JOURNAL OF ALGEBRA 82, 53–83 (1983)

Approximation Complexes of Blowing-Up Rings, II

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Communicated by D. A. Buchsbaum

Received November 17, 1981

0. INTRODUCTION

This paper is a sequel to [11], where we studied the relationships that hold between the arithmetical properties of the Rees ring, $\mathscr{R}(I)$, and the symmetric algebra, $\mathrm{Sym}(I)$, of an ideal I (and some of their fibers) and the depth properties of the Koszul homology modules, $H_i(I; R)$, on a set of generators of I. The connection between these objects is realized by certain differential graded algebras—the so-called approximation complexes—that are built out of ordinary Koszul complexes. These complexes show, however, different sensitivities, being acyclic in situations much broader than the usual context of regular sequences. It has been found that certain extensions thereof, d sequences and proper sequences, play here that role of "acyclic sequence."

The interplay between sequences and approximation complexes will be a basic point of view here, where we give:

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† The author was partially supported by a National Science Foundation grant.

0021-8693/83 $3.00

(a) A general result on the existence of systems of parameters which are d sequences, with applications to a local "on the nose" Cohen–Macaulayfication of local rings.

(b) Robust conditions under which the approximation complex labelled $\mathcal{M}$ in [11] is acyclic, in terms of d sequences and higher syzygietic conditions, and its relation to the theory of linear resolutions of regular local rings.

(c) Variations on the "main theorem" of [11] with a weakening of hypotheses, which allow for treating new examples of Cohen–Macaulay symmetric algebras and their specializations.

We now briefly describe the contents of each section.

In Section 1 we review the construction of the approximation complexes (and some of their properties) in a larger set-up, namely, we allow for coefficients in a module. This is not merely a natural generalization, it is in fact needed for the application of the theory to certain finite free resolutions (cf. Section 5).

The notion of a d sequence plays (for the approximation complexes) a role comparable to that of regular sequence in the theory of the ordinary Koszul complex. It was first introduced by Huneke [12], who has proved important results on the analytic properties of ideals generated by d sequences (cf. also [11, 20]). In Section 2 we consider such sequences (and a companion notion, proper sequence) and show that they occur quite often in arbitrary Noetherian rings. Precisely, every ideal of height n is shown to contain a d sequence of n elements which is a system of parameters (Proposition 2.7). It is used (Theorem 2.8) to obtain a coarse form of Cohen–Macaulayfication.

The Section 3 is devoted to a comparison among various homologies. Namely, let $I = (x_1, ..., x_n)$ be an ideal of R and let M be an R module. We view $\mathscr{R}(I; M)$ and $\mathrm{gr}_I(M)$, respectively, the Rees module of M relative to I and the associated graded module, as modules over the polynomial ring $R[e_1, ..., e_n]$ and compare the following Koszul homology modules: $H_i(\mathbf{x}; M)$, $H_i(\mathbf{e}; \mathscr{R}(I; M))$, and $H_i(\mathbf{e}; \mathrm{gr}_I(M))$. The main result (Theorem 3.7) states that the acyclicity of the associated $\mathcal{M}$ complex is decided at the level of the homologies above. It also sets the groundwork for Section 4, where we prove one of our main results (Theorem 4.1): Given an ideal I and a module M, then $\mathcal{M}(I; M)$ is acyclic essentially when (locally and possibly after a base change) I is generated by a d sequence with respect to M.

The purpose of Section 5 is to exploit the earlier technical developments in describing the complex $\mathcal{M}(\mathfrak{m}; M)$, where $\mathfrak{m}$ is the maximal ideal of the regular local ring R. In this case, $\mathcal{M}(\mathfrak{m}; M)$ is closely related to a minimal free resolution of M. Namely, such a minimal resolution $\mathscr{F}$ admits a natural filtration on each of its terms; the associated graded complex $\mathrm{gr}(\mathscr{F})$ is then a complex of free $\mathrm{gr}_{\mathscr{m}}(R)$ modules. Our first relevant result in this section is that the complexes $\mathscr{M}(\mathscr{m}; M)$ and $\mathrm{gr}(\mathscr{F})$ are isomorphic (Theorem 5.1).

We next consider modules that admit a linear resolution. In a precise way, we say that the R module M has a linear resolution if $\mathrm{gr}(\mathscr{F})$ as above is acyclic (this coincides with the standard notion when R and M have an underlying graded structure). Applying Theorem 5.1 along with the results of Section 4, we obtain a criterion for modules to admit linear resolutions in terms of d sequences. In addition it yields information on the Hilbert function of M and the depth of the associated graded module $\mathrm{gr}_{\mathscr{m}}(M)$ (Corollary 5.3). It is also used to show that the complex $\mathscr{M}$, in contrast to the ordinary Koszul complex, is not rigid (Example 5.5b). We also single out the characterization of ideals generated by (uniform) standard bases that admit linear resolutions (Theorem 5.7).

Finally, in Section 6 we examine various behaviours of the blowing-up rings with respect to Cohen-Macaulayness. First, we give a slightly improved version of the main theorem of [11], by means of writing hypotheses on the depth of the Koszul homology which are perhaps more attainable than the ones in [11, (2.6)]. Despite the lost clarity in such a replacement, we regain it in better understanding the balance between the new hypothesis and the standing hypothesis on the local number of generators (Theorem 6.1). Parallel assumptions can be made in order that the symmetric algebra of an ideal be Cohen–Macaulay, but not necessarily equal to the Rees algebra (Theorem 6.10). We also state a variant of that main theorem, having in mind the replacement of conditions that warrant acyclicity of the approximation complexes by others directly in terms of the objects that concern us (Theorem 6.4). To prove this and other statements we found it useful to rewrite some of the technical results of [21], by pushing them further (e.g., Propositions 6.5 and 6.8 and Corollary 6.7). Finally, we give conditions under which the symmetric algebra specializes well under Cohen–Macaulayness (Theorem 6.15). This is in the vein of recent results on the specialization of Rees algebras (cf. [7]).

1. Preliminaries

Throughout, all rings are commutative, Noetherian, and modules are finitely generated. For terminology and basic results on Noetherian rings, especially on Cohen–Macaulay rings, our sources are [15, 17].

Let us recall the construction of the approximation complexes associated to a sequence $\mathbf{x} = \{x_1, \dots, x_n\}$ of elements in a ring R (cf. [11, 20, 21]). Furthermore, it will be necessary to extend that definition by attaching coefficients from an R module M.

To the sequence $\mathbf{x}$ and the R module M there is an associated double complex $\{\mathscr{L}_{r,s}, \partial, \partial'\}$: $\mathscr{L}_{r,s} = M \otimes_R K_r \otimes_R S_s$, with $K_r = \Lambda^r(F) =$ rth exterior power of F and $S_s = \mathrm{Sym}_s(F) =$ sth symmetric power of F, where F is a free R module with basis $\{e_1,..., e_n\}$—one generator for each element in the sequence $\mathbf{x}$. We shall also denote $S = \mathrm{Sym}(F) = R[e_1,..., e_n]$, or, for emphasis, $S = R[T_1,..., T_n]$. The differentials ∂ and ∂' are, respectively, obtained by viewing $\mathscr{L}$ as either $\mathscr{K}^R(\mathbf{x}; M) \otimes_R S = \mathscr{K}^S(\mathbf{x}; M \otimes_R S)$, the Koszul complex associated to the sequence $\mathbf{x}$ in the ring S and the module $M \otimes_R S$; or as $\mathscr{K}^S(\mathbf{e}; M \otimes_R S)$, the Koszul complex associated to the sequence $\mathbf{e} = \{e_1,..., e_n\}$ in the ring S and the module $M \otimes_R S$. It is easy to verify that ∂ and ∂' commute.

Denote by Z the submodule of cycles of the Koszul complex $\mathscr{K}(\mathbf{x}; M)$, and by H its homology. The double complex $\mathscr{L}$ gives rise to several complexes among which we single out

$$\mathscr{Z} = \mathscr{Z}(\mathbf{x}; M) = \{Z \otimes_R S, \partial'\}$$

and

$$\mathscr{M} = \mathscr{M}(\mathbf{x}; M) = \{H \otimes_R S, \partial'\}.$$

The complexes are called the *approximation complexes* associated with $\mathbf{x}$ and M.

The $\mathscr{M}$ complex, as well the $\mathscr{Z}$ complex, are graded complexes over S. The tth homogeneous part $\mathscr{M}_t$ of $\mathscr{M}$ is a complex of finitely generated R modules of the form

$$\mathscr{M}_t: \qquad 0 \to H_n \otimes S_{t-n} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} H_1 \otimes S_{t-1} \xrightarrow{\partial'} H_0 \otimes S_t \to 0.$$

In a similar manner $\mathscr{Z}_t$ is a complex of finitely generated R modules. For certain uses, however, we must view $\mathscr{M}$ and $\mathscr{Z}$ as complexes over the polynomial ring S. Thus, for instance, the $\mathscr{Z}$ complex may be written

$$\mathscr{Z}: \qquad 0 \to Z_n \otimes S[-n] \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} Z_1 \otimes S[-1] \xrightarrow{\partial'} Z_0 \otimes S \to 0,$$

where, as usual, $S[k]$ denotes the graded S module with S_{k+m} for its component in degree m.

