 0 \) or if \( d_4 \geq 0, d_2 > 0 \) and \( d_2^2 - 4d_1d_4 > 0 \), then for any \( \xi \in \mathbb{R} \), \( \varphi_2(\xi) < 0 \);
4. if \( d_4 \geq 0, d_2 < 0 \) and \( d_2^2 - 4d_1d_4 > 0 \), then \( \varphi_2(\xi) \) is an unbounded function.

Note that if we let

\[
d_3 = -\frac{d_2}{2d_1}, \quad d_5^2 = \frac{d_2^2 - 4d_1d_4}{4d_1^2}, \quad d_6^2 = 4d_1,
\]

(21) then (19) can be rewritten as

\[
\varphi_1(\xi) = \frac{1}{d_3 + d_5 \cosh d_6(\xi + \xi_0)}, \quad \xi \in \mathbb{R}.
\]

(22)

Substituting (22) into (14), and noticing that if \( a(\xi) \) is a solution of Eq. (13), then \(-a(\xi)\) is also a solution of Eq. (13), we have the following lemma.
Lemma 1. Suppose that $d_1 > 0$. If $d_4 < 0$ or if $d_4 \geq 0$, $d_2 < 0$ and $d_2^2 - 4d_1d_4 > 0$, then Eq. (13) has bounded analytic solutions of the form

$$a(\xi) = \pm \left[ \frac{1}{d_3 + d_5 \cosh d_6(\xi + \xi_0)} \right]^{\frac{1}{2}}. \quad (23)$$

Furthermore, we get the following theorem.

Theorem 1. Suppose that $d_1 > 0$. If $d_4 < 0$ or if $d_4 \geq 0$, $d_2 < 0$ and $d_2^2 - 4d_1d_4 > 0$, then Eq. (1) has solitary waves $u(x, t) = e^{-i\omega t}e^{i\psi(x - vt)}a(x - vt)$, where $a(\xi)$ and $\psi(\xi)$ are given by (23) and (12), respectively.

Theorem 1 implies that when $d_1 > 0$, $d_4 < 0$, Eq. (1) has solitary waves of the form (5) no matter whether $d_2 < 0$ or $d_2 \geq 0$. It is also true for Eq. (2) in this case. Theorem 1 obtained in [10] does not include the result that Eq. (2) has solitary waves of the form (5) in the case of $d_2 \geq 0$.

3. Orbital stability of solitary waves

In this section, we only consider the positive solutions in (23) (the negative solutions in (23) can be discussed similarly). Therefore, we assume that $a(\xi) > 0$ in the remainder of this paper.

Now, we consider the following initial value problem of Kundu equation

$$u_t = iu_{xx} + i(c_3|u|^2 + c_5|u|^4)u + \alpha|u|^2u_x + \beta u^2 \bar{u}_x, \quad (24)$$

$$u(0, x) = u_0(x), \quad x \in \mathbb{R}. \quad (25)$$

The complex space $X = H^1(\mathbb{R})$ with real inner product

$$(u, v) = \text{Re} \int_{\mathbb{R}} (u_x \bar{v}_x + u \bar{v}) \, dx, \quad \forall u, v \in X, \quad (26)$$

is chosen as the function space. The dual space of $X$ is denoted by $X^* = H^{-1}(\mathbb{R})$ and the natural isomorphism $I : X \rightarrow X^*$ is defined by

$$\langle Iu, v \rangle = (u, v), \quad (27)$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between $X$ and $X^*$,

$$\langle f, u \rangle = \text{Re} \int_{\mathbb{R}} f \bar{u} \, dx. \quad (28)$$

From (26)–(28), we have

$$I = -\frac{\partial^2}{\partial x^2} + 1. \quad (29)$$

Let $T_1, T_2$ be one-parameter groups of unitary operator on $X$ defined by

$$T_1(s_1)\phi(\cdot) = e^{-s_1 i} \phi(\cdot), \quad \forall \phi(\cdot) \in X, \ s_1 \in \mathbb{R}, \quad (30)$$

$$T_2(s_2)\phi(\cdot) = \phi(\cdot - s_2), \quad \forall \phi(\cdot) \in X, \ s_2 \in \mathbb{R}. \quad (31)$$

Obviously, $T_1'(0) = -i$, $T_2'(0) = -\frac{\partial}{\partial x}$. 

According to [16] (or [19–21]), we know that for any \( u_0 \in H^1(R) \), Eq. (24) has a unique solution \( u \in C([0, T_{\text{max}}], H^1(R)) \) satisfying \( u(0) = u_0 \).

From the above definitions, we write the solitary waves of Eq. (24) in Theorem 1 as \( T_1(\omega t)T_2(\nu t)\hat{a}_{\omega,\nu}(x) \), where \( \hat{a}_{\omega,\nu} \) is defined by (7) and (23). In what follows, we mainly study the orbital stability of solitary waves \( T_1(\omega t)T_2(\nu t)\hat{a}_{\omega,\nu}(x) \). Note that Eq. (24) has the symmetries of phase and translation, we define the orbital stability as follows (see also [22]).

**Definition.** The solitary wave \( T_1(\omega t)T_2(\nu t)\hat{a}_{\omega,\nu}(x) \) is called orbitally stable if for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) with the following property. If \( \| u_0 - \hat{a}_{\omega,\nu}\|_X < \delta \) and \( u(t) \) is a solution of Eq. (24) in the interval \([0, t_0]\) with \( u(0) = u_0 \), then \( u(t) \) can be continued to a solution in \( 0 \leq t < \infty \) and

\[
\sup_{0 \leq t < \infty} \inf_{s_1 \in R} \inf_{s_2 \in R} \| u(t) - T_1(s_1)T_2(s_2)\hat{a}_{\omega,\nu}\|_X < \varepsilon.
\]

Otherwise, \( T_1(\omega t)T_2(\nu t)\hat{a}_{\omega,\nu}(x) \) is called orbitally unstable.

It is difficult for us to transform Eq. (24) into a standard form of Hamiltonian system when \( \alpha \) and \( \beta \) are equal to zero asynchronously. (In [10], Eq. (2) cannot be changed into a standard form of Hamiltonian system as well.) Therefore, the abstract theory on orbital stability of solitary waves of nonlinear Hamiltonian system in [17] and [18] cannot be applied directly to study Eq. (24). According to the “stability theorem” in the introduction of [18], the assumptions 1–3 which make that theorem hold, and the deductions in Sections 3–4 of [18], we know that when a local solution of initial value problem of an equation exists, we only need to find \( E(u), Q_\sigma(u) \) such that assumptions 2–3 hold. The results on orbital stability can be obtained even the equation cannot been changed into a standard form of Hamiltonian system. According to the above analysis, we construct three new invariants of motion as follows

\[
E(u) = \frac{1}{2} \int_R \left( |u_x|^2 + \frac{\beta(\alpha + \beta)}{6} |u|^6 - G(|u|^2) + \frac{\alpha + \beta}{2} \text{Im}(|u|^2u\bar{u}_x) \right) dx,
\]

(32)

where \( G(u) = \int_0^u g(s) \, ds \), \( g(s) = c_3s^2 + c_5s^4 \),

\[
Q_1(u) = \frac{1}{2} \int_R |u|^2 \, dx,
\]

(33)

\[
Q_2(u) = \int_R \left[ \frac{1}{2} \text{Im}(u\bar{u}_x) - \frac{\beta}{4} |u|^4 \right] \, dx.
\]

(34)

It is easy to check that \( E(u), Q_1(u) \) and \( Q_2(u) \) are \( C^2 \) functionals defined on the complex space \( X \). Let their derivatives be \( (E'(u), \nu), (Q_1'(u), \nu) \) and \( (Q_2'(u), \nu) \), respectively, where \( E', Q_1', Q_2' : X \to X^* \), and their second derivatives be \( (E''(u)\omega, \nu), (Q_1''(u)\omega, \nu) \) and \( (Q_2''(u)\omega, \nu) \), respectively.

