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Families of group presentations related to topology

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Abstract

We study some algebraic properties of a class of group presentations depending on a finite number of integer parameters. This class contains many well-known groups which are interesting from a topological point of view. We find arithmetic conditions on the parameters under which the considered groups cannot be fundamental groups of hyperbolic 3-manifolds of finite volume. Then we investigate the asphericity for many presentations contained in our family.

1. Introduction

In this paper we shall consider a class of cyclically presented groups $G_n^\varepsilon(m,k,h)$, where $\varepsilon = (a,b,r,s) \in \mathbb{Z}^4$, $n \geqslant 2$, and the integer parameters m, k and h are taken modulo n. The groups $G_n^\varepsilon(m,k,h)$ have generators $x_1, \ldots, x_n$ and defining relations

$$x_i^a x_{i+k}^b x_{i+h+m}^a = \left(x_{i+h}^r x_{i+m}^r\right)^s \tag{1.1}$$

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for $i = 1, \ldots, n$ (subscripts mod n). This class of groups contains well-known groups considered by several authors, and it is related to the topology of closed connected orientable 3-manifolds. We first illustrate some examples of this connection.

(1) If $a = b = s = 1$, $r = 2$ and $h = 0$, then the groups $G_n^\varepsilon(m, k, h)$ have defining relations

$$x_i x_{i+m} = x_{i+k}.$$

This class of groups was introduced in [5], and subsequently studied in [1] and [6]. It contains many well-known groups, e.g., the *Fibonacci groups* $F(2, n)$, for $m = 1$ and $k = 2$, the *Sieradski groups* $S(n)$, for $m = 2$ and $k = 1$, and the *Gilbert–Howie groups*, for $k = 1$ (see [8], [21] and [9], respectively). For $n \geqslant 4$ even, the group $F(2, n)$ is the fundamental group of a closed connected orientable 3-manifold. This manifold can be represented as the $n/2$-fold cyclic covering of the 3-sphere branched over the figure-eight knot [13]. Moreover, the manifold has a hyperbolic structure for every $n \geqslant 8$ (even) [11]. On the other hand, if n is odd, then $F(2, n)$ cannot be the fundamental group of any hyperbolic 3-orbifold (in particular, 3-manifold) of finite volume [15]. For every $n \geqslant 2$, the group $S(n)$ is the fundamental group of the n-fold cyclic covering of the 3-sphere branched over the trefoil knot [4]. Arithmetic conditions on the parameters n and m for which the Gilbert–Howie groups are aspherical can be found in [9].

(2) If $h = k = s = 0$ and $m = 1$, then the groups $G_n^\varepsilon(m, k, h)$ have defining relations

$$x_i^p = x_{i+1}^q$$

where $p = a + b$ and $q = -a$ are integers. This class of groups was studied by Heil in [10]. He proved that if $|p| \neq |q|$, $|p| \neq 1$ and $|q| \neq 1$, then for every $n \geqslant 3$ the group is not the fundamental group of any 3-manifold (see Proposition 1 of [10]).

(3) If $h = k = r = s = 1$, $m = 0$ and $b = 1 - a$, then the groups $G_n^\varepsilon(m, k, h)$ have defining relations

$$x_i^a = x_{i+1} x_i x_{i+1}^{-1}.$$

For $a = 2$, these groups were first introduced by Higman in [12] (see also [18, pp. 546–548]). In [20] Schafer proved that for $n = 4$ the Higman group is not a 3-manifold group. Setting $y_i = x_i^{-1}$, the initial group have the defining relations

$$y_i^{-a} = y_{i+1}^{-1} y_i^{-1} y_{i+1}.$$

For $a = -2$, these groups were first considered by Mennicke in [17]. Therefore, we call the groups in (3) the *Higman–Mennicke groups*, denoted by $HM_n(a)$.

(4) If $h = m = s = 1$, $k = 0$ and $b = -2a$, then the groups $G_n^\varepsilon(m, k, h)$ have defining relations

$$x_i^a x_{i+1}^{2r} = x_{i+2}^a.$$

These groups form a subclass of the *fractional Fibonacci groups* studied in [14], and are denoted by $F^{2r/a}(n)$. For $n \geqslant 4$ even and $2r$ coprime to a, $F^{2r/a}(n)$ is the fundamental group of a closed connected orientable 3-manifold. This manifold can be obtained by Dehn surgery with rational coefficients $2r/a$ and $-2r/a$ on the components of an oriented link in the 3-sphere. The link is formed by a chain of n (even) unknotted circles, each one of them is linked with exactly two adjacent components with alternating crossings. If further $a = 1$, then these manifolds are examples of *generalised Fibonacci manifolds* [16]. Moreover, it was proved in [16] that such manifolds are hyperbolic for almost all r.

(5) If $h = k = m = s = 1$ and $a = 2r$, then the groups $G_n^\varepsilon(m, k, h)$ have defining relations

$$x_{i+1}^{p+q} = x_i^{-q} x_{i+1}^{q} x_{i+2}^{-q}$$

where $q = 2r$ and $p = b - 2r$. If p and q are coprime, then these groups are fundamental groups of closed connected orientable 3-manifolds which are examples of *Takahashi manifolds* (see [19] and [22]). Such manifolds can be represented as n-fold branched coverings of the lens space $L(p + 2q, q)$ (including the 3-sphere when $p + 2q = \pm 1$). Setting $y_i = x_i^{-1}$ gives the defining relations

$$y_{i+1}^{-p-q} = y_i^{q} y_{i+1}^{-q} y_{i+2}^{q}.$$

Taking the inverse relation we get

$$y_i^{p+q} = y_{i+1}^{-q} y_i^{q} y_{i-1}^{-q}.$$

If $p = 4r - 1$ and $q = 1 - 2r$, then we obtain the defining relations of the groups $G_n^\varepsilon(m, k, h)$, for $h = k = m = -s = -1$ and $a = -b = 2r - 1$, that is,

$$y_{i+1}^{2r-1} y_i^{-2r+1} y_{i-1}^{2r-1} = y_i^{2r}.$$

For $r \geqslant 1$, these groups are fundamental groups of the n-fold cyclic coverings of the 3-sphere branched over the 2-bridge knot $(8r - 3)/2$ (see [7]). In particular, if $r = 1$, then the knot $5/2$ is the figure-eight knot, so we again obtain the Fibonacci manifolds. Furthermore, the manifolds are hyperbolic for $r \geqslant 2$, $n \geqslant 3$ and $r = 1$, $n \geqslant 4$.