We shall now points out some properties of these complexes, as developed in [11, 20, 21].

(1.1) The homologies of the complexes $\mathscr{Z}(\mathbf{x}; M)$ and $\mathscr{M}(\mathbf{x}; M)$ do not depend on the sequence $\mathbf{x}$ but only on the ideal I it generates ([21]). This is due to an underlying mapping cone construction in the complex $\mathscr{L}$ and contrasts sharply with what occurs in the ordinary Koszul homology modules. For this reason we shall often denote these complexes by $\mathscr{Z}(I; M)$

and $\mathcal{M}(I; M)$, or, if I is fixed in a discussion, by $\mathcal{Z}(M)$ and $\mathcal{M}(M)$. If M is fixed as well, as above we shall denote the complexes simply by $\mathcal{Z}$ and $\mathcal{M}$.

(1.2) A major source of interest for introducing these complexes occur when one takes $M = R$. The $\mathcal{Z}$ and $\mathcal{M}$ complexes are then differential graded algebras with $H_0(\mathcal{Z}) = \mathrm{Sym}(I)$ and $H_0(\mathcal{M}) = \mathrm{Sym}(I/I^2)$, $I = (\mathrm{x})$. At the same time they permit comparisons of these rings to the more standard blowing-up rings represented by the Rees algebra $\mathcal{R}(I) = \bigoplus_j I^j T^j$ and the associated graded ring $\mathrm{gr}_I(R) = \bigoplus_j I^j/I^{j+1}$, respectively.

For $M \neq R$, the meaning of $H_0(\mathcal{Z}(I; M))$ and $H_0(\mathcal{M}(I; M))$ is not so clear-cut. There are, however, natural surjections

$$\alpha\colon \quad H_0(\mathcal{Z}(I; M)) \to \mathcal{R}(I; M) = \bigoplus_j I^j M = \text{associated Rees module,}$$

and

$$\beta\colon \quad H_0(\mathcal{M}(I; M)) \to \mathrm{gr}_I(R) = \bigoplus_j I^j M/I^{j+1} M = \text{associated graded module.}$$

(1.3) The single complex $\{\mathcal{L}, \partial'\}$ is acyclic. This fact triggers several relationships between the two approximation complexes. In particular, for each integer t, we have the long exact homology sequence (cf. [11, (2.1)])

$$\cdots \to H_r(\mathcal{Z}_{t+1}) \to H_r(\mathcal{Z}_t) \to H_r(\mathcal{M}_t) \to H_{r-1}(\mathcal{Z}_{t+1}) \to \cdots.$$

If $H_1(\mathcal{M}) = 0$, we obtain the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_0(\mathcal{Z}_{t+1}) & \longrightarrow & H_0(\mathcal{Z}_t) & \longrightarrow & H_0(\mathcal{M}_t) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_{t+1}} & & \downarrow{\scriptstyle \alpha_t} & & \downarrow{\scriptstyle \beta_t} & & \\
0 & \longrightarrow & I^{t+1}M & \longrightarrow & I^t M & \longrightarrow & I^t M/I^{t+1}M & \longrightarrow & 0.
\end{array}$$

Since α and β are isomorphism in degree 0, by induction and the snake lemma, it follows that α and β are isomorphisms.

The condition $H_1(\mathcal{M}) = 0$ is, however, much too strong when compared to $\alpha =$ isomorphism. For the case $M = R$, [16] contains sharp necessary and sufficient conditions for α to be an isomorphism.

(1.4) There are still other approximation complexes with similar (but often less malleable) properties. Another we shall briefly use is $\mathcal{M} * (I; M)$, which is obtained when $H(I; M)$ is replaced by $H^*(\mathbf{x}; M) = Z(\mathbf{x}; M)/Z^*(\mathbf{x}; M)$, where $Z^*(I; M)$ consists of the cycles of $\mathcal{K}(\mathbf{x}; M)$ with coefficients in IM. It gives to the exact sequence of complexes

$$0 \to \delta(\mathbf{x}; M) \otimes S \to \mathcal{M}(I; M) \to \mathcal{M} * (I; M) \to 0. \qquad (*)$$

The $\delta_i(\mathbf{x}; M)$'s were used as measures of syzygieticity in [20] and [11]. Thus, for instance, $\delta_1(\mathbf{x}; R) = \ker(\mathrm{Sym}_2(I) \to I^2)$.

In Section 3 we shall consider the δ_i's of the Rees ring itself and they will provide a flexible tool for studying acyclicity in the complex $\mathcal{M}$. Now we point out that while the $\delta_i(\mathbf{x}; M)$ may depend on the sequence $\mathbf{x}$, rather than just on the ideal I it generates, the first nonvanishing $\delta_i(\mathbf{x}; M)$ is invariantly defined. Indeed, consider the tails of the complexes $(*)$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \delta_i(\mathbf{x}; M) & \longrightarrow & H_i(\mathbf{x}; M) & \longrightarrow & H_i^*(\mathbf{x}; M) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \delta_{i-1}(\mathbf{x}; M) \otimes S_1 & \longrightarrow & H_{i-1}(\mathbf{x}; M) \otimes S_1 & \longrightarrow & H_{i-1}^*(\mathbf{x}; M) \otimes S_1 & \longrightarrow & 0. \end{array}$$

If $\delta_{i-1}(x; M) = 0$, then

$$\delta_i(\mathbf{x}; M) = \ker(H_i(\mathcal{M}_i(\mathbf{x}; M)) \to H_i^*(\mathbf{x}; M))),$$

and the assertion will follow from (1.1).

2. Sequences

There are some notions of sequences which play in the setting of the approximation complexes roles comparable to those of regular sequences in the theory of the ordinary Koszul complex. In this section we recall the notion of a d sequence and of a proper sequence, remark on some of their properties, and derive a coarse form of Cohen–Macaulayfication.

We begin with the definition of the various sequences we shall be interested in. For a more systematic discussion of their relationships, see [11].

Definition 2.1. *Suppose* $\mathbf{x} = \{x_1, \ldots, x_n\}$ is a sequence of elements in a ring R, and M is an R-module. The sequence $\mathbf{x}$ is called a

(a) *d sequence* with respect to M, if

(a$_1$) $\mathbf{x}$ is a minimal generating set of the ideal $I = (\mathbf{x}) = (x_1, \ldots, x_n)$,

(a$_2$) $(x_1, \ldots, x_i)M :_M x_{i+1}x_k = (x_1, \ldots, x_i)M :_M x_k$, for $i = 0, \ldots, n-1$ and $k \geqslant i+1$;

(b) *proper sequence* with respect to M, if $x_{i+1}H_j(x_1, \ldots, x_i; M) = 0$ for $i = 0, \ldots, n-1$, $j > 0$, where $H_j(x_1, \ldots, x_i; M)$ denotes the Koszul homology associated to the initial subsequence $\{x_1, \ldots, x_i\}$.

Remark 2.2. The notion of a d sequence was introduced by Huneke, who has constructed many interesting classes of examples and proved some fundamental properties ([12]). Proper sequences were defined in [11] and are critical for the acyclicity of the complexes $\mathcal{M}$ and $\mathcal{Z}$.

In the enlarged context here (that of R modules) the minimality condition (a_1) does not play a major role. For this reason we sometimes refer to the following condition (equivalent to (a_2)) as a d-sequence also:

(a_2^*) $(x_1,..., x_i)M :_M x_{i+1} \cap IM = (x_1,..., x_i)M$ for $i = 0,..., n-1$, where $I = (\mathbf{x})$.

Remark 2.3. (i) For $M = R$ d sequences show features common to regular sequences and generating sets of projective ideals. Thus, for instance,

(i_1) If $\dim(R) = r$, then every d sequence has at most $r+1$ elements.

(i_2) For a Dedekind domain every ideal is generated by a d sequence.

(ii) For other modules, however, the bound (i_1) cases to exist, as when (R, m) is local and $M = R/m$.

Remark 2.4. There is another way to express condition (b). Using the recursive form of the Koszul homology modules (cf. [17]), (b) can also be written

(b^*) $\mathbf{x}$ is a proper sequence if $I \cdot H_j(x_1,..., x_i; M) = 0$, for $j > 0$ and $i = 0,..., n$ and $H_j(x_1,..., x_i; M)$ with the earlier meaning.

Contrary to d sequences, even for $M = R$, there may not exist dimension dependent bounds for the lengths of minimally generated proper sequences. Thus, if (R, m) is a local Artinian ring and $m^2 = 0$, then m is generated by a proper sequence.

Remark 2.5. A link between d sequences and proper sequences is provided by the following notion ([9]):

(c) $\mathbf{x} = \{x_1,..., x_n\}$ is a *relative regular sequence* with respect to M, if $(x_1,..., x_i) IM :_M x_{i+1} \cap IM = (x_1,..., x_i)M$ for $i = 0,..., n-1$, $I = (\mathbf{x})$.

One has then the implications (cf. [9, 11, 12]).

$$d \text{ sequence} \Rightarrow \text{relative regular sequence} \Rightarrow \text{proper sequence}.$$

Remark 2.6. One way to, possibly, get regular sequences from d sequences and proper sequences emerges directly from definition (b^*): Passing to the localization R_{x_n} it follows that $H_1(x_1,..., x_{n-1}; M)_{x_n} = 0$. Thus, if $(x_1,..., x_{n-1}) M_{x_n} \neq M_{x_n}$, then one obtains a regular M_{x_n} sequence of length $n-1$ in the ideal $(x_1,..., x_{n-1}) R_{x_n}$.