By computation, we have

\[
E'(u) = -u_{xx} + \frac{\beta(\alpha + \beta)}{2} |u|^4 u - g(|u|^2)u + i(\alpha + \beta)|u|^2u_x,
\]

\[
Q_1'(u) = u,
\]

\[
Q_2'(u) = -iu_x - \beta|u|^2u.
\]
We can prove that $E(u)$, $Q_1(u)$ and $Q_2(u)$ are invariant under $T_1$ and $T_2$, which means for any $s_1, s_2 \in \mathbb{R}$, we have

\[
\begin{align*}
E(T_1(s_1)T_2(s_2)u) &= E(u), \\
Q_1(T_1(s_1)T_2(s_2)u) &= Q_1(u), \\
Q_2(T_1(s_1)T_2(s_2)u) &= Q_2(u),
\end{align*}
\]  

(35)

and for any $t \in \mathbb{R}$, $u(t)$ is a flow of (24),

\[
\begin{align*}
E(u(t)) &= E(u(0)), \\
Q_1(u(t)) &= Q_1(u(0)), \\
Q_2(u(t)) &= Q_2(u(0)).
\end{align*}
\]  

(36)

(35) can be obtained by direct substitution. We will verify that (36) holds next. First, we prove

\[
\text{Re} \int_{\mathbb{R}} |u|^2 u \bar{u}_x \, dx = \text{Re} \int_{\mathbb{R}} |u|^4 u \bar{u}_x \, dx = \text{Re} \int_{\mathbb{R}} |u|^6 u \bar{u}_x \, dx = 0.
\]

Let $u = u_1 + iu_2$, where $u_1, u_2$ are real functions and $u_1, u_2 \in H^1(\mathbb{R})$, then

\[
\text{Re} \int_{\mathbb{R}} |u|^2 u \bar{u}_x \, dx = \int_{\mathbb{R}} (u_1^2 + u_2^2)(u_1 u_{1x} + u_2 u_{2x}) \, dx
\]

\[
= \int_{\mathbb{R}} (u_1^2 u_2 u_{2x} + u_1 u_2^2 u_{1x}) \, dx
\]

\[
= \int_{\mathbb{R}} u_1^2 u_2 u_{2x} \, dx - \int_{\mathbb{R}} u_1^2 u_2 u_{2x} \, dx = 0,
\]

and

\[
\text{Re} \int_{\mathbb{R}} |u|^4 u \bar{u}_x \, dx = \int_{\mathbb{R}} (u_1^4 + 2u_1^2 u_2^2 + u_2^4)(u_1 u_{1x} + u_2 u_{2x}) \, dx
\]

\[
= \int_{\mathbb{R}} (u_1^4 u_2 u_{2x} + 2u_1^2 u_2^2 u_{1x} + 2u_1^2 u_2^2 u_{2x} + u_1 u_2^4 u_{1x}) \, dx.
\]

Due to

\[
\int_{\mathbb{R}} (2u_1^2 u_2^2 u_{1x} + 2u_1^2 u_2^2 u_{2x}) \, dx = \frac{1}{2} \int_{\mathbb{R}} u_2^2 \, du_1^4 + \frac{1}{2} \int_{\mathbb{R}} u_1^2 \, du_2^4
\]

\[
= - \int_{\mathbb{R}} (u_1^4 u_2 u_{2x} + u_1 u_2^4 u_{1x}) \, dx,
\]

we have \(\text{Re} \int_{\mathbb{R}} |u|^4 u \bar{u}_x \, dx = 0\) and
Moreover, further we obtain

\[ I = \int_R (u_1^6 u_2 u_{2x} + 3u_1^5 u_2^2 u_{1x} + 3u_1^4 u_2^3 u_{2x} + u_1^3 u_2^4 u_{1x}) \, dx \]

and

\[ II = \int_R (3u_1^4 u_2^3 u_{2x} + 3u_1^3 u_2^4 u_{1x}) \, dx \]

Let \( I = \int_R (u_1^6 u_2 u_{2x} + 3u_1^5 u_2^2 u_{1x} + 3u_1^4 u_2^3 u_{2x} + u_1^3 u_2^4 u_{1x}) \, dx \), and \( II = \int_R (3u_1^4 u_2^3 u_{2x} + 3u_1^3 u_2^4 u_{1x}) \, dx \), then

\[ I = -1. \]

Further, we obtain \( I = 0 \) and \( \text{Re} \int_R |u|^6 u_x u_x \, dx = I + II = 0 \).

Now, we prove that \( E(u) \) given by (32) is an invariant of motion. By computations, we have

\[
\frac{dE(u)}{dt} = \left\langle E'(u), u_t \right\rangle
\]

\[
= \left\langle \left( -u_{xx} + \frac{\beta(\alpha + \beta)}{2} |u|^4 u - g(|u|^2) u + i(\alpha + \beta)|u|^2 u_x \right), \right. \\
\left. \left( iu_{xx} + ig(|u|^2) u + \alpha|u|^2 u_x + \beta u^2 u_x \right) \right\rangle.
\]

Moreover,

\[
\left\langle -u_{xx}, iu_{xx} \right\rangle = \text{Re} \int_R (-u_{xx} \cdot \bar{u}_{xx}) \, dx = \text{Re} \int_R i(u_{xx} \bar{u}_{xx}) \, dx = 0.
\]
\[ \langle -u_{xx}, ig(|u|^2)u \rangle + \langle -g(|u|^2)u, iu_{xx} \rangle = \text{Re} \int_R (-u_{xx} \bar{i} \cdot g(|u|^2) \bar{u}) \, dx + \text{Re} \int_R (-g(|u|^2)u \bar{i} \cdot \bar{u}_{xx}) \, dx \]
\[ = \text{Re} \int_R i(u_{xx} g(|u|^2) \bar{u}) \, dx - \text{Re} \int_R i(g(|u|^2)u \bar{i} \cdot u_{xx}) \, dx \]
\[ = 0, \]
\[ \langle -u_{xx}, \alpha |u|^2 u_x \rangle + \langle -u_{xx}, \beta u^2 \bar{u}_x \rangle + \langle i(\alpha + \beta) |u|^2 u_x, iu_{xx} \rangle = -\text{III} = 0, \]
\[ \text{III} = \text{Re} \int_R \beta (u_1 + iu_2)(u_{1xx} - iu_{2xx})(u_1 + iu_2)(u_{1x} - iu_{2x}) - (u_1 - iu_2)(u_{1x} + iu_{2x}) \, dx \]
\[ = -2\beta \int_R (u_2^2 u_{1x} u_{1xx} - u_1 u_2 u_{1xx} u_{2x} - u_1 u_2 u_{1x} u_{2xx} + u_1^2 u_{2x} u_{2xx}) \, dx \]
\[ = -2\beta \left( \int_R u_2^2 u_{1x} u_{1xx} \, dx - \int_R u_1 u_2 u_{2x} u_{1xx} \, dx - \int_R u_1 u_2 u_{1x} u_{2xx} \, dx + \int_R u_1^2 u_{2x} u_{2xx} \, dx \right) \]
\[ = -2\beta \left( -\int_R u_2^2 u_{1x} u_{1xx} \, dx + \int_R u_1 u_2 u_{1x} u_{2xx} \, dx + \int_R u_1 u_2 u_{1xx} u_{2x} \, dx - \int_R u_1^2 u_{2x} u_{2xx} \, dx \right) \]
\[ = \text{III} = 0. \]