2. Algebraic properties

In this section we present some algebraic properties of the groups $G_n^\varepsilon(m, k, h)$, where $\varepsilon = (a, b, r, s) \in \mathbb{Z}^4$, $n \geqslant 2$, and k, h and m are reduced mod n. We consider repetitions within the family and prove that in some cases our groups decompose into non-trivial free products.

Lemma 2.1. *There are isomorphisms*

$$G_n^\varepsilon(m, k, h) \cong G_n^\varepsilon(n - m, k + 2n - h - m, n - h).$$

Proof. Let us denote $y_i = x_i^{-1}$ for $i = 1, \dots, n$. Taking the inverse relation of (1.1) gives

$$\bigl(x_{i+h+m}^{-1}\bigr)^a \bigl(x_{i+k}^{-1}\bigr)^b \bigl(x_i^{-1}\bigr)^a = \bigl(\bigl(x_{i+m}^{-1}\bigr)^r \bigl(x_{i+h}^{-1}\bigr)^r\bigr)^s$$

hence

$$y_{i+h+m}^a y_{i+k}^b y_i^a = \bigl(y_{i+m}^r y_{i+h}^r\bigr)^s.$$

Setting $j = i + h + m$ we can write the system of the defining relations in the form

$$y_j^a y_{j+k-h-m}^b y_{j-h-m}^a = \bigl(y_{j-h}^r y_{j-m}^r\bigr)^s$$

where $j = 1, \dots, n$ (subscripts mod n). Since the lower indices are taken mod n we can write

$$y_j^a y_{j+k+2n-h-m}^b y_{j+2n-h-m}^a = \bigl(y_{j+n-h}^r y_{j+n-m}^r\bigr)^s$$

which defines the groups $G_n^\varepsilon(n-m, k+2n-h-m, n-h)$. $\square$

Lemma 2.2. *If n and k are coprime, then the group $G_n^\varepsilon(m,k,h)$ is isomorphic to $G_m^\varepsilon(\pm mk', 1, \pm hk')$, where $kk' \equiv \pm 1 \pmod{n}$.*

Proof. Let n and k be coprime. Then we can re-order the generators of $G_n^\varepsilon(m,k,h)$ by defining

$$y_i = x_{1+(i-1)k}$$

for $i = 1, \dots, n$. Of course, the set $\{y_1, \dots, y_n\}$ coincides with the set $\{x_1, \dots, x_n\}$. The relations of $G_n^\varepsilon(m,k,h)$ can be written in the form

$$x_{1+(j-1)k}^a x_{1+jk}^b x_{1+(j-1)k\pm(h+m)kk'}^a = \bigl(x_{1+(j-1)k\pm hkk'}^r x_{1+(j-1)k\pm mkk'}^r\bigr)^s$$

hence

$$y_j^a y_{j+1}^b y_{j\pm(h+m)k'}^a = \bigl(y_{j\pm hk'}^r y_{j\pm mk'}^r\bigr)^s$$

for $j = 1, \dots, n$ (subscripts mod n). These relations define $G_n^\varepsilon(\pm mk', 1, \pm hk')$. $\square$

Lemma 2.3. *If $\gcd(n,h) = 1$ or $\gcd(n,m) = 1$, then there are isomorphisms $G_n^\varepsilon(m,k,h) \cong G_n^\varepsilon(\pm mh', \pm kh', 1)$ or $G_n^\varepsilon(m,k,h) \cong G_n^\varepsilon(1, \pm km', \pm hm')$, where $hh' \equiv \pm 1 \pmod{n}$ or $mm' \equiv \pm 1 \pmod{n}$, respectively.*

The proof of Lemma 2.3 is analogous to that of Lemma 2.2.

Lemma 2.4. *For any positive integer ℓ, the group $G_{n\ell}^\varepsilon(m\ell, k\ell, h\ell)$ is isomorphic to the free product of ℓ copies of $G_n^\varepsilon(m,k,h)$.*

Proof. For each $j = 1, \dots, \ell$, let G^ε_j be the subgroup of $G^\varepsilon_{n\ell}(m\ell, k\ell, h\ell)$ generated by the elements

$$x_j, x_{j+\ell}, \dots, x_{j+(n-1)\ell}$$

which may be not all distinct. Then G^ε_j is isomorphic to $G^\varepsilon_n(m,k,h)$. Of course, if $j \neq j'$, then the sets of generators of the groups G^ε_j and $G^\varepsilon_{j'}$ are disjoint. From the presentation of the group $G^\varepsilon_{n\ell}(m\ell, k\ell, h\ell)$, it follows that it is isomorphic to the free product $G^\varepsilon_1 * \cdots * G^\varepsilon_\ell$. $\square$

By Lemma 2.4 we shall only consider groups $G^\varepsilon_n(m,k,h)$ whose parameters (taken mod n) satisfy $0 \leqslant m, k, h, m+h < n$ and $\gcd(n,m,k,h) = 1$ (also without an explicit mention).