Before we make use of these remarks, let us show that d sequences, especially for certain straight-up types, are rather ubiquitous. For simplicity we consider the case of $M=R$ since the extension to arbitrary modules is immediate.

PROPOSITION 2.7. *Let I be an ideal of height n of the Noetherian ring R. Then I contains a d sequence of length (and height) n.*

Proof. Let $\{P_1,..., P_r\}$ be the minimal prime ideals of I. We shall use the following notation: For an R module M, denote by $z(M)$ the union of the associated primes of M which are properly contained in one of the P_i.

We may assume $n>0$. Let $y_1 \in I\backslash z(R)$, and stabilize it, that is, pick m large enough so that $0:y_1^m=0:y_1^{m+1}$. Put $x_1=y_1^m$; if $n=1$, $\{x_1\}$ will do.

If $n>1$, consider the ideal $J_1=0:(0:x_1)$. We claim that J_1 is not contained in any prime ideal associated to R and properly contained in some P_i. Otherwise, for such a prime P we would have: If x_1 is regular in R_P, $0:x_1=0$, which is impossible; if x_1 is not regular in R_P, P contains some associated prime of R, which conflicts with the choice of y_1, and hence of x_1. Thus we may pick $y_2 \in I \cap J_1 \backslash \{z(R) \cup z(R/(x_1))\}$ and stabilize it, meaning now take m large enough so that $0:y_2^m=0:y_2^{m+1}$ and $(x_1):y_2^m=(x_1):y_2^{m+1}$. Put $x_2=y_2^m$.

In general, if $x_1,..., x_s$ have been chosen and $n>s$, one picks x_{s+1} in the following manner. Again it is easy to see that $J_s=(x_1,..., x_{s-1}):((x_1,..., x_{s-1}):x_s)$ is not contained in any prime properly inside one of the P_i. Now pick y_{s+1} in $I \cap J_1 \cap J_2 \cap \cdots \cap J_s \backslash \{z(R) \cup z(R/(x_1)) \cup \cdots \cup z(R/(x_1,..., x_s))\}$, and stabilize it. This means picking m large enough to that for each initial subsequence $\{x_1,..., x_l\}$, $l \leqslant s$, $(x_1,..., x_l):y_{s+1}^m=(x_1,..., x_l):y_{s+1}^{m+1}$. Finally put $x_{s+1}=y_{s+1}^m$.

We claim that $\{x_1,..., x_n\}$ is a d sequence. Since the chosen elements will generate an ideal of height n, the minimal condition is ensured. Note also that any sequence $\{x_1,..., x_s, x_n\}$, $s<n-1$, is, by induction, a d sequence since it contains at most $n-1$ elements and the construction leading to it is the same that leads to the sequence $\{x_1,..., x_s, x_{s+1}\}$. Therefore the only step to check is the equality

$$(x_1,..., x_{n-2}):x_{n-1}x_n=(x_1,..., x_{n-2}):x_n.$$

Let t be an element in the first ideal; then $tx_n \in (x_1,..., x_{n-2}):x_{n-1}$ and, since $x_n \in J_{n-1}$, we have $tx_n^2 \in (x_1,..., x_{n-2})$. The assertion now follows from the stabilization step. ∎

Remark. A similar construction was used by Bass [3] to find bounds for various finitistic dimensions of Noetherian rings.

We shall make use of Proposition 2.7 to obtain a coarse form of Cohen–Macaulayfication. Recalling the problem (see [8] for a discussion), for a ring R, by the Cohen–Macaulayfication of Spec(R) we mean a pair $\{X, f\}$ of a Cohen–Macaulay schema X and a proper mapping $f: X \to \operatorname{Spec}(R)$ which is an isomorphism at the Cohen–Macaulay points of Spec(R). Some special cases have been dealt with in [4, 8], and we shall consider another approach to one of such cases.

THEOREM 2.8. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension n, and let $\mathbf{x} = \{x_1, ..., x_{n-1}, x_n = z\}$ be a system of parameters which is a d sequence (such systems always exist by Proposition 2.7). Let S be the subring of R_z generated by R and the x_i/z, $i < n$, and let P be the prime ideal $(\mathfrak{m}, x_1/z, ..., x_{n-1}/z)$. Then S_P is a Cohen–Macaulay ring of dimension n.*

For the proof we shall need the following result of Huneke [13], the converse of which was proved by Kühl [16].

PROPOSITION 2.9. *Let R be a local ring and let $\mathbf{x} = \{x_1, ..., x_n\}$ be a d sequence. Let $I = (\mathbf{x})$ and denote by $\boldsymbol{\xi} = \{\xi_1, ..., \xi_n\}$ the sequence of 1-forms of $\operatorname{gr}_I(R)$ corresponding to $x_1, ..., x_n$. Then $\boldsymbol{\xi}$ is a d sequence.*

Proof of Theorem 2.8. Write $I = (\mathbf{x})$ and let C be the Rees algebra $C = \mathscr{R}(I) = \sum_j I^j t^j$. Denote the elements $x_1 t, ..., x_{n-1} t$, $x_n t = zt$, respectively by $\zeta_1, ..., \zeta_{n-1}, \zeta$. Note that in C_ζ the elements x_n is regular and $C_\zeta/(x_n) = \operatorname{gr}_I(R)_\zeta$.

Since $\{x_1, ..., x_n\}$ is a d sequence, by Proposition 2.9 the coresponding 1-forms $\{\xi_1, ..., \xi_{n-1}, \xi_n = \xi\}$ in $\operatorname{gr}_I(R)$ is also a d sequence. Thus, by Proposition 2.6 and the choice of the x_i's, $\xi_1, ..., \xi_{n-1}$ form a regular sequence in $\operatorname{gr}_I(R)_\zeta = \operatorname{gr}_I(R)_\xi$. Altogether we have that $\{x_n, \zeta_1, ..., \zeta_{n-1}\}$ is a regular sequence in C_ζ.

Finally, as $C_\zeta = S[\zeta, \zeta^{-1}]$ (S is the 0th homogeneous piece of C_ζ) the rest of the assertion follows from standard facts on the dimension of blowing-up rings. ∎

3. THE SYZYGIETIC REDUCTION GROUPS

In this section we study the relationship between the approximation complexes associated to a sequence $\mathbf{x}$ and the Koszul homology of the corresponding 1-forms in $\mathscr{R}(I)$ and $\operatorname{gr}_I(R)$, $I = (\mathbf{x})$. The vehicle for this comparison will be an infinitesimal version of certain syzygietic measures introduced in [11].

To shift the emphasis from sequences to ideals, we shall, despite the ambiguity, denote the Koszul complex associated to $\mathbf{x}$ and the R module M by $\mathscr{K}(I; M)$. Consider the natural filtration of the complex $\mathscr{K}(I; M)$:

$$\mathscr{F}_j K_i(I; M) = I^{j-i} K_i(I; M), \qquad \text{for} \quad j \geqslant i,$$
$$= K_i(I; M), \qquad \text{for} \quad j < i.$$

DEFINITION 3.1. The modules $\delta_i^j(I; M) = H_i(\mathscr{F}_{i+j} \mathscr{K}(I; M))$, $i \geqslant 0, j \geqslant 0$, are called the *syzygietic reduction groups* of I with respect to M.

There is a more concrete way of defining these groups. Let $\mathscr{K} = \mathscr{K}(I; M)$, $Z =$ cycles of $\mathscr{K}$, $B =$ boundaries of $\mathscr{K}$. We have

$$\delta_i^j(I; M) = Z_i \cap I^j \mathscr{K} / I^{j-1} B_i = \operatorname{Im}(H_i(I; I^j M) \to H_i(I; I^{j-1} M)).$$

This allows us to interpret the δ's as the Koszul homology of the corresponding Rees module. Indeed, let

$$S = R[e_1, \ldots, e_n] \to \mathscr{R}(I) = \sum I^j t^j$$

be an R algebra presentation of $\mathscr{R}(I)$. The ideal $S_+ = (\mathbf{e})$ operates on $\mathscr{R}(I)$ (resp. $\operatorname{gr}_I(R)$) as the irrelevant ideal of $\mathscr{R}(I)$ (resp. $\operatorname{gr}_I(R)$). Let $\mathscr{R}(I; M) = \bigoplus_j I^j M$ be the corresponding Rees module. Now $\mathscr{R}(I; M)$ is a graded $\mathscr{R}(I)$ module and $\mathscr{R}(I; M) \otimes_R (R/I) = \operatorname{gr}_I(M) = \bigoplus_j I^j M / I^{j+1} M$.

Finally, to the Koszul complex $\mathscr{K}(S_+; \mathscr{R}(I; M)) = \mathscr{K}(\mathbf{e}; \mathscr{R}(I; M))$ we attach the grading

$$K_i(S_+; \mathscr{R}(I; M))_{j+i} = K_i(\mathbf{x}; I^j M).$$

In this manner all the differentials of $\mathscr{K}(S_+; \mathscr{R}(I; M))$ have degree zero.