Therefore, \text{III} = 0. Furthermore, we obtain
\[ \langle -u_{xx}, \alpha |u|^2 u_x \rangle + \langle -u_{xx}, \beta u^2 \bar{u}_x \rangle + \langle i(\alpha + \beta) |u|^2 u_x, iu_{xx} \rangle = -\text{III} = 0, \]
\[ \frac{\beta(\alpha + \beta)}{2} |u|^4 u, iu_{xx} \rangle + \langle i(\alpha + \beta) |u|^2 u_x, \beta u^2 \bar{u}_x \rangle \]
\[ = -\frac{\beta(\alpha + \beta)}{2} \text{Re} \int_R i(u^2 \bar{u}) \, d\bar{u}_x + \beta(\alpha + \beta) \text{Re} \int_R i|u|^2 u_x^2 \bar{u}^2 \, dx \]
It follows from the above computations that

\[
\frac{dE(u)}{dt} = \langle E'(u), u \rangle = 0, \quad \text{i.e., for any } t \in R, \ E(u(t)) = E(u(0)). \text{ Similarly, we can verify that } Q_1(u(t)) = Q_1(u(0)) \text{ and } Q_2(u(t)) = Q_2(u(0)) \text{ for any } t \in R.
\]

We can also prove that \( \hat{a} \) given by (7) and (23) satisfies

\[
E'(\hat{a}(x)) - \omega Q_1'(\hat{a}(x)) - \nu Q_2'(\hat{a}(x)) = 0.
\]  

(37)
In fact,

\[ E'(\hat{a}(x)) - \omega Q'_1(\hat{a}(x)) - \nu Q'_2(\hat{a}(x)) \]

\[ = -\hat{a}_{xx} + \frac{\beta(\alpha + \beta)}{2} |u|^4 \hat{a} - g(|a|^2) \hat{a} + i(\alpha + \beta)A \hat{a} - \omega \hat{a} + i\nu \hat{a}_x + \beta \nu |a|^2 \hat{a} \]

\[ \equiv \frac{\beta(\alpha + \beta)}{2} a^4 \hat{a} + \beta \nu a^2 \hat{a} - 2\beta ia_x \hat{a} + 2\beta i a^2 \hat{a}_x \]

\[ \equiv e^{i\psi} \left[ \frac{\beta(\alpha + \beta)}{2} a^5 + \beta \nu a^3 - 2\beta i a^2 \hat{a}_x + 2\beta i a^2 (i\psi'a + a') \right] = 0. \]

For any \( u, \phi \in X \), by computation, we have

\[ E''(u)\phi = -\phi_{xx} + \frac{\beta(\alpha + \beta)}{2} (3|u|^4 \phi + 2|u|^2 u^2 \hat{\phi}) - g(|u|^2) \phi \]

\[ - g'(|u|^2) (\bar{u}\phi + u\bar{\phi})u + i(\alpha + \beta) (|u|^2 \phi_x + \phi \bar{u}_x + \bar{\phi} uu_x), \]

\[ Q_1''(u)\phi = \phi, \]

\[ Q_2''(u)\phi = -i \phi_x - 2\beta |u|^2 \phi - \beta u^2 \bar{\phi}. \]

Defining an operator from \( X \) to \( X^* \) by

\[ H_{0,\nu} = E''(\hat{a}) - \omega Q_1''(\hat{a}) - \nu Q_2''(\hat{a}), \]  

\[ \tag{38} \]

then for any \( \phi \in X \), we have

\[ H_{0,\nu} \phi = -\phi_{xx} + \frac{\beta(\alpha + \beta)}{2} (3|a|^4 \phi + 2|a|^2 a^2 \hat{\phi}) - g(|a|^2) \phi \]

\[ - g'(|a|^2) (\hat{a}\phi + \hat{a}\phi)\hat{a} + i(\alpha + \beta) (|a|^2 \phi_x + \phi \hat{a}_x + \bar{\phi} \hat{a}_x) \]

\[ - \omega \phi + \nu \phi_x + 2\nu \beta |a|^2 \phi + \beta v a^2 \bar{\phi}. \]  

\[ \tag{39} \]

It is easy to see that \( H_{0,\nu} \) is self-adjoint, which shows that \( I^{-1}H_{0,\nu} \) is a bounded self-conjugate operator on \( X \). The spectrum of \( H_{0,\nu} \) consists of the real numbers \( \lambda \) such that \( H_{0,\nu} - \lambda I \) is not invertible. \( \lambda = 0 \) belongs to the spectrum of \( H_{0,\nu} \).

From \( T'_1(0) = -i \) and \( T'_2(0) = -\frac{\partial}{\partial x} \), we obtain

\[ T'_1(0)\hat{a}(x) = -ia(x)e^{i\psi(x)} = -i\hat{a}(x), \]

\[ T'_2(0)\hat{a}(x) = -(a'(x) + i\psi'(x)a)e^{i\psi(x)} = -\hat{a}_x(x). \]

Consequently, it is easy to obtain that

\[ H_{0,\nu} T'_1(0)\hat{a}(x) = 0, \]

\[ \tag{40} \]

\[ H_{0,\nu} T'_2(0)\hat{a}(x) = 0. \]

\[ \tag{41} \]
In fact,
\[
H_{\omega,\nu} T'_1(0) \hat{a}(x) = H_{\omega,\nu}(-i \hat{a}(x)) \\
= i \hat{a}_{xx} + \frac{\beta(\alpha + \beta)}{2} (-3i|\alpha|^4 \hat{a} + 2i|\alpha|^2 \hat{a}^2 \hat{a}) + ig(|\alpha|^2) \hat{a} \\
- g'(|\alpha|^2)(-i\hat{a} + i\hat{a})\hat{a} + i(\alpha + \beta)(-i|\alpha|^2 \hat{a}_x - i\hat{a}\hat{a}_x + i\hat{a}\hat{a}_x) \\
+ i\omega \hat{a} + \nu \hat{a}_x - 2i\nu\beta |\alpha|^2 \hat{a} + i\beta \nu \hat{a} \hat{a}_x \\
= -i[E'(\hat{a}) - \omega Q'_1(\hat{a}) - \nu Q'_2(\hat{a})] = 0,
\]
and
\[
H_{\omega,\nu} T'_2(0) \hat{a}(x) = H_{\omega,\nu}(-\hat{a}_x(x)) \\
= \hat{a}_{xxx} - \frac{\beta(\alpha + \beta)}{2} (3|\alpha|^4 \hat{a}_x + 2|\alpha|^2 \hat{a}^2 \hat{a}_x) + g(|\alpha|^2) \hat{a}_x \\
+ g'(|\alpha|^2)(\hat{a}_{xx} + \hat{a} \hat{a}_x) \hat{a} - i(\alpha + \beta)(|\alpha|^2 \hat{a}_{xx} + \hat{a}_x \hat{a}_x + \hat{a} \hat{a}_x) \\
+ \omega \hat{a}_x - i\nu \hat{a}_{xx} - 2\nu \beta |\alpha|^2 \hat{a}_x - \beta \nu \hat{a} \hat{a}_x \\
= -\frac{\partial}{\partial x}[E'(\hat{a}(x)) - \omega Q'_1(\hat{a}(x)) - \nu Q'_2(\hat{a}(x))] = 0.
\]

Let \( Z = \{k_1 T'_1(0) \hat{a}(x) + k_2 T'_2(0) \hat{a}(x) | k_1, k_2 \in \mathbb{R} \} \), from (40) and (41), we know that \( Z \) is contained in the kernel of \( H_{\omega,\nu} \).