Lemma 2.5. *For a given group $G^\varepsilon_n(m,k,h)$, denote $u = \gcd(n,k,h)$, $\bar{u} = \gcd(n,k)$, $v = \gcd(u, k-h-m)$, and $\bar{v} = \gcd(\bar{u}, k-m, k-h)$. If $\gcd(u,v) > 1$ (respectively $\gcd(\bar{u},\bar{v}) > 1$), then $G^\varepsilon_n(m,k,h)$ decomposes into a non-trivial free product.*

Proof. Suppose for example $\rho = \gcd(u,v) > 1$. Then the integers n, m, k and h have ρ as a common divisor. So the statement follows from Lemma 2.4. The other case is analogous. $\square$

Theorem 2.6. *Suppose that $\rho = \gcd(n, k-h-m)$ divides k' and there exist positive integers α, β, γ and δ such that*

$$\begin{aligned}
\alpha + \beta(k-h-m) &\equiv 1-m \pmod{n},\\
\gamma + \delta(k-h-m) &\equiv 1-h \pmod{n},\\
\alpha + \beta k' &\equiv 1+m' \pmod{n},\\
\gamma + \delta k' &\equiv 1+h' \pmod{n}
\end{aligned}$$

where $1 \leqslant \alpha, \gamma \leqslant \rho$ and $1 \leqslant \beta, \delta \leqslant n/\rho$. Then $G^\varepsilon_n(m,k,h)$ is isomorphic to $G^\varepsilon_n(m',k',h')$.

Proof. By Lemma 2.1, the group $G^\varepsilon_n(m,k,h)$ has a finite presentation with generators $y_1, \dots, y_n$, and defining relations

$$y_i^a y_{i+k-h-m}^b y_{i-h-m}^a = \left(y_{i-h}^r y_{i-m}^r\right)^s$$

for $i = 1, \dots, n$. We set $\ell = n/\rho$, where $\rho = \gcd(n, k-h-m)$. Then we separate the generators $y_1, \dots, y_n$ into ρ sets $A_1, \dots, A_\rho$ of ℓ elements each one, where

$$A_j = \{y_j, y_{j+k-h-m}, \dots, y_{j+(\ell-1)(k-h-m)}\}$$

for $j = 1, \dots, \rho$. This gives a partition of the relations into ρ sets $R_1, \dots, R_\rho$ of ℓ elements each one, where R_j is formed by

$$y_j^a y_{j+k-h-m}^b y_{j-h-m}^a = \left(y_{j-h}^r y_{j-m}^r\right)^s,$$
$$y_{j+k-h-m}^a y_{j+2(k-h-m)}^b y_{j+k-2(h+m)}^a = \left(y_{j+k-m-2h}^r y_{j+k-h-2m}^r\right)^s,$$
$$\vdots$$
$$y_{j+(\ell-1)(k-h-m)}^a y_{j+\ell(k-h-m)}^b y_{j+(\ell-1)k-\ell(h+m)}^a = \left(y_{j+(\ell-1)(k-m)-\ell h}^r y_{j+(\ell-1)(k-h)-\ell m}^r\right)^s.$$

Observe that $y_{j+\ell(k-h-m)} = y_j$ for every $j = 1, \dots, \rho$ because $\ell(k-h-m) = (n/\rho)(k-h-m)$ is congruent to zero mod n. Therefore, for each relation of R_j the first two terms on the left side belong to A_j. Let us consider the presentation of $G_n^\varepsilon(m', k', h')$ with generators $z_1, \dots, z_n$, and defining relations

$$z_i^a z_{i+k'}^b z_{i+h'+m'}^a = \left(z_{i+h'}^r z_{i+m'}^r\right)^s.$$

We separate the generators $z_1, \dots, z_n$ into ρ sets $B_1, \dots, B_\rho$ of ℓ elements each one, where

$$B_j = \{z_j, z_{j+k'}, \dots, z_{j+(\ell-1)k'}\}$$

for every $j = 1, \dots, \rho$. As above, we obtain a partition of the defining relations of $G_n^\varepsilon(m', k', h')$ into ρ sets $S_1, \dots, S_\rho$ of ℓ elements each one, where S_j is formed by

$$z_j^a z_{j+k'}^b z_{j+h'+m'}^a = \left(z_{j+h'}^r z_{j+m'}^r\right)^s,$$
$$z_{j+k'}^a z_{j+2k'}^b z_{j+k'+h'+m'}^a = \left(z_{j+k'+h'}^r z_{j+k'+m'}^r\right)^s,$$
$$\vdots$$
$$z_{j+(\ell-1)k'}^a z_{j+\ell k'}^b z_{j+\ell k'+h'+m'}^a = \left(z_{j+(\ell-1)k'+h'}^r z_{j+(\ell-1)k'+m'}^r\right)^s.$$

Since ρ divides k', we have $z_{j+\ell k'} = z_j$ for every $j = 1, \dots, \rho$. Therefore, for each relation of S_j the first two terms on the left side belong to B_j. Let us define the correspondence ψ from $G_n^\varepsilon(m, k, h)$ onto $G_n^\varepsilon(m', k', h')$ by its action on the generators, i.e.,

$$\psi(y_{j+\tau(k-h-m)}) := z_{j+\tau k'}$$

for $1 \leqslant j \leqslant \rho$ and $0 \leqslant \tau \leqslant \ell - 1$. We check that each defining relation of $G_n^\varepsilon(m, k, h)$ goes under ψ to a defining relation of $G_n^\varepsilon(m', k', h')$, hence ψ is a group homomorphism. Let us consider the first relation of R_1, that is,

$$y_1^a y_{1+k-h-m}^b y_{1-h-m}^a = \left(y_{1-h}^r y_{1-m}^r\right)^s.$$

By hypothesis there exist positive integers α, β, γ and δ such that

$$\alpha + \beta(k-h-m) \equiv 1 - m \pmod{n}$$

and

$$\gamma + \delta(k-h-m) \equiv 1-h \pmod{n}.$$

Therefore, the relation above can be written in the form

$$y_1^a y_{1+k-h-m}^b y_{\alpha+\gamma-1+(\beta+\delta)(k-h-m)}^a = \left(y_{\gamma+\delta(k-h-m)}^r y_{\alpha+\beta(k-h-m)}^r\right)^s.$$

The image of this relation under ψ is

$$z_1^a z_{1+k'}^b z_{\alpha+\gamma-1+(\beta+\delta)k'}^a = \left(z_{\gamma+\delta k'}^r z_{\alpha+\beta k'}^r\right)^s.$$