PROPOSITION 3.2. *With the notation above we have*

(a) $$H_i(S_+; \mathscr{R}(I; M))_{j+i} = \delta_i^j(I; M), \qquad \textit{for} \quad j > 0,$$
$$= Z_i(I; M), \qquad \textit{for} \quad j = 0.$$

(b) *There exists a long exact sequence*

$$\cdots \to \delta_i^{j+1}(I; M) \to \delta_i^j(I; M) \to H_i(S_+; \operatorname{gr}_I(M))_{j+i} \to \delta_{i-1}^{j+1}(I; M) \to \cdots.$$

Proof. Condition (a) follows directly from the definitions, while (b) is just the long exact homology sequence associated to the isomorphism

$$\mathscr{K}(S_+; \operatorname{gr}_I(M)) = \operatorname{gr}_I(\mathscr{K}(I; M)) = \bigoplus_j \mathscr{F}_j \mathscr{K}(I; M) / \mathscr{F}_{j+1} \mathscr{K}(I; M). \quad \blacksquare$$

Remark 3.3. Note that for the Rees module the modified Koszul homology modules (cf. (1.4)) $H_i^*(S_+; \mathscr{R}(I; M))_{j+i} = 0$ for $j > 0$. This implies, according to property (1.4), that while the $\delta_i^j(I;M)$ may depend on the generating set chosen for I, if $\delta_i^j(I;M)$ vanishes for some $i \geqslant 0$, $j \geqslant 0$, then $\delta_{i+1}^j(I;M)$ will not depend on the chosen generators for I.

It makes sense then to have

DEFINITION 3.4. The ideal I is j *syzygietic* with respect to the module M, if

$$\delta_i^k(I;M)=0 \qquad \text{for all } i \text{ and all } k \text{ with} \quad 1 \leqslant k \leqslant j.$$

Remark 3.5. (a) This notion was introduced by Huneke [14], who showed that an ideal generated by a d sequence is j syzygietic for all $j>0$. (Its proof holds for modules as well.)

(b) For any ideal I and any module M there exists an integer $n_0 \geqslant 0$ such that I is j syzygietic with respect to I^nM for all j and all $n \geqslant n_0$ (cf. [1]).

(c) It is clear from the definition that if I is j syzygietic with respect to M for all j, then so is I with respect to IM.

PROPOSITION 3.6. *The following conditions are equivalent*:

(a) *I is j syzygietic with respect to M for all $j>0$.*

(b) $H_i(S_+;\mathscr{R}(I;M))=Z_i(I;M)(i)$ *for all i.*

(c) $H_i(S_+;\mathrm{gr}_I(M))=H_i(I;M)(i)$ *for all i.*

Proof. The equivalence of (a) and (b) is contained in Proposition 3.2(a), while that of (a) and (c) follows from Proposition 3.2(b) and the above fact that $\mathscr{F}_j\mathscr{K}(I;M)$ is acyclic for $j \gg 0$. ∎

THEOREM 3.7. *The complex $\mathscr{M}(I;M)$ is acyclic if and only if I is j syzygietic with respect to M for all j.*

Proof. Assuming that $\mathscr{M}(I;M)$ is acyclic, we have (cf. Property (1.3)) an exact sequence of graded S modules

$$0 \to H_n(I;M)\otimes S[-n] \to \cdots \to H_0(I;M)\otimes S \to \mathrm{gr}_I(M) \to 0.$$

We claim that $H_i(S_+;\mathrm{gr}_I(M)) \cong H_i(\mathscr{M}(I;M))/S_+\mathscr{M}(I;M))$. Since this last module is just $H_i(I;M)(i)$, I will be j syzygietic by Proposition 3.6.

To see the above isomorphism we use the following fact:

LEMMA. *Suppose that A is graded algebra and*

$$\cdots \to M_{l+1} \to M_l \to \cdots \to M_0 \to N \to 0$$

is an exact sequence of graded A modules. Let $\mathbf{x}$ *be a sequence of homogeneous elements of A such that*

$$H_i(\mathbf{x}; M_l) = 0 \qquad \textit{for} \quad i > 0, \quad l \geqslant 0.$$

Then there exist isomorphisms of graded modules

$$H_i(\mathbf{x}; N) \cong H_i(M_*/(\mathbf{x})\, M_*).$$

For the converse, let ∂_0' be the composition

$$H_0(I; M) \otimes S \to H_0(\mathscr{M}(I; M)) \xrightarrow{\beta} \operatorname{gr}_I(M)$$

and denote by $\mathscr{N}$ the augmented complex

$$0 \to H_n(I; M) \otimes S[-n] \xrightarrow{\partial_n'} \cdots \xrightarrow{\partial_1'} H_0(I; M) \otimes S \xrightarrow{\partial_0'} \operatorname{gr}_I(M) \to 0.$$

We are going to prove, by induction on i, that $H_i(\mathscr{N}) = 0$ for $i \geqslant -1$. (Here $N_{-1} = \operatorname{gr}_I(M)$.)

This is obvious if $i = -1$. Suppose then $i \geqslant 0$ and that $H_j(\mathscr{N}) = 0$ for $j < i$. We have an exact sequence

$$0 \to U_i \to N_i \to \cdots \to N_0 \to N_{-1} \to 0,$$

with $U_i = \operatorname{Ker}(\partial_i')$. Using that $H_j(S_+; N_l) = 0$ for $j > 0$ and $l \geqslant 0$, we obtain

$$H_{i+1}(S_+; N_{-1}) = \operatorname{Ker}(H_0(S_+; U_i) \to H_0(S_+; N_i)).$$

Since $H_0(S_+; N_i)_j = 0$ for $j \neq i$, we have the isomorphism

$$H_{i+1}(S_+; N_{-1})_j = H_0(S_+; U_i)_j \qquad \text{for} \quad j \neq i.$$

We now use that I is syzygietic with respect to M. According to Proposition 3.6(c), $H_k(S_+; N_{-1})$ is concentrated in degree k, and therefore, taking into account the above, $H_0(S_+; U_i)_j = 0$ for $j \neq i,\ i+1$. As the surjection $U_i \to H_i(\mathscr{N})$ induces the surjection $H_0(S_+; U_i) \to H_0(S_+; H_i(\mathscr{N}))$, we also have $H_0(S_+; H_i(\mathscr{N}))_j = 0$ for $j \neq i,\ i+1$.

For $j = i > 0$, we have $H_i(\mathscr{N})_i = \delta_i^1(I; M) = 0$, while $H_0(\mathscr{N})_0 = 0$ since $N_0 \to N_{-1}$ is an isomorphism in degree 0.

For $j = i + 1$, we employ the commutative diagram with exact rows (cf. Proposition 3.6)

$$\begin{array}{ccccccc}
H_0(S_+; N_{i+1})_{i+1} & \to & H_0(S_+; N_i)_{i+1} & \to & H_0(S_+; H_i(\mathscr{N}))_{i+1} & \to & 0 \\
\wr\downarrow & & \wr\downarrow & & & & \\
\delta_{i+1}^0(I; M) & \to & H_{i+1}(S_+; \operatorname{gr}_I(M))_{i+1} & \to & \delta_i^1(I; M) & = & 0
\end{array}$$

and therefore $H_0(S_+; H_i(\mathcal{N}))_{i+1} = 0$. The assertion now follows from Nakayama's lemma. ∎

To conclude this section, we point out the following consequences of the acyclicity of the $\mathcal{M}$ complex for the depth if the irrelevant ideals of the associated graded modules.

PROPOSITION 3.8. *If the complex $\mathcal{M}(I; M)$ is acyclic, then I – depth M = S_+-depth* $\mathrm{gr}_I(M)$.

PROPOSITION 3.9. *Let (R, m) be a local ring and let I be a primary ideal such that $\mathcal{M}(I; R)$ is acyclic. Then depth R = depth* $\mathrm{gr}_I(R)$.

Remark. Proposition 3.9 applies in particular I generated by systems of parameters in a Buchsbaum ring, or suitable chosen—via Proposition 2.7—systems of parameters in arbitrary local rings. Such systems are d sequences and the corresponding $\mathcal{M}$ complexes are acyclic by Remark 3.5(a) and Theorem 3.7, or by [11, (5.6)].

4. THE ACYCLICITY OF THE $\mathcal{M}$ COMPLEX

We shall give here the sequential necessary and sufficient conditions for the acyclicity of the $\mathcal{M}$ complex. Essentially it is shown that d sequences play for the $\mathcal{M}$ complex the same role as regular sequences in the theory of the ordinary Koszul complexes.

THEOREM 4.1. *Let (R, m) be a Noetherian local ring with infinite residue field. Let I be an ideal and let M be a finitely generated R module. Then the following conditions are equivalent*:

(a) *$\mathcal{M}(I; M)$ is acyclic.*

(b) *I is generated by a d sequence with respect to M.*

(c) *I is generated by a proper sequence with respect to M and* $H_0(\mathcal{M}(I; M)) = \mathrm{gr}_I(M)$.

(d) *I is generated by a relative regular sequence with respect to M and* $H_0(\mathcal{M}(I; M)) = \mathrm{gr}_I(M)$.

(e) *I is generated by a sequence $\{x_1, ..., x_n\}$ satisfying*

$$((x_1, ..., x_i)\, IM :_M x_{i+1}) \cap I^{k+1}M = (x_1, ..., x_i)\, I^k M \qquad \textit{for} \quad i = 0, ..., n-1$$

and all k.