**Assumption 1.** Spectral decomposition of operator \( H_{\omega,\nu} \): the space \( X \) can be decomposed as a direct sum
\[
X = N + Z + P,
\]
where \( Z \) is the space defined above, \( N \) is a finite-dimensional subspace such that
\[
\langle H_{\omega,\nu} u, u \rangle < 0, \quad \forall \Omega \neq u \in N,
\]
\( P \) is a closed subspace, and for any \( u \in P \), there exists \( \delta > 0 \) independent of \( u \), such that
\[
\langle H_{\omega,\nu} u, u \rangle \geq \delta \|u\|^2_X, \quad \forall u \in P.
\]

We define \( d(\omega, \nu) : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) as
\[
d(\omega, \nu) = E(\hat{a}_{\omega,\nu}) - \omega Q'_1(\hat{a}_{\omega,\nu}) - \nu Q'_2(\hat{a}_{\omega,\nu}),
\]
and \( d''(\omega, \nu) \) as the Hessian matrix of function \( d(\omega, \nu) \), which has a symmetric bilinear form. In addition, we use \( p(d'') \) to express the numbers of positive eigenvalue of \( d'' \) and \( n(H_{\omega,\nu}) \) to express the numbers of negative eigenvalue of \( H_{\omega,\nu} \).

[16] indicates the existence of the local solution of initial value problem of Kundu equation. Based on the above discussions, we know that Eq. (24) has three invariants of motion satisfying (35) and (36). Moreover, we prove that the solitary waves of Eq. (24) satisfy (37). Furthermore, we give the definition of the operator \( H_{\omega,\nu} \). According to the "stability theorem" in the introduction of [18] or Theorem 4.1 in [18], we can obtain the following abstract orbital stability theorem for the solitary waves of Eq. (24).
Theorem 2. Suppose that there exist three functionals $E(u), Q_1(u), Q_2(u)$ satisfying (35) and (36), and solitary waves $T_1(\omega t)T_2(\nu t)\hat{a}_{\omega,\nu}(x)$ satisfying (37). Moreover, suppose that the operator $H_{\omega,\nu}$ given by (38) satisfies Assumption 1. If $d(\omega, \nu)$ is non-degenerate and $p(d^r) = n(H_{\omega,\nu})$, then solitary waves $T_1(\omega t)T_2(\nu t)\hat{a}_{\omega,\nu}(x)$ are orbitally stable.

From Theorem 2, the results about orbital stability of solitary waves for Eq. (24) can be obtained.

Theorem 3. For any fixed real constants $c_3, c_5, \alpha = 2s_2 + r, \beta = s_2 + r$, if $\omega, \nu$ satisfy $4\omega + \nu < 0$ and one of the following conditions holds:

(a) $16c_5 + (\alpha + \beta)(3\alpha - 5\beta) = 0, 2c_3 + (\alpha - \beta)\nu < 0$,
(b) $16c_5 + (\alpha + \beta)(3\alpha - 5\beta) > 0, 2c_3 + (\alpha - \beta)\nu \geq 0$,
(c) $16c_5 + (\alpha + \beta)(3\alpha - 5\beta) < 0, 2c_3 + (\alpha - \beta)\nu > 0$, and $3(2c_3 + (\alpha - \beta)\nu)^2 - (16c_5 + (\alpha + \beta)(3\alpha - 5\beta))(4\omega + \nu^2) > 0$,
(d) $16c_5 + (\alpha + \beta)(3\alpha - 5\beta) > 0, 2c_3 + (\alpha - \beta)\nu < 0$, and $3(\alpha - \beta)^2 - (16c_5 + (\alpha + \beta)(3\alpha - 5\beta)) \geq \sqrt{3(\alpha - \beta)^2 - (4\omega + \nu^2)}(16c_5 + (\alpha + \beta)(3\alpha - 5\beta)) \geq \sqrt{3(\alpha - \beta)^2 - (4\omega + \nu^2)}(16c_5 + (\alpha + \beta)(3\alpha - 5\beta))$,

then the solitary waves $e^{-i\alpha t}\hat{a}(x - \nu t)$ of Eq. (24) are orbitally stable.

Since Eq. (24) reduces to Eq. (2) as $r = 0$, we have the following corollary for Eq. (2).

Corollary 1. For any fixed real constants $c_3, c_5, s_2$, if $\omega, \nu$ satisfy $4\omega + \nu < 0$ and one of the following conditions holds:

(a) $16c_5 + 2s_2 = 0, 2c_3 + s_2\nu > 0$,
(b) $16c_5 + 2s_2 > 0, 2c_3 + s_2\nu \geq 0$,
(c) $16c_5 + 2s_2 < 0, 2c_3 + s_2\nu > 0$, and $3(2c_3 + s_2\nu)^2 - (16c_5 + 2s_2^2)(4\omega + \nu^2) > 0$,
(d) $16c_5 + 2s_2 > 0, 2c_3 + s_2\nu < 0$, and $16c_5 \leq -\frac{\sqrt{3s_2^2 - (4\omega + \nu^2)(16c_5 + 2s_2^2)}}{2(2c_3 + s_2\nu)}$,

then the solitary waves of Eq. (2) are orbitally stable.

When $c_3 = c_5 = 0, s_2 = -1, r = 0$, Eq. (24) reduces to Eq. (6), we have the following corollary for Eq. (6).

Corollary 2. For any wave speed $\nu \neq 0$, if $\nu^2 < -4\omega$, then the solitary waves of Eq. (6) are orbitally stable.

Proof. When $\nu^2 < -4\omega$, from Theorem 3(b), it is easy to prove that the solitary waves of Eq. (6) are orbitally stable as $\nu < 0$, while from Theorem 3(d), it is easy to prove that the solitary waves of Eq. (6) are orbitally stable as $\nu > 0$.

Similarly, we can obtain the orbital stability of solitary waves for Eqs. (3)–(4) from Theorem 3. \qed

Corollary 3. For any wave speed $\nu \neq 0$, if $\nu^2 < -4\omega$, then solitary waves of Eq. (3) are orbitally stable.

Corollary 4. Suppose that $\delta \neq 0$. for any wave speed $\nu \neq 0$, if $\nu^2 < -4\omega$, then solitary waves of Eq. (4) are orbitally stable.

4. Proof of Theorem 3

By the above discussions, we know that if the conditions in Theorem 1 hold, then Eq. (24) has solitary waves of the form $e^{-i\alpha t}\hat{a}(x - \nu t)$. In this section, we mainly prove that these solitary waves
are orbitally stable. According to (32)–(38), we only need to prove that Assumption 1 and \( n(H_{\omega, \upsilon}) = p(d'') \) hold. Consequently, solitary waves of Eq. (24) are orbitally stable from Theorem 2.

4.1. Proof of Assumption 1

First of all, we study the decomposition of \( H_{\omega, \upsilon} \) and the spectral properties of each part. For any \( \phi(x) \in X \), let

\[
\phi(x) = e^{i\psi(x)} z(x), \quad z(x) = z_1(x) + iz_2(x), \quad z_1(x) = \text{Re} \, z(x),
\]

then

\[
H_{\omega, \upsilon} \phi = [L_{11}z_1 + L_{12}z_2 + i(L_{21}z_1 + L_{22}z_2)] e^{i\psi},
\]

where

\[
L_{11} = -\frac{\partial}{\partial x^2} + (\psi')^2 - g(a^2) - 2g'(a^2)a^2 - \omega - \upsilon \psi' + \frac{5\beta(\alpha + \beta)}{2} a^4 - 3(\alpha + \beta)a^2\psi' + 3\upsilon \beta a^2,
\]

\[
L_{12} = \psi'' - \frac{\alpha + \beta}{2} a^2 \frac{\partial}{\partial x},
\]

\[
L_{21} = -\psi'' + \frac{\alpha + \beta}{2} a^2 \frac{\partial}{\partial x} + 2(\alpha + \beta)aa',
\]

\[
L_{22} = -\frac{\partial}{\partial x^2} + (\psi')^2 - g(a^2) - \omega - \upsilon \psi' - \alpha a^2\psi' + \beta a^2\psi'.
\]

Furthermore, we have

\[
\langle H_{\omega, \upsilon} \phi, \phi \rangle = \langle L_{11}z_1, z_1 \rangle + \langle L_{22}z_2, z_2 \rangle - \langle (\alpha + \beta)a^2z'_2, z_1 \rangle
\]

\[
+ \left\langle \frac{(\alpha + \beta)^2}{4} a^4z_1 + (\alpha + \beta)aa'z_2, z_1 \right\rangle,
\]