Using the hypotheses

$$\alpha + \beta k' \equiv 1+m' \pmod{n},$$
$$\gamma + \delta k' \equiv 1+h' \pmod{n}$$

we get the relation

$$z_1^a z_{1+k'}^b z_{1+m'+h'}^a = \left(z_{1+h'}^r z_{1+m'}^r\right)^s.$$

This is the first relation of S_1, i.e., a defining relation of $G_n^\varepsilon(m',k',h')$. To complete the proof, it suffices to observe that all the defining relations of $G_n^\varepsilon(m,k,h)$ (respectively $G_n^\varepsilon(m',k',h')$) arise from the first one under cyclic permutations of the suffices. Therefore, ψ is a group homomorphism. It is easily seen that ψ is invertible, so it is an isomorphism. □

If $h = h' = 0$, then the conditions of Theorem 2.6 become

$$\alpha + \beta(k-m) \equiv 1-m \pmod{n},$$
$$\gamma + \delta(k-m) \equiv 1 \pmod{n},$$
$$\alpha + \beta k' \equiv 1+m' \pmod{n},$$
$$\gamma + \delta k' \equiv 1 \pmod{n}$$

where $\rho = \gcd(n, k-m)$ divides k'. So we can choose $\gamma = 1$ and $\delta = n/\rho$. This gives the following result which extends [1, Theorem 2.1], for which ε is $(1,1,2,1)$.

Corollary 2.7. *Suppose that $\rho = \gcd(n, k-m)$ divides k' and there exist positive integers α and β such that*

$$\alpha + \beta(k-m) \equiv 1-m \pmod{n},$$
$$\alpha + \beta k' \equiv 1+m' \pmod{n}$$

where $1 \leqslant \alpha \leqslant \rho$ *and* $1 \leqslant \beta \leqslant n/\rho$. *Then* $G_n^\varepsilon(m,k,0)$ *is isomorphic to* $G_n^\varepsilon(m',k',0)$ *for every* $\varepsilon = (a,b,r,s) \in \mathbb{Z}^4$.

As a particular case of Corollary 2.7 we obtain a result which extends [9, Lemma 2.1], for which ε is $(1,1,2,1)$.

Corollary 2.8. *Let* n *and* m *be positive integers such that* $m < n$ *and* n *is coprime to* $m-1$. *Let* m' *be an integer such that* $0 \leqslant m' < n$ *and* $(m-1)m' \equiv m \pmod{n}$. *Then* $G_n^\varepsilon(m,1,0)$ *and* $G_n^\varepsilon(m',1,0)$ *are isomorphic for any* $\varepsilon = (a,b,r,s) \in \mathbb{Z}^4$.

Proof. Apply Corollary 2.7 for $\rho = \alpha = k = k' = 1$ and $\beta = m'$. $\square$

Example. There are isomorphisms

$$G_7^\varepsilon(2,6,3) \cong G_7^\varepsilon(3,2,1) \cong G_7^\varepsilon(1,3,5) \cong G_7^\varepsilon(6,4,2).$$

Let $n = 7$, $(m,k,h) = (2,6,3)$, $(m',k',h') = (3,2,1)$, $(m'',k'',h'') = (1,3,5)$, and $(m''',k''',h''') = (6,4,2)$. Then we have $\rho = \gcd(n, k-h-m) = \gcd(7,1) = 1$. We can take $\alpha = \gamma = 1$, $\beta = 5$ and $\delta = 4$ to satisfy the conditions of Theorem 2.6.

The following arises in a natural way from the arguments discussed above:

Problem 2.9. Find a finite system of arithmetic conditions on the parameters which completely determines the isomorphism type of the group $G_n^\varepsilon(m,k,h)$.

3. Groups $G_n^\varepsilon(m,k,h)$ with n odd

The following is our main result.

Theorem 3.1. *Suppose that* n *and* b *are odd and* n *is coprime with* $2k-h-m$. *Then the group* $G_n^\varepsilon(m,k,h)$ *cannot be the fundamental group of any hyperbolic* 3*-orbifold* (*in particular,* 3*-manifold*) *of finite volume.*

Proof. Let $G_n^\varepsilon = G_n^\varepsilon(m,k,h)$ be the fundamental group of a hyperbolic 3-dimensional orbifold (in particular, 3-manifold) of finite volume. Then there is a faithful representation

$$f : G_n^\varepsilon \to \mathrm{Isom}\bigl(\mathbb{H}^3\bigr)$$

such that $F_n^\varepsilon = f(G_n^\varepsilon)$ is a hyperbolic group, that is, a discrete group of finite covolume. Of course, F_n^ε admits the automorphism θ which cyclically permutes the generators, i.e., $\theta(x_i) = x_{i+1}$ (subscripts mod n). By abuse of language we denote the generators of G_n^ε and F_n^ε with the same symbols. By the Mostow rigidity theorem there exists an isometry $t \in \mathrm{Isom}(\mathbb{H}^3)$ such that $\theta(u) = t^{-1}ut$ for every $u \in F_n^\varepsilon$. Let us consider the split extension of F_n^ε by the cyclic group generated by t, and denote it by E_n^ε. Then E_n^ε is the fundamental group of a hyperbolic 3-dimensional orbifold of finite volume. Since $\theta^n = 1$, t^n commutes with all elements of the non-elementary Kleinian group F_n^ε. So t^n belongs to the center of F_n^ε which is trivial by [2]. Therefore, t is of order n', where n' divides n. Substituting $x_{i+1} = t^{-1}x_i t = t^{-i}x_1 t^i$ in the initial relation of F_n^ε:

$$x_1^a x_{1+k}^b x_{1+h+m}^a = \left(x_{1+h}^r x_{1+m}^r\right)^s$$

yields

$$x_1^a t^{-k} x_1^b t^{k-h-m} x_1^a t^{h+m} = \left(t^{-h} x_1^r t^{h-m} x_1^r t^m\right)^s. \tag{3.1}$$