Proof. By property (1.3) we know that if $\mathcal{M}(I; M)$ is acyclic, then

$H_0(\mathcal{M}(I; M)) = \mathrm{gr}_I(M)$. Taking Remark 3.5(a) and Theorem 3.7 into account we see that (b) $\Rightarrow$ (d). The implication (d) $\Rightarrow$ (c) is trivial, while (c) $\Rightarrow$ (a) is proved in the same manner as [11, (5.5)]. Hence it remains to show that (a) $\Rightarrow$ (e) and (e) $\Rightarrow$ (b).

The implication (e) $\Rightarrow$ (b) has been observed by Kühl [16]. We are going to show that for all $j \geqslant i$ we have

$$((x_1,..., x_i)M :_M x_{i+1}) \cap (x_1,..., x_{j+1})M = (x_1,..., x_j)M.$$

This, of course, then implies that $\{x_1,..., x_n\}$ is a d sequence with respect to M.

Let $a \in ((x_1,..., x_i)M :_M x_{i+1}) \cap (x_1,..., x_{j+1})M$; then $ax_{j+1} \in ((x_1,..., x_i)IM :_M x_{i+1}) \cap I^2M$. By assumption (e) it then follows that

$$ax_{j+1} \in (x_1,..., x_i)IM \subset (x_1,..., x_j)IM,$$

and hence

$$a \in ((x_1,..., x_j)IM :_M x_{j+1}) \cap IM = (x_1,..., x_j)M.$$

Note that we have used the equalities in (e) only for $k = 0, 1$.

To show the implication (a) $\Rightarrow$ (e) we first prove a series of lemmas. Denote by I^* the irrelevant ideal $\mathrm{gr}_I(R)_+$ of $\mathrm{gr}_I(R)$, and by ξ the 1-form corresponding to the element $x \in I$.

LEMMA 4.2. *I^* can be generated by a sequence of* 1*-forms $\{\xi_1,..., \xi_n\}$ with the property*:

For all $i > 0$ and for all $k \in \{1,..., n\}$ we have $H_i(\xi_1,..., \xi_k)_j = 0$ for almost all j. $(\#)$

Here $H_i(\xi_1,..., \xi_k) = H_i(\xi_1,..., \xi_k; \mathrm{gr}_I(M))$. This is an easy consequence of the following lemma whose proof is standard.

LEMMA. *Let $(R, \mathfrak{m})$ be a Noetherian local ring with infinite residue field and let N be a finitely graded* $\mathrm{gr}_I(R)$*-module. Then there exist a* 1*-form $\xi \in I^*$ such that*

$$\mathrm{Ker}(N \xrightarrow{\xi} N)_j = 0, \qquad \textit{for almost all } j.$$

LEMMA 4.3. *If $\mathcal{M}(I; M)$ is acyclic and $\{\xi_1,..., \xi_n\}$ is a sequence generating I^* and satisfying $(\#)$, then for all $k \in \{1,..., n\}$ and $i > 0$ we have $H_i(\xi_1,..., \xi_k)_j = 0$ for $j \neq i$.*

Proof. We proceed by induction on $l=n-k$. For $l=0$ the assertion follows from Proposition 3.6 and Theorem 3.7.

Suppose now that $l>0$ and consider the long exact sequence of Koszul homology

$$H_{i+1}(\xi_1,..., \xi_{k+1})_{j+1} \to H_i(\xi_1,..., \xi_k)_j \to H_i(\xi_1,..., \xi_k)_{j+1} \to H_i(\xi_1,..., \xi_{k+1})_{j+1}.$$

By the induction hypothesis we have $H_i(\xi_1,..., \xi_{k+1})_j=0$ for $j\neq i$. It follows that

$$H_i(\xi_1,..., \xi_k)_j = H_i(\xi_1,..., \xi_k)_{j+1} \qquad \text{for} \quad j>i.$$

Since $\{\xi_1,..., \xi_n\}$ satisfies ($\#$) we have $H_i(\xi_1,..., \xi_k)_j=0$ for $j \gg 0$. Thus we conclude $H_i(\xi_1,..., \xi_k)_j=0$ for $j>i$. As, always, $H_i(\xi_1,..., \xi_k)_j=0$ for $j<i$, the assertion follows. ∎

LEMMA 4.4. *A sequence $\{\xi_1,..., \xi_n\}$ generating I^* and satisfying $H_i(\xi_1,..., \xi_k)_j=0$ for $j\neq i$, for all $i>0$ and all $k\in\{1,..., n\}$, is a d sequence with respect to* $\mathrm{gr}_I(M)$.

Proof. Write $M^*=\mathrm{gr}_I(M)$, and let $a\in((\xi_1,..., \xi_{i-1})M^* : \xi_i)\cap I^*M^*$ be a homogeneous element. Then $a\xi_i=\sum_1^{i-1} b_j\xi_j$, with $b_j\in M^*$ homogeneous. Since $a\in I^*M^*$, we may assume that all $b_j\in I^*M^*$. Therefore the element $z=ae_i-\sum_1^{i-1} b_je_j$ represents a homology class of $H_1(\xi_1,..., \xi_i)_j$, where $j>1$. By assumption $H_1(\xi_1,..., \xi_i)_j=0$ for $j>1$, which implies that z is a boundary of $\mathscr{K}(\xi_1,..., \xi_i)$ and thus $a\in(\xi_1,..., \xi_i)M^*$ as required. ∎

LEMMA 4.5. *Suppose $\{\xi_1,..., \xi_n\}$ is a sequence as in Lemma 4.4. Choose $x_1,..., x_n\in I$ with $\xi_i=x_i+I^2$. Then the sequence $\{x_1,..., x_n\}$ satisfies*

$$(x_1,..., x_i)IM\cap I^{k+1}M\subset(x_1,..., x_i)I^kM$$

for $i=1,..., n$ and all $k\geqslant 0$.

Proof. From the exact sequence of complexes

$$0\to\mathscr{F}_{k+1}\mathscr{K}(x_1,..., x_i)\to\mathscr{F}_k\mathscr{K}(x_1,..., x_i)\to\mathscr{K}(\xi_1,..., \xi_i)_k\to 0$$

we obtain the exact sequence

$$H_1(\xi_1,..., \xi_i)_k\to H_0(\mathscr{F}_{k+1}\mathscr{K}(x_1,..., x_i))\to H_0(\mathscr{F}_k\mathscr{K}(x_1,..., x_i)).$$

By the choice of our sequence that the last map is an injection for $k>1$. The injectivity of this map says, however, that

$$(x_1,..., x_i)I^{k-1}M\cap I^{k+1}M=(x_1,..., x_i)I^kM \qquad \text{for all} \quad k>1. \quad (\#\#)$$

We now prove the assertion of the lemma by induction on k. For $k=0, 1$ the assertion is trivial. Suppose $k>1$. Using (##) we have

$$\begin{aligned}(x_1,\dots,x_i)\,IM\cap I^{k+1}M &= ((x_1,\dots,x_i)\,IM\cap I^kM)\cap I^{k+1}M\\ &\subset (x_1,\dots,x_i)\,I^{k-1}M\cap I^{k+1}M\\ &= (x_1,\dots,x_i)\,I^kM.\end{aligned}$$

We are now ready to prove (a) $\Rightarrow$ (e). Choose a sequence of elements $\{x_1,\dots,x_n\}$ generating I and such that for all $i>0$ and all $k\in\{1,\dots,n\}$ we have

$$H_i(\xi_1,\dots,\xi_k)_j=0 \qquad \text{for} \quad j\neq i,$$

where $\xi_i=x_i+I^2$.

By Lemmas 4.2 and 4.3 we can find such a sequence. We now show that this sequence satisfies the conditions of (e). Let $a\in((x_1,\dots,x_i)\,IM:x_{i+1})\cap I^kM$. By Lemma 4.5 we get

$$ax_{i+1}\in(x_1,\dots,x_i)\,IM\cap I^{k+1}M\subset(x_1,\dots,x_i)\,I^kM.$$

We may then write $ax_{i+1}=\sum_1^i b_jx_j$, $b_j\in I^kM$. Letting $a^*=a+I^{k+1}M$, $b_j^*=b_j+I^{k+1}M$, we have $a^*\xi_{i+1}=\sum_1^i b_j^*\xi_j$.

Since by Lemma 4.4 $\{\xi_1,\dots,\xi_n\}$ is a d sequence with respect to $M^*=\mathrm{gr}_I(M)$, we have

$$a^*\in(\xi_1,\dots,\xi_i)\,M^*\cap I^{*k}M^*\subset(\xi_1,\dots,\xi_i)\,I^{*k-1}M^*.$$

It follows that $a\in(x_1,\dots,x_i)\,I^{k-1}M+I^{k+1}M$. Writing $a=a'+a''$, $a'\in(x_1,\dots,x_i)\,I^{k-1}M$ and $a''\neq I^{k+1}M$, we have $a''x_{i+1}\in(x_1,\dots,x_i)\,IM+I^{k+2}M$, and hence, as before, we get $a''\in(x_1,\dots,x_i)\,I^kM+I^{k+2}M$. It follows that $a\in(x_1,\dots,x_i)\,I^{k-1}M+I^{k+2}M$.