(47)

where

\[
\tilde{L}_{11} = -\frac{\partial}{\partial x^2} + (\psi')^2 - g(a^2) - 2g'(a^2)a^2 - \omega - \upsilon \psi'
\]

\[
- 3\alpha a^2\psi' + 3\beta a^2\psi' - \frac{(\alpha + \beta)}{2} a^2(\upsilon - 2\psi' + \alpha a^2 - \beta a^2).
\]

(48)

due to

\[
\langle H_{\omega, \upsilon} \phi, \phi \rangle = \langle L_{11}z_1, z_1 \rangle + \langle L_{12}z_2, z_1 \rangle + \langle L_{21}z_1, z_2 \rangle + \langle L_{22}z_2, z_2 \rangle,
\]

\[
\langle L_{11}z_1, z_1 \rangle + \langle L_{12}z_2, z_1 \rangle + \langle L_{21}z_1, z_2 \rangle
\]

\[
= \langle L_{11}z_1, z_1 \rangle + \left( -\frac{\alpha + \beta}{2} a^2z'_2, z_1 \right) + \left( \frac{\alpha + \beta}{2} a^2z'_1, z_2 \right) + \langle 2(\alpha + \beta)aa'z_1, z_2 \rangle,
\]

and
\[
\left\langle \frac{\alpha + \beta}{2} a^2 \dot{z}_1, \dot{z}_2 \right\rangle = \frac{\alpha + \beta}{2} \int_R a^2 z_2 \, dz_1 = -\frac{\alpha + \beta}{2} \int_R z_1(2a \dot{a} z_2 + 2a^2 \dot{z}_2) \, dx
\]
\[
= \left\langle -(\alpha + \beta) a \dot{a} z_2, z_1 \right\rangle + \left\langle -\frac{\alpha + \beta}{2} a^2 \dot{z}_2, z_1 \right\rangle.
\]

It follows from Eq. (10) that
\[
L_{22} a(x) = 0. \tag{49}
\]

Differentiating Eq. (10) with respect to \(x\), and combining (12) and (48), we have
\[
\bar{L}_{11} a'(x) = 0. \tag{50}
\]

In view of (23), we know that \(a'(x)\) only has a simple zero point \(x = 0\) and it changes the sign only once. Therefore, by Sturm–Liouville theorem, it is easy to see that zero is the second eigenvalue of \(\bar{L}_{11}\) and \(\bar{L}_{11}\) only has one negative eigenvalue \(-\lambda_{11}^2\), whose corresponding eigenfunction is denoted by \(\chi_{11}\), namely,
\[
\bar{L}_{11} \chi_{11} = -\lambda_{11}^2 \chi_{11}. \tag{51}
\]

In addition, \(\bar{L}_{11}\) can be rewritten as
\[
\bar{L}_{11} = -\frac{a}{\alpha x^2} + d_1 + M_1(x), \tag{52}
\]

where
\[
M_1(x) = \frac{(\alpha + \beta)^2}{16} a^4 - g(a^2) - 2g'(a^2) + 3\alpha^2 \psi' - 3\beta a^2 \psi'
\]
\[
- \frac{(\alpha + \beta)}{2} a^2 (\nu - 2 \psi' + \alpha a^2 - \beta a^2).
\]

From (23), it can be seen that \(a^2 \to 0\) as \(|x| \to \infty\), therefore, we have
\[
M_1(x) \to 0 \quad (|x| \to \infty). \tag{53}
\]

Based on (52)–(53) and Weyl’s essential spectral theorem obtained in [23], we have
\[
\sigma_{\text{ess}}(\bar{L}_{11}) = [d_1, +\infty), \quad d_1 > 0. \tag{54}
\]

Thus, we have the spectral properties of \(\bar{L}_{11}\) as follows.

**Proposition 1.** \(\bar{L}_{11}\) only has a simple negative eigenvalue and its kernel is spanned by \(a'(x)\). Moreover, the rest of its spectrums are positive and bounded away from zero.

It follows from Proposition 1 and [17] that for any real function \(z_1 \in H^1(R)\), if it satisfies
\[
\langle z_1, a' \rangle = \langle z_1, \chi_{11} \rangle = 0, \tag{55}
\]
then there exists a positive constant \(\delta_1 > 0\) independent of \(z_1\), such that
\[
\langle \bar{L}_{11} z_1, z_1 \rangle \geq \delta_1 \|z_1\|_{L^2}^2. \tag{56}
\]
Furthermore, according to (52)–(56), we obtain the following lemma (see Appendix in [24]).

**Lemma 2.** For any real function \( z_1 \in H^1(\mathbb{R}) \) satisfying (55), there exists a positive \( \delta_1 > 0 \) which is independent of \( z_1 \), such that

\[
\langle \tilde{L}_{11} z_1, z_1 \rangle \geq \delta_1 \| z_1 \|_{H^1}^2.
\] (57)

Next, we discuss \( L_{22} \).

From (23) and (49), it is easy to see that \( a(x) \) has a fixed sign and zero is the first eigenvalue of \( L_{22} \). Note that

\[
L_{22} = -\frac{\partial}{\partial x^2} + d_1 + M_2(x),
\] (58)

where \( M_2(x) = \frac{(\alpha + \beta)^2}{16} a^4 - g(a^2) - \alpha a^2 \psi' + \beta a^2 \psi' \).

Since \( a^2 \to 0 \) as \( |x| \to \infty \), we have that

\[
M_2(x) \to 0 \quad (|x| \to \infty).
\] (59)

Moreover,

\[
\sigma_{\text{ess}}(L_{22}) = [d_1, +\infty), \quad d_1 > 0.
\] (60)

Therefore, we have the following spectral properties of \( L_{22} \).

**Proposition 2.** The kernel of \( L_{22} \) is spanned by \( a(x) \). Moreover, the rest of its spectrum is positive and bounded away from zero.

Similarly, by Proposition 2 and (58)–(60), we have

**Lemma 3.** For any real function \( z_2 \in H^1(\mathbb{R}) \) satisfying

\[
\langle z_2, a \rangle = 0,
\] (61)

there exists a positive number \( \delta_2 > 0 \) such that

\[
\langle L_{22} z_2, z_2 \rangle \geq \delta_2 \| z_2 \|_{H^1}^2,
\] (62)

where \( \delta_2 \) is independent of \( z_2 \).

In the following, we verify that Assumption 1 holds and \( n(H_{\omega, \nu}) = 1 \).

For any \( \phi(x) \in X \), let

\[
\phi(x) = e^{i \psi(x)} (z_1(x) + iz_2(x)), \quad z_2(x) = a(x)z_3(x),
\] (63)

where \( z_1, z_2, z_3 \) are real functions and \( z_1, z_2 \in H^1(\mathbb{R}) \). Note that
\begin{align}
\langle L_{22} z_2(x), z_2(x) \rangle &= \langle -z_2'', z_2 \rangle + \langle d_1 z_2, z_2 \rangle + \langle M_2 z_2, z_2 \rangle \\
&= \langle -a' z_3 + d_1 a z_3 + M_2 a z_3, a z_3 \rangle + \langle -2a' z_3' - a z_3', a z_3 \rangle \\
&= \langle L_{22} a, a z_3' \rangle - \langle (a^2 z_3')', z_3 \rangle \\
&= -\int_R z_3 d(a^2 z_3') \\
&= \int_R (a z_3') (a z_3') \, dx \\
&= \langle a z_3', a z_3' \rangle. 
\end{align}

(64)