Obviously, the split extension E_n^ε has a finite presentation with generators x_1 and t, and relations $t^{n'} = 1$ and (3.1). Let us consider the subgroup $(E_n^\varepsilon)^{(2)}$ generated by the squares of the elements in E_n^ε. If n (and hence n') is odd, then $t \in (E_n^\varepsilon)^{(2)}$. The element on the right side of (3.1) belongs to $(E_n^\varepsilon)^{(2)}$ as

$$\left(t^{-h} x_1^r t^{h-m} x_1^r t^m\right)^s = \left(t^{-h}\left(x_1^r t^{h-m}\right)^2 t^{2m-h}\right)^s \in \left(E_n^\varepsilon\right)^{(2)}.$$

For the left side of (3.1) we have

$$\begin{aligned} &x_1^a t^{-k} x_1^b t^{k-h-m} x_1^a t^{h+m} \\ &\quad = x_1^{-b}\left(x_1^a\right)^2 t^k \left(t^{-k} x_1^{b-a}\right)^2 \left(x_1^a t^{k-h-m}\right)^2 t^{2(h+m)-k} \in \left(E_n^\varepsilon\right)^{(2)}. \end{aligned}$$

Since $(x_1^a)^2 t^k (t^{-k} x_1^{b-a})^2 (x_1^a t^{k-h-m})^2 t^{2(h+m)-k} \in (E_n^\varepsilon)^{(2)}$, it follows that x_1 belongs to $(E_n^\varepsilon)^{(2)}$ when b is odd. Therefore, the hypotheses imply $E_n^\varepsilon = (E_n^\varepsilon)^{(2)}$, i.e., E_n^ε is a subgroup of the group $\mathrm{PSL}(2,\mathbb{C})$ of orientation-preserving isometries of $\mathbb{H}^3$. Let us denote by $P(A)$ the image in $\mathrm{PSL}(2,\mathbb{C})$ of a matrix $A \in \mathrm{SL}(2,\mathbb{C})$ under the 2-fold covering

$$P : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\{\pm \mathrm{I}_2\}.$$

Since t is of order n', we can assume without loss of generality that

$$t = P\begin{pmatrix} \varphi & 0 \\ 0 & \varphi^{-1} \end{pmatrix}$$

where φ is a primitive root of the unity in $\mathbb{C}$ of degree $2n'$. Let

$$x_1 = P\begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

with $xw - yz = 1$. Since F_n^ε is of finite covolume, we have $yz \neq 0$. For any j we have

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix}^j = \begin{pmatrix} S_j & yR_j \\ zR_j & T_j \end{pmatrix}$$

whose determinant is

$$S_j T_j - yzR_j^2 = 1. \tag{3.2}$$

Now we substitute the above matrices in relation (3.1). From the element on the left side we get

$$\begin{aligned}
A &= \begin{pmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{pmatrix} \\
&= \begin{pmatrix} x & y \\ z & w \end{pmatrix}^a \begin{pmatrix} \varphi^{-k} & 0 \\ 0 & \varphi^k \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix}^b \begin{pmatrix} \varphi^{k-h-m} & 0 \\ 0 & \varphi^{h+m-k} \end{pmatrix} \\
&\quad \times \begin{pmatrix} x & y \\ z & w \end{pmatrix}^a \begin{pmatrix} \varphi^{h+m} & 0 \\ 0 & \varphi^{-h-m} \end{pmatrix} \\
&= \begin{pmatrix} S_a & yR_a \\ zR_a & T_a \end{pmatrix} \begin{pmatrix} \varphi^{-k} & 0 \\ 0 & \varphi^k \end{pmatrix} \begin{pmatrix} S_b & yR_b \\ zR_b & T_b \end{pmatrix} \begin{pmatrix} \varphi^{k-h-m} & 0 \\ 0 & \varphi^{h+m-k} \end{pmatrix} \\
&\quad \times \begin{pmatrix} S_a & yR_a \\ zR_a & T_a \end{pmatrix} \begin{pmatrix} \varphi^{h+m} & 0 \\ 0 & \varphi^{-h-m} \end{pmatrix} \\
&= \begin{pmatrix} \varphi^{-k} S_a & y\varphi^k R_a \\ z\varphi^{-k} R_a & \varphi^k T_a \end{pmatrix} \begin{pmatrix} \varphi^{k-h-m} S_b & y\varphi^{h+m-k} R_b \\ z\varphi^{k-h-m} R_b & \varphi^{h+m-k} T_b \end{pmatrix} \begin{pmatrix} \varphi^{h+m} S_a & y\varphi^{-h-m} R_a \\ z\varphi^{h+m} R_a & \varphi^{-h-m} T_a \end{pmatrix}
\end{aligned}$$

hence

$$\begin{aligned}
a_1^1 &= S_a^2 S_b + yz\varphi^{2k} R_a R_b S_a + yz\varphi^{2(h+m-k)} R_a R_b S_a + yz\varphi^{2(h+m)} R_a^2 T_b, \\
a_2^1 &= y\varphi^{-2(h+m)} R_a S_a S_b + y^2 z\varphi^{2(k-h-m)} R_a^2 R_b + y\varphi^{-2k} R_b S_a T_a + yR_a T_a T_b, \\
a_1^2 &= zR_a S_a S_b + z\varphi^{2k} R_b S_a T_a + yz^2\varphi^{2(h+m-k)} R_a^2 R_b + z\varphi^{2(h+m)} R_a T_a T_b, \\
a_2^2 &= yz\varphi^{-2(h+m)} R_a^2 S_b + yz\varphi^{2(k-h-m)} R_a R_b T_a + yz\varphi^{-2k} R_a R_b T_a + T_a^2 T_b.
\end{aligned}$$