The conclusion now follows by induction and Krull's intersection theorem. ▮

Remark. Kühl [16] has derived a similar result for proper sequences: Under the same conditions on the local ring R, an ideal I is generated by a proper sequence with respect to the module M is and only if the complex $\mathscr{Z}(I;M)$ is acyclic.

5. Linear Resolutions

In this section we are mainly concerned with the study of the $\mathscr{M}$ complex associated with the maximal ideal of a local ring.

Let $(R, \mathfrak{m}, k)$ be a regular local ring and let M be a finitely generated R module. Let

$$\mathscr{M}(\mathfrak{m}; M)\colon 0 \to H_n(\mathfrak{m}; M) \otimes S[-n] \to \cdots \to H_0(\mathfrak{m}; M) \otimes S \to 0$$

be the $\mathscr{M}$ complex built with respect to a minimal system of generators $\mathbf{x} = \{x_1,..., x_n\}$ of $\mathfrak{m}$. Recall that $S = R[e_1,..., e_n]$ and thus $\mathscr{M}(\mathfrak{m}; M)_i = H_i(\mathfrak{m}; M) \otimes R[e_1,..., e_n] \cong k[e_1,..., e_n]^{b_i}$, $b_i = \dim_k H_i(\mathfrak{m}; M)$.

Thus $\mathscr{M}(\mathfrak{m}; M)$ may be considered to be a complex of free $\mathrm{gr}_{\mathfrak{m}}(R)$-modules, since $\mathrm{gr}_{\mathfrak{m}}(R) = k[e_1,...,e_n]$, where the isomorphism is given by $x_i + \mathfrak{m}^2 \to e_i$, $i = 1,..., n$.

To the module M we may assign a different complex of free $\mathrm{gr}_{\mathfrak{m}}(R)$-modules. Let

$$(\mathscr{G}, d)\colon \cdots \to G_2 \xrightarrow{d_2} G_1 \xrightarrow{d_1} G_0 \to 0$$

be a minimal free R resolution of M. We define a natural filtration on $(\mathscr{G}, d)$ by setting

$$\begin{aligned} \mathscr{F}_j G_i &= G_i, && \text{for} \quad j < i, \\ &= \mathfrak{m}^{j-i} G_i, && \text{for} \quad j \geqslant i. \end{aligned}$$

The associated graded complex

$$\mathrm{gr}(G)\colon \cdots \to \mathrm{gr}_{\mathfrak{m}}(G_2) \xrightarrow{\mathrm{gr}(d_2)} \mathrm{gr}_{\mathfrak{m}}(G_1) \xrightarrow{\mathrm{gr}(d_1)} \mathrm{gr}_{\mathfrak{m}}(G_0) \to 0$$

is then a complex of free $\mathrm{gr}_{\mathfrak{m}}(R)$ modules.

THEOREM 5.1. $\mathscr{M}(\mathfrak{m}; M) \cong \mathrm{gr}(\mathscr{G})$.

Proof. We proceed by induction on $\mathrm{pd}(M)$, the projective dimension of M. If M is free, the assertion is clear. Suppose now $\mathrm{pd}(M) > 0$, and consider the exact sequence

$$0 \to N \to G_0 \to M \to 0, \qquad N = d_1(G_1).$$

Since G_0 is free and $N \subset \mathfrak{m}G_0$, we get isomorphisms

$$H_{i+1}(\mathfrak{m}; M) \cong_\sigma H_i(\mathfrak{m}; N) \qquad \text{for all} \quad i \geqslant 0.$$

Here σ is the connection homomorphism in the long exact homology sequence.

We check readily that for all $i \geqslant 0$ the following commutes:

$$\begin{array}{ccc} H_{i+2}(m; M) \otimes S & \xrightarrow{\partial'} & H_{i+1}(m; M) \otimes S \\ \sigma\otimes id \downarrow & & \downarrow \sigma\otimes id \\ H_{i+1}(m; N) \otimes S & \xrightarrow{\partial'} & H_i(m; N) \otimes S. \end{array}$$

Using this fact and the induction hypothesis, all that remains is to prove that

$$\mathrm{gr}(G_1) \xrightarrow{\mathrm{gr}(d_1)} \mathrm{gr}(G_0)$$

is isomorphic to

$$H_1(m; M) \otimes S \xrightarrow{\partial'} H_0(m; M) \otimes S.$$

To see this we choose bases (f_k), $k = 1,..., b_1$, of G_1 and (g_l), $l = 1,..., b_0$, of G_0. We then have $d_1(f_k) = \sum \alpha_{kl} g_l$, where each α_{kl} can be written as $\alpha_{kl} = \sum a^i_{kl} x_i$. If we now put $a_{ki} = \sum a^i_{kl} g_l$, then $d_1(f_k) = \sum a_{ki} x_i$.

Let f^*_k (resp. g^*), denote the initial forms of f_k (resp. g_l), in $\mathrm{gr}(\mathscr{G})$ and denote by $\tilde{r}$ the residue class modulo m of an element $r \in R$.

We then obtain $\mathrm{gr}(d_1)(f^*_k) = \sum_l (\sum_i \tilde{a}^i_{kl} e_i) g^*$ (recall that $e_i = x_i + m^2$).

In order to describe ∂' we observe that the elements $[\sum_i \rho(a_{ki}) e_i]$, $k = 1,..., b_1$, form a basis of $H_1(m; M)$ and the elements $[\rho(g_l)]$, $l = 1,..., b_0$, form a basis of $H_0(m; M)$. (Here [ ] denotes the homology class of a cycle and $\rho: G_0 \to M$ is the augmentation map.)

We now compute ∂' explicitly

$$\begin{aligned} \partial'\left[\sum_i \rho(a_{ki}) e_i\right] &= \sum_i [\rho(a_{ki})] \otimes e_i = \sum_i \sum_l \tilde{a}^i_{kl} [\rho(g_l)] \otimes e_i \\ &= \sum_l ([\rho(g_l)] \otimes \sum_i \tilde{a}^i_{kl} e_i. \end{aligned}$$

The assertion follows. ∎

DEFINITION. The R module M has a *linear resolution* if the associated complex $\mathrm{gr}(\mathscr{G})$ is acyclic.

COROLLARY 5.2. (a) *If the R module M has a linear resolution so does mM.*

(b) *For an R module M, there exists an integer n_0 so that $m^n M$ has a linear resolution for $n \geqslant n_0$.*

Proof. Both assertion follow from Remark 3.5 and Theorems 3.7, and 5.1. ∎

Putting Theorem 5.1 together with the results of Sections 3 and 4, we obtain one of our main results:

COROLLARY 5.3. *Let $(R, \mathfrak{m})$ be a regular local ring with infinite residue field, and let M be a finitely generated R module. The following conditions are equivalent:*

(a) *The maximal ideal $\mathfrak{m}$ is generated by a d sequence with respect to M.*

(b) *M has a linear resolution.*

If the equivalent conditions hold, then

$$\mathscr{M}(\mathfrak{m}; M)\colon 0 \to H_n(\mathfrak{m}; M) \otimes S[-n] \to \cdots \to H_0(\mathfrak{m}; M) \otimes S \to 0$$

is a minimal free homogeneous $\mathrm{gr}_{\mathfrak{m}}(R)$ resolution of $\mathrm{gr}_{\mathfrak{m}}(M)$, and the Hilbert function $H_M(z)$ of M is given by the formula

$$H_M(z) = P(z)/(1-z)^n, \qquad P(z) = \sum (-1)^{i+1} b_i z^i, \quad b_i = \dim_k H_i(\mathfrak{m}; M).$$

Moreover, depth $M =$ depth $\mathrm{gr}_{\mathfrak{m}}(M)$.

Now let $(R, \mathfrak{m}, k)$ be any local ring (not necessarily regular). We are going to give a description of $\mathscr{M}(\mathfrak{m}; R)$. The homology of $\mathscr{M}(\mathfrak{m}; R)$ does not change if we pass to the $\mathfrak{m}$-adic completion of R. Hence we may assume that R is complete and thus choose a presentation $R = A/I$, where $(A, \mathfrak{n}, k)$ is a regular local ring and $I \subset \mathfrak{n}^2$.

Let $\mathscr{F}$ be a minimal free resolution of R as an A module.

COROLLARY 5.4.

$$H_i(\mathscr{M}(\mathfrak{m}; R)) \cong H_i(\mathrm{gr}(\mathscr{F})).$$

EXAMPLE 5.5. (a) Let $I \subset k[x_{ij}]$ be the ideal generated by the maximal minors of an $m \times n$ of indeterminates. Let $A =\ [[x_{ij}]]$, $R = A/IA$. Since the resolution of this ideal is known to be linear ([6]), we see that

$$H_i(\mathscr{M}(\mathfrak{m}; R)) = 0 \qquad \text{for} \quad i \qquad \text{and} \qquad H_1(\mathscr{M}(\mathfrak{m}; R)) = I.$$

(b) Let $I \subset k[x_{ij}]$ be ideal generated by the $2n \times 2n$ Pfaffians of a skew-symmetric $(2n+1) \times (2n+1)$ matrix in the indeterminates x_{ij}. Again let $A = k[[x_{ij}]]$, $R = A/IA$. Using the structure theorem of [5], we see that

$$H_1(\mathscr{M}(\mathfrak{m}; R)) = I, \qquad H_2(\mathscr{M}(\mathfrak{m}; R)) = 0, \qquad \text{and} \qquad H_3(\mathscr{M}(\mathfrak{m}; R)) = \mathrm{gr}_{\mathfrak{m}}(R).$$

This example shows that, the $\mathcal{M}$ complex is not rigid, in contrast to the ordinary Koszul complex.