Then, it follows from (47) and (63)–(64) that

\begin{align}
\langle H_{\omega, \upsilon} \phi, \phi \rangle &= \langle \bar{L}_{11} z_1, z_1 \rangle + \langle a z_3', a z_3' \rangle \\
&= \langle \bar{L}_{11} z_1, z_1 \rangle + \int_R \left( \langle a z_3' \rangle^2 + \frac{\alpha + \beta}{2} a^2 z_1 \right)^2 - 2 \langle a z_3' \rangle \left( \frac{\alpha + \beta}{2} a^2 z_1 \right) \, dx \\
&= \langle \bar{L}_{11} z_1, z_1 \rangle + \int_R \left( \frac{\alpha + \beta}{2} a^2 z_1 - a z_3' \right)^2 \, dx. 
\end{align}

(65)

Choose

\begin{equation}
\chi_- = (\chi_{11} + i \chi_{12}) e^{i \psi(x)},
\end{equation}

(66)

and

\begin{equation}
\chi_{12} = \alpha \chi_{13} = a \left( \frac{\alpha + \beta}{2} \int_{-\infty}^x a(s) \chi_{11}(s) \, ds + k_1 \right).
\end{equation}

(67)

where $k_1$ is an arbitrary real constant, then from (63) and (65)–(67), we have

\begin{equation}
\langle H_{\omega, \upsilon} \chi_-, \chi_- \rangle = \langle \bar{L}_{11} \chi_{11}, \chi_{11} \rangle = -\lambda_{11}^2 < 0.
\end{equation}

(68)

Choose $k_1$ such that

\begin{equation}
\langle \chi_{12}, a \rangle = 0,
\end{equation}

(69)

and set

\begin{equation}
N = \{ k \chi_- | k \in \mathbb{R} \},
\end{equation}

(70)

then we have (43) holds from (68) and (70).
Let
\[ \chi_1 = \left( a'(x) + ia(x) \left( k_2 + \frac{\alpha + \beta}{4} a^2(x) \right) \right) e^{i\psi(x)}, \]
(71)
\[ \chi_{12} = ia(x) e^{i\psi(x)}, \]
(72)
and choose \( k_2 \) such that
\[ \left( k_2 + \frac{\alpha + \beta}{4} a^2 \right) a, a \right) = 0, \]
(73)
then subspace \( Z \) can be rewritten as
\[ Z = \{ k_3 \chi_1 + k_4 \chi_2 \mid k_3, k_4 \in \mathbb{R} \}. \]
(74)
Denoted subspace \( P \) by
\[ P = \{ p \in X \mid p = (p_1 + ip_2) e^{i\psi(x)}, \langle p_1, \chi_{11} \rangle = \langle p_1, a' \rangle = \langle p_2, a \rangle = 0 \}, \]
(75)
then by the method in Appendix 1 of [10], we have the following lemma.

**Lemma 4.** For any \( p \in P \), there exists a constant \( \delta > 0 \) independent of \( p \) such that
\[ \langle H_{\omega, \upsilon} p, p \rangle \geq \delta \| p \|^2_{H^1}. \]
(76)
For any \( \phi(x) = e^{i\psi(x)}(z_1 + iz_2) \in X \), we choose
\[ a_1 = \langle z_1, \chi_{11} \rangle, \quad b_1 = \frac{\langle z_1, a' \rangle}{\| a' \|^2_{L^2}}, \quad b_2 = \frac{\langle z_2, a \rangle}{\| a \|^2_{L^2}}, \]
(77)
then \( \phi(x) \) can be rewritten as
\[ \phi(x) = a_1 \chi_- + b_1 \chi_1 + b_2 \chi_2 + p. \]
(78)
From (50)–(51) and (69)–(73), we obtain that \( \phi(x) \) is expressed uniquely by (78). Then, it follows from (70) and (74)–(78) that Assumption 1 holds and \( n(H_{\omega, \upsilon}) = 1 \).

4.2. **Proof of** \( p(d'') = n(H_{\omega, \upsilon}) = 1 \)

In this subsection, we focus on proving that \( p(d'') = 1 \) under the conditions in Theorem 3. Consequently, \( p(d'') = n(H_{\omega, \upsilon}) = 1 \).

From
\[ d(\omega, \upsilon) = E(\hat{\omega}) - \omega Q_1(\hat{\omega}) - \upsilon Q_2(\hat{\omega}), \]
we have
\[ d_\omega = -Q_1(\hat{a}), \quad d_\nu = -Q_2(\hat{a}), \]
\[ d_{\omega\nu} = d_{\nu\omega} = -\left( Q'_1(\hat{a}) \right) = -\left( \hat{a}, \frac{\partial \hat{a}}{\partial \omega} \right) = -\frac{1}{2} \frac{\partial}{\partial \omega} (a, a), \]
\[ d_{\nu\nu} = d_{\nu\nu} = -\left( Q'_2(\hat{a}) \right) = \left( i\hat{a}_x + \beta a^2 \hat{a}, \frac{\partial \hat{a}}{\partial \nu} \right) \]
\[ = \text{Re} \int_{\mathbb{R}} \left( -\psi' a + i a' + \beta a^3 \right) \left( \frac{\partial a}{\partial \nu} - i \frac{\partial \psi}{\partial \nu} \right) dx \]
\[ = \int_{\mathbb{R}} \left( -\psi' a + \beta a^3 \right) \frac{\partial a}{\partial \nu} + a a' \frac{\partial \psi}{\partial \nu} \right] dx \]
\[ = \int_{\mathbb{R}} \left( -\left( \frac{\nu + \alpha + \beta}{4} a + \beta a^3 \right) \frac{\partial a}{\partial \nu} \right) dx + \frac{1}{2} \int_{\mathbb{R}} \frac{\partial \psi}{\partial \nu} da^2 \]
\[ = \int_{\mathbb{R}} \left( -\left( \frac{\nu a}{4} \right) \frac{\partial a}{\partial \nu} \right) dx - \int_{\mathbb{R}} \frac{1}{4} a^2 dx - \frac{\alpha - \beta}{2} \int_{\mathbb{R}} a^3 \frac{\partial a}{\partial \nu} dx \]
\[ = -\frac{\nu}{4} \frac{\partial}{\partial \nu} (a, a) - \frac{1}{4} (a, a) - \frac{\alpha - \beta}{8} \frac{\partial}{\partial \nu} \int_{\mathbb{R}} a^4 dx. \]

Therefore, we obtain
\[ d'' = \begin{pmatrix} d_{\omega\omega} & d_{\omega\nu} \\ d_{\omega\nu} & d_{\nu\nu} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \frac{\partial}{\partial \omega} (a, a) & -\frac{1}{2} \frac{\partial}{\partial \nu} (a, a) \\ -\frac{1}{2} \frac{\partial}{\partial \nu} (a, a) & d_{\nu\nu} \end{pmatrix}. \]

Moreover,
\[ \det(d'') = \frac{\nu}{8} \frac{\partial}{\partial \omega} (a, a) \frac{\partial}{\partial \nu} (a, a) + \frac{1}{8} (a, a) \frac{\partial}{\partial \omega} (a, a) \]
\[ + \frac{\alpha - \beta}{16} \frac{\partial}{\partial \nu} (a, a) \int_{\mathbb{R}} a^4 dx - \frac{1}{4} \left( \frac{\partial}{\partial \nu} (a, a) \right)^2. \tag{79} \]

Note that
\[ a^2(x) = \frac{1}{d_3 + d_5 \cosh d_6 x}, \]
\[ d_3 = -\frac{d_2}{2d_1}, \quad d_5^2 = d_2^2 - 4d_1 d_4, \quad d_6^2 = 4d_1, \]
\[ d_1 = -\omega - \frac{\nu^2}{4}, \quad d_2 = -\frac{1}{2} c_3 - \frac{\alpha - \beta}{4} \nu, \]
\[ d_4 = -\frac{1}{3} \left( c_5 + \frac{(\alpha + \beta)(3\alpha - 5\beta)}{16} \right), \]
it is obviously that

\[
\begin{align*}
\frac{\partial d_1}{\partial \omega} &= -1, & \frac{\partial d_1}{\partial \nu} &= -\frac{1}{2}, \\
\frac{\partial d_2}{\partial \nu} &= -\frac{1}{4}(\alpha - \beta), & \frac{\partial d_2}{\partial \omega} &= 0, \\
\frac{\partial d_4}{\partial \omega} &= \frac{\partial d_4}{\partial \nu} = 0.
\end{align*}
\]

We will prove that \( p(d'') = 1 \) from the following four cases. That is, we will prove that \( \det(d'') < 0 \).