From the element on the right side of (3.1) we obtain

$$\begin{aligned}
A &= \begin{pmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{pmatrix} \\
&= \left(\begin{pmatrix} \varphi^{-h} & 0 \\ 0 & \varphi^h \end{pmatrix} \begin{pmatrix} S_r & yR_r \\ zR_r & T_r \end{pmatrix} \begin{pmatrix} \varphi^{h-m} & 0 \\ 0 & \varphi^{m-h} \end{pmatrix} \begin{pmatrix} S_r & yR_r \\ zR_r & T_r \end{pmatrix} \begin{pmatrix} \varphi^m & 0 \\ 0 & \varphi^{-m} \end{pmatrix} \right)^s \\
&= \left(\begin{pmatrix} \varphi^{-h} S_r & y\varphi^{-h} R_r \\ z\varphi^h R_r & \varphi^h T_r \end{pmatrix} \begin{pmatrix} \varphi^h S_r & y\varphi^{h-2m} R_r \\ z\varphi^{2m-h} R_r & \varphi^{-h} T_r \end{pmatrix} \right)^s \\
&= \begin{pmatrix} S_r^2 + yz\varphi^{2(m-h)} R_r^2 & y\varphi^{-2m} R_r S_r + y\varphi^{-2h} R_r T_r \\ z\varphi^{2h} R_r S_r + z\varphi^{2m} R_r T_r & yz\varphi^{2(h-m)} R_r^2 + T_r^2 \end{pmatrix}^s \\
&= \begin{pmatrix} \overline{S}_s & (y\varphi^{-2m} R_r S_r + y\varphi^{-2h} R_r T_r)\overline{R}_s \\ (z\varphi^{2h} R_r S_r + z\varphi^{2m} R_r T_r)\overline{R}_s & \overline{T}_s \end{pmatrix}.
\end{aligned}$$

Equating the corresponding elements of the resulting matrix (and using $yz \neq 0$) we obtain

$$\begin{cases} \varphi^{-2(h+m)} R_a S_a S_b + yz\varphi^{2(k-h-m)} R_a^2 R_b + \varphi^{-2k} R_b S_a T_a + R_a T_a T_b \\ \quad = (\varphi^{-2m} R_r S_r + \varphi^{-2h} R_r T_r)\overline{R}_s, \\ R_a S_a S_b + \varphi^{2k} R_b S_a T_a + yz\varphi^{2(h+m-k)} R_a^2 R_b + \varphi^{2(h+m)} R_a T_a T_b \\ \quad = (\varphi^{2h} R_r S_r + \varphi^{2m} R_r T_r)\overline{R}_s. \end{cases}$$

Multiplying the first (respectively second) equation by φ^{2h} (respectively φ^{-2m}) yields

$$\begin{cases} \varphi^{-2m} R_a S_a S_b + yz\varphi^{2(k-m)} R_a^2 R_b + \varphi^{2(h-k)} R_b S_a T_a + \varphi^{2h} R_a T_a T_b \\ \quad = (\varphi^{2(h-m)} R_r S_r + R_r T_r)\overline{R}_s, \\ \varphi^{-2m} R_a S_a S_b + \varphi^{2(k-m)} R_b S_a T_a + yz\varphi^{2(h-k)} R_a^2 R_b + \varphi^{2h} R_a T_a T_b \\ \quad = (\varphi^{2(h-m)} R_r S_r + R_r T_r)\overline{R}_s. \end{cases}$$

Subtracting the equations we get

$$\varphi^{2(h-k)} R_b\bigl(S_a T_a - yzR_a^2\bigr) - \varphi^{2(k-m)} R_b\bigl(S_a T_a - yzR_a^2\bigr) = 0$$

hence

$$\bigl(\varphi^{2(h-k)} - \varphi^{2(k-m)}\bigr)R_b = 0,$$

by using (3.2). Since F_n^ε is of finite covolume and $x_1^b \in F_n^\varepsilon$, we have $yzR_b^2 \neq 0$. Thus the last equation gives

$$\varphi^{2(2k-h-m)} = 1.$$

But φ is a primitive root of the unity in $\mathbb{C}$ of degree $2n'$, and n' is coprime to $2k-h-m$. This gives a contradiction. Therefore, G_n^ε cannot be the fundamental group of a hyperbolic 3-orbifold (respectively 3-manifold) of finite volume. □

Corollary 3.2. *Suppose that the automorphism θ which cyclically permutes the generators of $G_n^\varepsilon(m,k,h)$ is exactly of order n. If n and b are odd and n does not divide $2k-h-m$, then $G_n^\varepsilon(m,k,h)$ cannot be the fundamental group of any hyperbolic 3-orbifold (respectively 3-manifold) of finite volume.*

The conditions of Corollary 3.2 are satisfied for example by the Fibonacci groups $F(2,n) = G_n^\varepsilon(m,k,h)$, where $\varepsilon = (a,b,r,s) = (1,1,2,1)$, $m=1$, $k=2$, $h=0$, and n is odd and greater than 3. (If $n=3$, then $F(2,n)$ is a finite group.) As special cases of Theorem 3.1 and Corollary 3.2, one can obtain the results on the non-hyperbolicity of certain groups of Fibonacci type proved in [1,6,15]. As a further result, we have

Corollary 3.3. *Let $HM_n(a)$ be the Higman–Mennicke group with generators $x_1,\dots,x_n$ and defining relations $x_i^a = x_{i+1}x_i x_{i+1}^{-1}$ for $i = 1,\dots,n$ (subscripts* mod n). *If a is even* *and n is odd, then $HM_n(a)$ cannot be the fundamental group of a hyperbolic* 3*-orbifold* (*respectively* 3*-manifold*) *of finite volume.*

The following arises in a natural way from the above results:

Problem 3.4. Determine all values of the parameters $\varepsilon = (a, b, r, s)$, m, k and h for which $G_n^\varepsilon(m, k, h)$ is the fundamental group of closed connected orientable 3-manifolds for infinitely many n. Then classify the topological and geometric structures of such manifolds.