Quite generally we have that $\mathcal{M}_1(m; R) \to \mathcal{M}_0(m; R)$ is the zero mapping. Hence the following conditions are equivalent:

(a) $H_1(\mathcal{M}(m; R)) = 0$.

(b) $\mathcal{M}(m; R)$ is acyclic.

(c) R is regular local ring.

Thus for a nonregular local ring (R, m), we cannot have much better exactness of $\mathcal{M}(m; R)$ than $H_i(\mathcal{M}(m; R)) = 0$ for $i > 1$.

Using the result of [16] (cf. Theorem 4.1) and [11, (2.1)], we have

THEOREM 5.6. *Let (R, m) be a local ring with infinite residue field. The following conditions are equivalent*:

(a) *m is generated by a proper sequence.*

(b) *$H_i(\mathcal{M}(m; R)) = 0$ for $i > 1$, and the canonical homomorphism $H_1(\mathcal{M}(m; R)) \to \mathrm{Sym}(m)$ is injective.*

Suppose again that R has a presentation $R = A/I$, with (A, n, k) regular, $I \subset n^2$, and k infinite. In general $\mathrm{gr}_n(I) \neq \mathrm{Ker}(\mathrm{gr}_n(A) \to \mathrm{gr}_n(R))$. Making an extra assumption on $\mathrm{gr}_m(I)$ we get

THEOREM 5.7. *Suppose that there exists an integer $d \geqslant 1$ such that $\mathrm{gr}_n(I)[-d] = \mathrm{Ker}[\mathrm{gr}_n(A) \to \mathrm{gr}_m(R))$. Then the following conditions are equivalent*:

(a) *The maximal ideal m of R is generated by a proper sequence.*

(b) *I has a linear resolution.*

If the equivalent conditions hold, then

$$\begin{aligned} 0 \to H_n(m; R) \otimes S[-n-d+1] \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} H_1(m; R) \\ \otimes S[-d] \to H_0(m; R) \otimes S \to 0 \qquad (\#) \end{aligned}$$

is a minimal free homogeneous $\mathrm{gr}_n(A)$ resolution of $\mathrm{gr}_m(R)$, and depth $R =$ depth $\mathrm{gr}_m(R)$.

Remark. What the hypothesis on $\mathrm{gr}_n(I)$ means is that

$$\begin{aligned} I \cap n^\nu &= I, && \text{for} \quad \nu < d, \\ &= n^{\nu-d} I, && \text{for} \quad \nu \geqslant d. \end{aligned}$$

In other words, I is generated minimally by a standard basis $f_1,\ldots, f_m$ of degree $\nu(f_i)=d$ for all i.

Proof of Theorem 5.7. (a) $\Rightarrow$ (b) Let $\mathscr{G}$ be a minimal free A resolution of R. By Corollary 5.4 we have $H_i(\mathscr{M}(\mathfrak{m}; R)) \cong H_i(\mathrm{gr}(\mathscr{G}))$, and by (5.6) we know that $H_i(\mathscr{M}(\mathfrak{m}; R))=0$ for $i>1$. This shows that

$$\cdots \to \mathrm{gr}(G_2) \to \mathrm{gr}(G_1) \to 0$$

is acyclic, that is, I has a linear resolution.

(b) $\Rightarrow$ (a) If I has a linear resolution, then arguing as above we get that $H_i(\mathscr{M}(\mathfrak{m}; R))=0$ for $i>1$ and $H_1(\mathscr{M}(\mathfrak{m}; R)) \cong \mathrm{gr}_n(I)$. Using the isomorphism $\mathrm{gr}_n(I)[-d] = \mathrm{Ker}(\mathrm{gr}_n(A) \to \mathrm{gr}_m(R))$ we get that complex ($\#$) is a homogeneous $\mathrm{gr}_n(A)$ resolution of $\mathrm{gr}_m(R)$.

To prove that the maximal ideal of R is generated by a proper sequence we use the ideas of Section 4.

From the homogeneous resolution ($\#$) we get for all $i>0$,

$$H_i(\mathfrak{m}^*; \mathrm{gr}_m(R)) = H_i(\mathfrak{m}; R)(i+d-1), \qquad \text{where} \quad \mathfrak{m}^* = \mathrm{gr}_m(R)_+ .$$

Since k is infinite and the Koszul homology $H_i(\mathfrak{m}^*; \mathrm{gr}_m(R))$ of $\mathfrak{m}^*$ is concentrated in one degree, we find, as in Section 4, a sequence $\{\xi_1,\ldots,\xi_n\}$ generating $\mathfrak{m}^*$ such that for all $i>0$ and $k \in \{1,\ldots,n\}$ we have

$$H_i(\xi_1,\ldots,\xi_k)_j = 0 \qquad \text{for} \quad j \neq i+d-1. \tag{$\#\#$}$$

Choose $x_1,\ldots,x_n$ such that $\xi_i = x_i + \mathfrak{m}^2$ for $i=1,\ldots,n$. We claim that $\{x_1,\ldots,x_n\}$ is a proper sequence.

As before, $\mathscr{F}\cdot\mathscr{K}(x_1,\ldots,x_k; R)$ denotes the natural $\mathfrak{m}$-adic filtration of $\mathscr{K}(x_1,\ldots,x_k; R)$. From ($\#\#$) we conclude that for all $i>0$ and $k \in \{1,\ldots,n\}$ we have

$$\begin{aligned} H_i(\mathscr{F}_j\mathscr{K}(x_1,\ldots,x_k; R)) &= H_i(x_1,\ldots,x_k; R) && \text{for} \quad j \leqslant i+d-1, \\ &= 0, && \text{for} \quad j > i+d-1. \end{aligned}$$

Now suppose that $z \in K_i(x_1,\ldots,x_k; R)$, $i>0$, is a cycle. Then z is homologous to a cycle z' with $z' \in \mathscr{F}_{i+d-1}K_i(x_1,\ldots,x_k; R)$. Thus $x_{k+1}z'$, and hence also $x_{k+1}z$, is a boundary in $K_i(x_1,\ldots,x_k; R)$, which shows that $\{x_1,\ldots,x_n\}$ is proper. ∎

6. Cohen–Macaulay Symmetric Algebras

A. *High Depth Koszul Homology*

In this section we investigate the symmetric algebras $\mathrm{Sym}(I)$ and $\mathrm{Sym}(I/I^2)$ for their Cohen–Macaulay properties. A main point of [11] was that the Cohen–Macaulayness of the Koszul homology on a set of generators of I suffices (along with a condition on the local number of generators of I) to produce Cohen–Macaulay symmetric algebras. We quote the main theorem there but in a slightly modified version. In the sequel we shall denote the minimal number of generators of a module E by $v(E)$, and the height of an ideal I by $\mathrm{ht}(I)$.

THEOREM 6.1. *Let R be a Cohen–Macaulay ring and let I be an ideal of* $\mathrm{ht}(I) > 0$. *Assume*

(i) $v(I_P) \leqslant \mathrm{ht}(P)$, *for every prime $P \supset I$.*

(ii) $\mathrm{depth}(H_i)_P \geqslant \mathrm{ht}(P) - v(I_P) + i$, *for every prime $P \supset I$ and every $0 \leqslant i \leqslant v(I_P) - \mathrm{ht}(I_P)$, where the H_i denote the homology modules of the Koszul complex on a generating set for I. Then*:

(a) $\mathrm{Sym}(I) \cong \mathscr{R}(I)$ *and* $\mathrm{Sym}(I/I^2) \cong \mathrm{gr}_I(R)$.

(b) $\mathrm{Sym}(I)$ *and* $\mathrm{Sym}(I/I^2)$ *are Cohen–Macaulay rings.*

(c) *If, moreover, R is a Gorenstein ring and the H_i's are Cohen–Macaulay modules, then* $\mathrm{Sym}(I/I^2)$ *is Gorenstein.*

Remark. Condition (ii) is weaker than "H_i is Cohen–Macaulay" of [11, (2.6)]. Indeed, if H_i is C–M, $\mathrm{dept}(H_i)_P = \dim(R/I)_P = \mathrm{ht}(P) - v(I_P) + (v(I_P) - \mathrm{ht}(I_P) \geqslant \mathrm{ht}(P) - v(I_P) + i$.

The proof follows [11, (2.6)], so we just give an outline. It uses the following combination of the "acyclicity lemma" of [15, p. 103; 18].

PROPOSITION 6.2. *Let R be a local ring and let*

$$\mathscr{C}: \qquad 0 \to C_s \to \cdots \to C_1 \to C_0 \to 0$$

be a complex of finitely generated R modules.

(a) *If* (i) *depth $C_i \geqslant i$, and* (ii) *depth $H_i(\mathscr{C}) = 0$ or $H_i(\mathscr{C}) = 0$ for $i \geqslant 1$, then $\mathscr{C}$ is acylic.*

(b) *If, moreover, for some positive integer t, depth $C_i \geqslant t + i$ for each $i \geqslant 0$, then depth $H_0(\mathscr{C}) \geqslant t$.*

Sketch of the proof of Theorem 6.1. Since $\mathrm{depth}(H_i)_P \geqslant i$, the acyclicity of the complexes $\mathscr{Z}(I;R)$ and $\mathscr{M}(I;R)$ follow from [11, (2.5)] and the isomorphisms of (a) ensue.