**Case 1.** \( d_4 = 0, d_2 < 0 \).

In this case, we have

\[
\begin{align*}
a^2(x) &= \frac{1}{d_3 + d_5 \cosh d_6 x} = \frac{2d_1}{d_2} \frac{1}{(1 + \cosh d_6 x)}, \\
\langle a, a \rangle &= -\frac{2d_1}{d_2} \int_R \frac{1}{(1 + \cosh d_6 x)} \, dx = -\frac{2\sqrt{d_1}}{d_2}, \\
\frac{\partial}{\partial \omega} \langle a, a \rangle &= -\frac{\sqrt{d_1}}{2d_2} \left( (\alpha - \beta) - \nu \frac{d_2}{d_1} \right), \\
\frac{\partial}{\partial \nu} \langle a, a \rangle &= -\frac{1}{4} \left( \frac{\alpha - \beta}{d_1} \right), \\
\int_R a^4(x) \, dx &= \frac{4d_1^2}{d_2^2} \int_R \left( \frac{1}{1 + \cosh d_6 x} \right)^2 \, dx = \frac{4d_1^2}{d_2^2} \int_R \frac{4e^{2d_6 x}}{(e^{2d_6 x} + 2e^{d_6 x} + 1)^2} \, dx \\
&= \frac{16d_1^2}{d_2^2 d_6} \int_0^{+\infty} \frac{y}{(y + 1)^4} \, dy = \frac{4d_1 \sqrt{d_1}}{3d_2^2}, \\
\frac{\partial}{\partial \nu} \int_R a^4(x) \, dx &= \frac{d_1^3}{d_2^3} \left( \frac{\alpha - \beta}{d_1} - \nu \frac{d_2}{d_1} \right) - \frac{1}{3} \frac{(\alpha - \beta)^3 d_1^3}{d_2^3}.
\end{align*}
\]

Furthermore, from (79), we have

\[
\begin{align*}
\det(d'') &= \frac{1}{8} \langle a, a \rangle \frac{\partial}{\partial \omega} \langle a, a \rangle + \frac{1}{8} \frac{\partial}{\partial \nu} \langle a, a \rangle \left( \nu \frac{\partial}{\partial \omega} \langle a, a \rangle - 2 \frac{\partial}{\partial \nu} \langle a, a \rangle \right) \\
&+ \frac{\alpha - \beta}{16} \frac{\partial}{\partial \omega} \langle a, a \rangle \frac{\partial}{\partial \nu} \int_R a^4 \, dx \\
&= -\frac{1}{4d_2} - \frac{(\alpha - \beta)^2 d_1}{48d_2^4},
\end{align*}
\]

that it is negative because \( d_1 > 0 \) and \( d_2 \neq 0 \).
Consequently, $d''$ has exactly one negative and one positive eigenvalues. Namely, $p(d'') = 1$ holds for Case 1.

**Case 2.** $d_4 < 0$, $d_2 \leq 0$.

In this case, we have

$$d_5^2 - d_3^2 = \frac{d_2^2 - 4d_1d_4}{4d_1^2} - \frac{d_2^2}{4d_1^2} = -\frac{d_4}{d_1} > 0, \quad d_5^2 > d_3^2,$$

$$\langle a, a \rangle = \int_\mathbb{R} \frac{dx}{d_3 + d_5 \cosh d_5 x} = \frac{1}{\sqrt{-d_4}} \left( \frac{\pi}{2} - \arctan \frac{-d_2}{2\sqrt{-d_1d_4}} \right) > 0,$$

$$\frac{\partial}{\partial \omega} \langle a, a \rangle = \frac{d_2}{(d_2^2 - 4d_1d_4) \sqrt{d_1}} \leq 0,$$

$$\frac{\partial}{\partial \nu} \langle a, a \rangle = -\frac{\sqrt{d_1}}{2(d_2^2 - 4d_1d_4)} \left( (\alpha - \beta) - \nu \frac{d_2}{d_1} \right),$$

$$\int_\mathbb{R} a^4(x) \, dx = \int_\mathbb{R} \left( \frac{1}{d_3 + d_5 \cosh d_5 x} \right)^2 \, dx$$

$$= \frac{-\sqrt{d_1}}{d_4} - \frac{d_2}{2d_4\sqrt{-d_4}} \left( \frac{\pi}{2} - \arctan \frac{-d_2}{2\sqrt{-d_1d_4}} \right),$$

$$\frac{\partial}{\partial \nu} \int_\mathbb{R} a^4(x) \, dx = \frac{\nu}{4d_4 \sqrt{d_1}} + \frac{(\alpha - \beta)}{8d_4 \sqrt{-d_4}} \left( \frac{\pi}{2} - \arctan \frac{-d_2}{2\sqrt{-d_1d_4}} \right)$$

$$+ \frac{d_2 \sqrt{d_1}}{4d_4(d_2^2 - 4d_1d_4)} \left( \alpha - \beta - \nu \frac{d_2}{d_1} \right).$$

Then, we have

$$\text{det}(d'') = \frac{1}{8} \langle a, a \rangle \frac{\partial}{\partial \omega} \langle a, a \rangle + \frac{1}{8} \frac{\partial}{\partial \nu} \langle a, a \rangle \left( \nu \frac{\partial}{\partial \omega} \langle a, a \rangle - 2 \frac{\partial}{\partial \nu} \langle a, a \rangle \right)$$

$$+ \frac{\alpha - \beta}{16} \frac{\partial}{\partial \omega} \langle a, a \rangle \frac{\partial}{\partial \nu} \int_\mathbb{R} a^4 \, dx$$

$$= \frac{1}{8} \langle a, a \rangle \frac{\partial}{\partial \omega} \langle a, a \rangle + \frac{(\alpha - \beta)^2}{64d_4(d_2^2 - 4d_1d_4)}$$

$$\cdot \left[ 1 + \frac{d_2}{2\sqrt{-d_1d_4}} \left( \frac{\pi}{2} - \arctan \frac{-d_2}{2\sqrt{-d_1d_4}} \right) \right]$$

$$= 1 + \Pi.$$
Further, we have

\[ Y'_1(y) = -\frac{\pi}{2} + \arctan y + \frac{y}{1 + y^2}, \]

\[ Y'_1(0) = -\frac{\pi}{2}, \quad Y'_1(+\infty) = 0. \]  (81)

\[ Y''_1(y) = \frac{2}{1 + y^2} - \frac{2y^2}{(1 + y^2)^2} = \frac{2}{(1 + y^2)^2} > 0. \]  (82)

From (81) and (82), it is easy to see that \( Y'_1(y) < 0 \) for any \( y \in \mathbb{R} \). It follows from (80) that \( Y_1(y) > 0 \) for any \( y \in \mathbb{R} \). Since \( d_4 < 0, \) \( \Pi < 0 \). Accordingly, \( p(d'') = 1 \).