4. Asphericity

In this section we investigate the asphericity for groups $G_n^\varepsilon(m, k, h)$, where $a = b = 1$ and $s = 0$. These groups, denoted in short by $G_n = G_n(k, \ell)$, have generators $x_1, \dots, x_n$, and defining relations $x_i x_{i+k} x_{i+\ell} = 1$, where $\ell = h + m$. By Lemma 2.1 there are isomorphisms $G_n(k, \ell) \cong G_n(k - \ell, n - \ell)$. By Lemma 2.4 we assume $\gcd(n, k, \ell) = 1$. If $k = 0$ or $k = \ell$, then $G_n(k, \ell)$ is a cyclic group of order $|(-2)^n - 1|$. Then the parameters can be chosen so that $0 < k < \ell < n$ and $\gcd(n, k, \ell) = 1$. Form the split extension $E_n = E_n(k, \ell)$ by $\mathbb{Z}_n$. Here $\mathbb{Z}_n$ acts by cyclic permutation of the generators $x_1, \dots, x_n$. If $\mathbb{Z}_n$ is generated by σ, and we set $x = x_1$ in G_n, then E_n is generated by x and σ, and has the finite presentation

$$\langle x, \sigma\colon\ \sigma^n = 1,\ x\sigma^{-k}x\sigma^{k-\ell}x\sigma^{\ell} = 1\rangle.$$

We can regard E_n as a relative presentation in the sense of [3], i.e.,

$$\langle H, x\colon\ x\sigma^{-k}x\sigma^{k-\ell}x\sigma^{\ell} = 1\rangle,$$

where $H = \langle \sigma\colon\ \sigma^n = 1\rangle \cong \mathbb{Z}_n$.

Lemma 4.1. *If the relative presentation of $E_n(k, \ell)$ is aspherical, then the absolute presentation of $G_n(k, \ell)$ is also aspherical.*

Proof. Let $\mathbf{P}$ be a spherical picture over G_n (see [3] for more details on pictures over relative presentations and aspherical relative presentations). Replace each disc (Fig. 1) of $\mathbf{P}$ by the picture $\mathbf{Q}_i$ (Fig. 2) over E_n regarded as an absolute presentation. Here we have replaced each arc labelled x_i by a sequence of arcs with total label $\sigma^{-(i-1)}x\sigma^{i-1}$. The arcs of $\mathbf{Q}_i$ having both endpoints on the boundary can be made into floating circles. Thus they can be deleted from the resulting picture. Then the remaining arcs labelled σ are deleted and replaced by corner labels on the discs of the picture $\mathbf{Q}_i$ (Fig. 3). In this way we obtain a picture $\mathbf{Q}$ over the relative presentation of E_n. Since the relative presentation of E_n is aspherical, it must contain a dipole, i.e., a pair of oppositely oriented discs, connected by an arc of the picture, which carry inverse labels when read from the connecting arc (Fig. 4). It is easy to see that any such dipole in $\mathbf{Q}$ must arise from a pair of identical but oppositely oriented discs in $\mathbf{P}$ which were connected by an arc labelled x_i for some i.

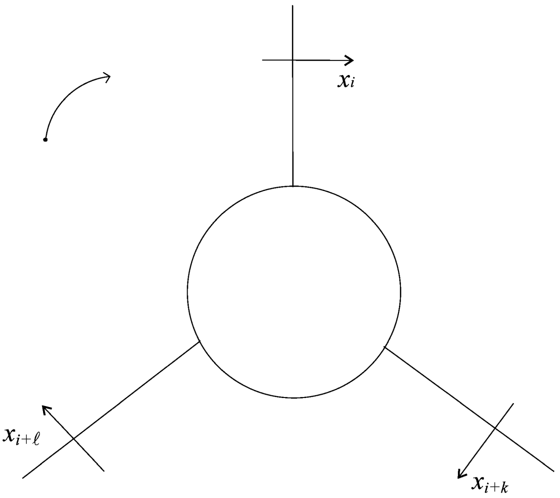

Fig. 1. A disc of the spherical picture **P** over the absolute presentation of $G_n(k, \ell)$.

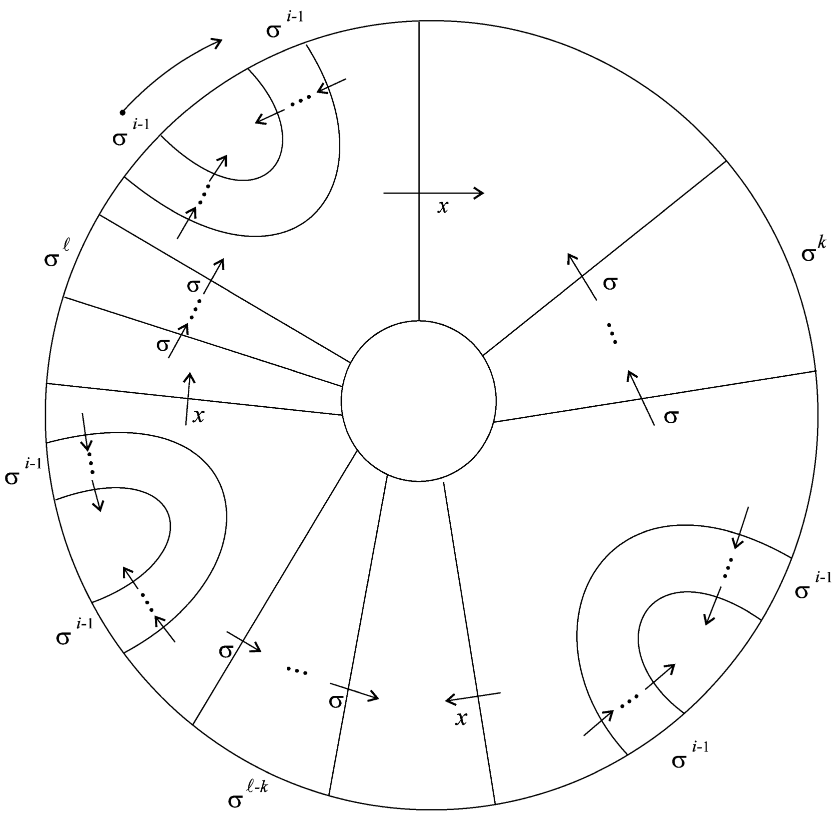

Fig. 2. The picture $\mathbf{Q}_i$ over the absolute presentation of $E_n(k, \ell)$.