**Case 3.** \( d_4 > 0, d_2 < 0, d_2^2 - 4d_1d_4 > 0. \)

In this case, we have

\[ d_2^2 - d_3^2 = \frac{d_2^2 - 4d_1d_4}{4d_1^2} - \frac{d_2^2}{4d_1} = -\frac{d_4}{d_1}, \quad d_2^2 < d_3^2. \]

\[ \langle a, a \rangle = \int_{\mathbb{R}} \frac{dx}{d_3 + d_5 \cosh d_6 x} = \int_{\mathbb{R}} \frac{2e^{d_6 x}}{d_5 e^{2d_6 x} + 2d_3 e^{d_6 x} + d_5} dx = \frac{1}{2\sqrt{d_4}} \ln \left( \frac{-d_2 + 2\sqrt{d_1d_4}}{-d_2 - 2\sqrt{d_1d_4}} \right). \]

\[ \frac{\partial}{\partial \omega} \langle a, a \rangle = \frac{d_2}{\sqrt{d_1}(d_2^2 - 4d_1d_4)}, \]

\[ \frac{\partial}{\partial \nu} \langle a, a \rangle = -\frac{\nu d_2}{2(d_2^2 - 4d_1d_4)} \left( (\alpha - \beta) - \frac{vd_2}{d_1} \right). \]

From

\[ \int_{\mathbb{R}} a^4(x) dx = \int_{\mathbb{R}} \left( \frac{1}{d_3 + d_5 \cosh d_6 x} \right)^2 dx = -\frac{\sqrt{d_4}}{d_4} - \frac{d_2}{4d_4\sqrt{d_4}} \ln \left( \frac{-d_2 + 2\sqrt{d_1d_4}}{-d_2 - 2\sqrt{d_1d_4}} \right). \]

we have

\[ \frac{\partial}{\partial \nu} \int_{\mathbb{R}} a^4(x) dx = \frac{\nu}{4d_4\sqrt{d_1}} + \frac{(\alpha - \beta)}{16d_4\sqrt{d_4}} \ln \left( \frac{-d_2 + 2\sqrt{d_1d_4}}{-d_2 - 2\sqrt{d_1d_4}} \right) \]

\[ + \frac{\sqrt{d_1}d_2}{4d_4(d_2^2 - 4d_1d_4)} \left( (\alpha - \beta) - \frac{vd_2}{d_1} \right). \]

Furthermore, we obtain

\[ \det(d'') = \frac{1}{8} \langle a, a \rangle \frac{\partial}{\partial \omega} \langle a, a \rangle + \frac{1}{8} \frac{\partial}{\partial \nu} \langle a, a \rangle \left( \nu \frac{\partial}{\partial \omega} \langle a, a \rangle - 2 \frac{\partial}{\partial \nu} \langle a, a \rangle \right) \]

\[ + \frac{\alpha - \beta}{16} \frac{\partial}{\partial \omega} \langle a, a \rangle \frac{\partial}{\partial \nu} \int_{\mathbb{R}} a^4 dx \]
Case 4. To prove that det
\[
= \left(\alpha - \beta\right)^2 \frac{d_2}{64d_4(d_2^2 - 4d_1d_4)} \left(1 + \frac{d_2}{4\sqrt{d_1d_4}} \ln\left(-\frac{d_2 + 2\sqrt{d_1d_4}}{-d_2 - 2\sqrt{d_1d_4}}\right)\right)
\]
\[
= I + II.
\]

It is obviously that I < 0. In what follows, we will prove II < 0.

Assume that \( y = -\frac{d_2}{2\sqrt{d_1d_4}} > 0 \), then we have \( y > 1 \) and II = \( \frac{(\alpha - \beta)^2}{64d_4(d_2^2 - 4d_1d_4)} Y_2(y) \), where \( Y_2(y) = 1 - \frac{y}{2} \ln\left(\frac{y+1}{y-1}\right) \). Evidently, \( Y_2(1) = -\infty \). By L'Hospital's rule of limit, we have
\[
\lim_{y \to \infty} Y_2(y) = 1 - \lim_{y \to \infty} \left(\frac{y-1}{y+1}\right) \left(\frac{y^2}{(y-1)^2}\right) = 0. \tag{83}
\]

Notice that
\[
Y_2'(y) = -\frac{1}{2} \ln\left(\frac{y+1}{y-1}\right) + \frac{y}{y^2 - 1}, \quad Y_2'(\infty) = 0. \tag{84}
\]
\[
Y_2''(y) = -\frac{2}{(y^2 - 1)^2} < 0, \quad \forall 1 < y < +\infty. \tag{85}
\]

From (84) and (85), we have \( Y_2'(y) > 0, \forall 1 < y < +\infty, \forall y \in \mathbb{R} \). From (83), it is easy to obtain that \( Y_2(y) < 0, \forall 1 < y < +\infty \). Since \( d_4 > 0, d_2^2 - 4d_1d_4 > 0, \) II < 0. Furthermore, we have \( p(d'') = 1 \).

Case 4. \( d_4 < 0, d_2 > 0 \).

Based on the above results, we have
\[
1 \langle a, a \rangle \frac{\partial}{\partial \omega} \langle a, a \rangle = \frac{d_2}{8(d_2^2 - 4d_1d_4)\sqrt{-d_1d_4}} \left(\frac{\pi}{2} - \arctan\left(-\frac{d_2}{2\sqrt{-d_1d_4}}\right)\right),
\]
\[
\det(d'') = \frac{1}{8} \langle a, a \rangle \frac{\partial}{\partial \omega} \langle a, a \rangle + \frac{(\alpha - \beta)^2}{64d_4(d_2^2 - 4d_1d_4)} \cdot \left[1 + \frac{d_2}{2\sqrt{-d_1d_4}} \left(\frac{\pi}{2} - \arctan\left(-\frac{d_2}{2\sqrt{-d_1d_4}}\right)\right)\right] = \frac{1}{64d_4(d_2^2 - 4d_1d_4)} \left[(\alpha - \beta)^2 + [(\alpha - \beta)^2 + 16d_4]\right] \cdot \frac{d_2}{2\sqrt{-d_1d_4}} \left(\frac{\pi}{2} + \arctan\left(\frac{d_2}{2\sqrt{-d_1d_4}}\right)\right).
\]

To prove that \( \det(d'') < 0 \), we only need to prove that
\[
[(\alpha - \beta)^2 + 16d_4] > \frac{-(\alpha - \beta)^2}{\frac{d_2}{2\sqrt{-d_1d_4}} \left(\frac{\pi}{2} + \arctan\left(\frac{d_2}{2\sqrt{-d_1d_4}}\right)\right)}.
\]

Since
\[
\frac{-(\alpha - \beta)^2}{\frac{d_2}{2\sqrt{-d_1d_4}} \left(\frac{\pi}{2} + \arctan\left(\frac{d_2}{2\sqrt{-d_1d_4}}\right)\right)} < \frac{-(\alpha - \beta)^2}{\frac{d_2}{2\sqrt{-d_1d_4}} \left(\frac{\pi}{2} + \frac{\pi}{2}\right)} = \frac{(\alpha - \beta)^2 2\sqrt{-d_1d_4}}{\pi d_2},
\]

\[
\left(\alpha - \beta\right)^2 \frac{d_2}{64d_4(d_2^2 - 4d_1d_4)} \left(1 + \frac{d_2}{4\sqrt{d_1d_4}} \ln\left(-\frac{d_2 + 2\sqrt{d_1d_4}}{-d_2 - 2\sqrt{d_1d_4}}\right)\right)
\]
\[
= I + II.
\]
the parameters only need to satisfy
\[(\alpha - \beta)^2 + 16d_4 \geq \frac{(\alpha - \beta)^22\sqrt{-d_1d_4}}{\pi d_2}, \tag{86}\]
which is equal to
\[-2(4c_5 - s_2r - r^2) \geq \frac{s_2^2\sqrt{-3(4\omega + \upsilon^2)(16c_5 + 3s_2^2 - 4s_2r - 4r^2)}}{2\pi(2c_3 + s_2\upsilon)}. \tag{87}\]
Under the condition (86) or (87), we have \(\det(d'') < 0\), which implies \(p(d'') = 1\).

Summarized the above results, we have \(p(d'') = 1\) under the conditions in Theorem 3. Furthermore, \(n(H_{\omega,\upsilon}) = p(d'')\). Thus, Theorem 3 is proved completely.

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