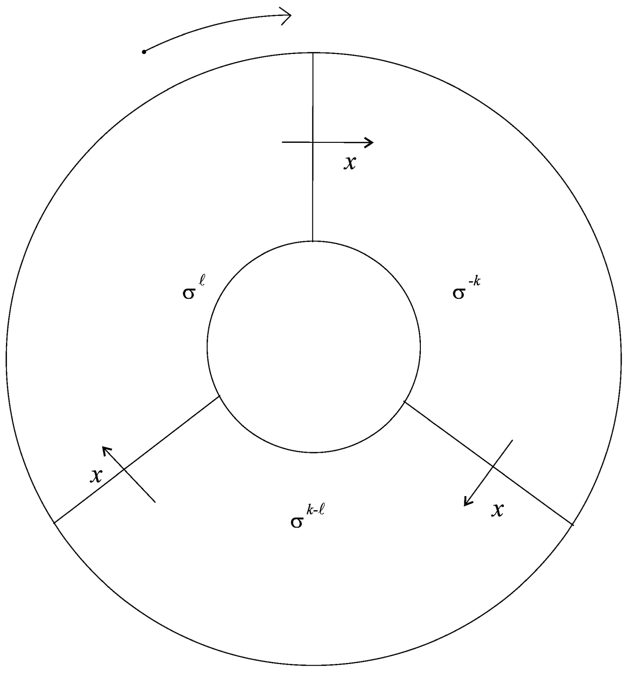

Fig. 3. A disc of the spherical picture **Q** over the relative presentation of $E_n(k,\ell)$.

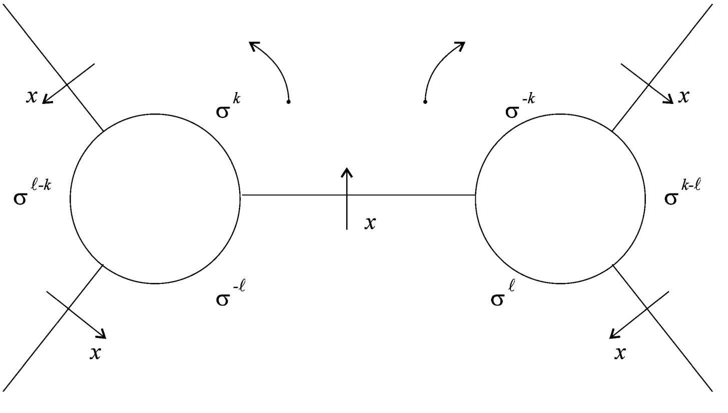

Fig. 4. A dipole in the spherical picture **Q**.

Furthermore, two bridge moves in **P** produce a cancelling pair of discs. Therefore, any non-empty spherical picture over G_n is equivalent to one having two fewer discs. This implies that the absolute presentation of G_n is aspherical. $\square$

To study the asphericity of the relative presentation of E_n we use the following result due to Bogley and Pride (see [3, Theorem 3.1]).

Theorem 4.2. *Let a_1, a_2 and a_3 be elements of a group H such that $\{a_1, a_2, a_3\}$ contains at least two elements. The relative presentation*

$$\langle H, x\colon\ xa_1xa_2xa_3 = 1\rangle$$

is aspherical if and only if neither of the following conditions holds:

(1) *For $i = 1, 2, 3$, $a_i a_{i+1}^{-1}$ has finite order $p_i > 0$ (subscripts $\bmod 3$), and*

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} > 1.$$

(2) *There exist $j \in \{1, 2, 3\}$, $p > 2$, and $0 \leqslant \alpha < p$ such that*

$$\operatorname{sgp}\bigl\{a_i a_{i+1}^{-1}\colon\ i = 1, 2, 3\bigr\}$$

is finite cyclic with generator $a_j a_{j+1}^{-1}$ of order p, and $a_{j+1} a_{j+2}^{-1} = (a_j a_{j+1}^{-1})^{\alpha}$, where either
(2.1) $\alpha = 1$;
(2.2) $p = \alpha + 2$ *or* $p = 2\alpha + 1$; *or*
(2.3) $p = 6$ *and* $\alpha = 2$ *or* 3.

We apply this result to our case when $a_1 = \sigma^{-k}$, $a_2 = \sigma^{k-\ell}$ and $a_3 = \sigma^{\ell}$. Then $a_1 a_2^{-1} = \sigma^{\ell-2k}$, $a_2 a_3^{-1} = \sigma^{k-2\ell}$ and $a_3 a_1^{-1} = \sigma^{k+\ell}$ have orders

$$p_1 = \frac{n}{\gcd(n, \ell - 2k)}, \quad p_2 = \frac{n}{\gcd(n, k - 2\ell)}, \quad \text{and} \quad p_3 = \frac{n}{\gcd(n, k + \ell)},$$

respectively. Then we have

Theorem 4.3. *Let $G_n(k, \ell)$ be the cyclically presented group with generators $x_1, \ldots, x_n$, and defining relations $x_i x_{i+k} x_{i+\ell} = 1$, for $i = 1, \ldots, n$ (subscripts $\bmod n$). Suppose that $0 < k < \ell < n$ and $\gcd(n, k, \ell) = 1$. Then $G_n(k, \ell)$ is aspherical if none of the following conditions is satisfied*:

(1) $\gcd(n, \ell - 2k) + \gcd(n, k - 2\ell) + \gcd(n, k + \ell) > n$,
(2) $n = 6\gcd(n, \ell - 2k)$ *and* 6 *divides* $2\ell - k$ *or* $k + \ell$,
(3) $n = 6\gcd(n, k - 2\ell)$ *and* 6 *divides* $-2k + \ell$ *or* $k + \ell$,
(4) $n = 6\gcd(n, k + \ell)$ *and* 6 *divides* $2\ell - k$ *or* $\ell - 2k$.

In this case, $G_n(k, \ell)$ is torsion free, and if it is non-trivial, then it is infinite.

The following arises in a natural way:

Problem 4.4. Find necessary and sufficient conditions on the parameters for the asphericity of the groups $G_n^{\varepsilon}(m, k, h)$ in the general case.

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