Next, to obtain $\mathrm{Sym}(I/I^2)$ Cohen–Macaulay we assume R local of dimension d and look at the complex $\mathscr{M}$ built on a minimal number of generators of I

$$0 \to H_s \otimes S \to \cdots \to H_0 \otimes S \to \mathrm{Sym}(I/I^2) \to 0,$$

where $s = n - \mathrm{ht}(I)$, $n = v(I)$. Using (ii), and counting with respect to the irrelevant ideal of $S = R[T_1, ..., T_n]$, we have the depth estimates

$$\mathrm{depth}(H_i \otimes S) \geqslant (d - n + i) + n = d + i, \qquad 0 \leqslant i \leqslant s.$$

Thus, by Proposition 6.2, $\mathrm{depth}(\mathrm{Sym}(I/I^2)) \geqslant d$. Since under (i) $\dim(\mathrm{Sym}(I/I^2) \leqslant d$ (cf. [21]), the assertion follows.

As for $\mathrm{Sym}(I)$, in a similar manner we obtain depth estimates for Z_i = cycles of the Koszul complex on a minimal generating set of I. They are

$$\mathrm{depth}\, Z_i \geqslant d - n + i + 1, \qquad 1 \leqslant i \leqslant n - 1$$

($Z_n = 0$ since $\mathrm{ht}(I) > 0$). Thus

$$\mathrm{depth}(Z_i \otimes S) \geqslant d + i + 1, \qquad 1 \leqslant i \leqslant n - 1,$$

and again by Proposition 6.2, $\mathrm{depth}\, \mathrm{Sym}(I) \geqslant d + 1 \geqslant \dim \mathscr{R}(I) = \dim \mathrm{Sym}(I)$.

Part (c) was proved in [11]. ∎

Remark. It is likely that the above relaxation on the depth of the Koszul homology modules may still yield (c). A related question discussed in [11] was that of finding ideals in regular local rings for which $\mathrm{Sym}(I)$ is Gorenstein. If I is a complete intersection at a minimal prime $P \supset I$ (e.g., if $\mathrm{Sym}(I) = \mathscr{R}(I)$), then $\mathrm{Sym}(I_P) \cong \mathrm{Sym}(I)_P$ is still Gorenstein but of type greater than one if I_P is generated by a regular sequence of more than two elements.

EXAMPLE 6.3. The modified $\mathscr{M}$ complex (cf. Section 1) may be used to derive depth properties of certain symmetric algebras. We will illustrate this with an example: Let $I \subset k[x, y, z, w]$ be the ideal of the smooth rational nonnormal quartic in P^3. I is generated by four elements and the following conditions hold: (i) $v(I_P) \leqslant \mathrm{ht}(P)$ for every prime $P \supset I$. (ii) As for the Koszul homology modules: depth $H_2 \geqslant 2$ since $H_2 = H_2(I;R)$ is the canonical module of R/I (cf. [10]); H_1, however, has torsion since there are

Plücker relations among the generators of I. For this reason we replace the complex $\mathscr{M}$ by $\mathscr{M}^*$ (cf. Section 1),

$$0 \to H_2^* \otimes S \to H_1^* \otimes S \to H_0 \otimes S \to \mathrm{Sym}(I/I^2) \to 0.$$

Since $H_2^* = H_2$, a simple counting of depths (relative to the maximal irrelevant ideal of S) now yields $\mathrm{depth}(\mathrm{Sym}(I/I^2)) \geqslant 4$. It follows from (i) (cf. [21]) that $\dim(\mathrm{Sym}(I/I^2)) \leqslant 4$. Thus $\mathrm{Sym}(I/I^2)$ is Cohen–Macaulay. It will follow from Theorem 6.4, however, that for this ideal $\mathrm{Sym}(I)$ is not Cohen–Macaulay.

The next result is akin to Proposition 6.2, but also emphasizes the module-theoretic properties (torsion especially) of the symmetric algebras.

THEOREM 6.4. *Let R be a Cohen–Macaulay ring and let I be an ideal of* $\mathrm{ht}(I) > 0$. *Assume*:

(i) $\mathrm{Sym}(I)$ *and R/I are Cohen–Macaulay rings.*

(ii) *I is generically a complete intersection.*

Then the following conditions are equivalent:

(a) $\mathrm{Sym}(I) \cong \mathscr{R}(I)$.

(b) $\mathrm{Sym}(I/I^2)$ *is Cohen–Macaulay.*

(c) $v(I_P) \leqslant \mathrm{ht}(P)$ *for every prime* $P \supset I$.

Moreover, the following conditions are equivalent:

(α) $v(I_P) \leqslant \mathrm{ht}(P) - 1$ *for every prime* $P \supset I$ *such that* $\mathrm{ht}(P) \geqslant \mathrm{ht}(I) + 1$.

(β) $\mathrm{Sym}(I/I^2)$ *is R/I-torsion free.*

Remark. In the latter set of equivalent conditions, if in addition I is a prime ideal, then $I^k = I^{(k)}$, $k \geqslant 0$, that is, the ordinary and symbolic powers of I coincide.

For the proof one needs to estimate the dimensions of symmetric algebras of modules. This question was partly dealt with in [21], and we recall here some consequences.

Let R be a Cohen–Macaulay ring and let E be a finitely generated R-module with a (generic) rank, that is, $E \otimes_R K$ is a free K module of rank $= \mathrm{rk}(E)$, where K is the total ring of quotients of R. Thus an ideal I of height > 0 has rank 1.

It is convenient to relate the dimension of $\mathrm{Sym}(E)$ to the details of a free presentation of E

$$R^m \xrightarrow{\phi} R^n \to E \to 0, \qquad \phi = (a_{ij}), \quad a_{ij} \in R.$$

$\mathrm{Sym}(E)$ may then be written as $R[T_1,..., T_n]/J$, where J is generated by the linear forms $l_j = \sum a_{ij} T_i$, $1 \leqslant j \leqslant m$. Note that $\mathrm{rk}(\phi) = n - \mathrm{rk}(E)$.

We shall denote by $I_t(\phi)$ the ideal generated by the $t \times t$ minors of the matrix ϕ: $I_t(\phi) = F_{n-t}(E) = (n-t)$th-fitting ideal of E.

PROPOSITION 6.5. *Let R be a Cohen-Macaulay local ring and let E be an R module with a rank. Then the following conditions are equivalent*:

(a) *grade* $I_t(\phi) \geqslant \mathrm{rk}(\phi) - t + 1$, $1 \leqslant t \leqslant \mathrm{rk}(\phi)$.

(b) $\dim \mathrm{Sym}(E) \leqslant \dim R + \mathrm{rk}(E)$

(c) $v(E_P) \leqslant \mathrm{ht}(P) + \mathrm{rk}(E)$ *for every prime ideal P.*

Proof. The equivalence of (a) and (b) is proved in [22, (2.2) and (2.3)], while the equivalence of (a) and (c) follows from Lemma 6.6. ∎

LEMMA 6.6. *Let R be a Cohen–Macaulay ring and let E be an R module with a rank. Let ϕ be the matrix of a presentation of E as above. Then the following conditions are equivalent for any positive integer k*:

(a) *grade* $I_t(\phi) \geqslant \mathrm{rk}(\phi) - t + k + 1$, $1 \leqslant t \leqslant \mathrm{rk}(\phi)$.

(b) $v(E_P) \leqslant \mathrm{ht}(P) + \mathrm{rk}(E) - k$, *for every prime P such that E_P is not R_P-free.*

Proof. First, for any prime P, we bring the original presentation to the form

$$R_P^l \oplus R_P^r \xrightarrow{id \oplus \phi'} R_P^l \oplus R_P^v \to E_P \to 0, \qquad v = v(E_P), \quad l = n - v.$$

The matrix ϕ' has entries in PR_P (hence $r = 0 \Leftrightarrow E_P$is R_P-free). From this one clearly has $I_t(\phi) \subset P \Leftrightarrow t \geqslant l + 1$.

(a) $\Rightarrow$ (b) Let P be such that E_P is not R_P-free. Then $v(E_P) > \mathrm{rk}(E)$, so $l + 1 = n - v(E_P) + 1 \leqslant n - \mathrm{rk}(E) = \mathrm{rk}(\phi)$ and $I_{l+1}(\phi) \subset P$. We then have from the assumption

$$\begin{aligned} \mathrm{ht}(P) \geqslant \mathrm{grade}\, I_{l+1}(\phi) &\geqslant \mathrm{rk}(\phi) - (l+1) + (k+1) \\ &= n - \mathrm{rk}(E) - l + k \\ &= (n-l) - \mathrm{rk}(E) + k = v(E_p) - \mathrm{rk}(E) + k. \end{aligned}$$

(b) $\Rightarrow$ (a) Given $1 \leqslant t \leqslant \mathrm{rk}(\phi)$, pick $P \supset I_t(\phi)$ such that $\mathrm{ht}(P) = \mathrm{ht}(I_t(\phi))$. In particular $I_{\mathrm{rk}(\phi)}(\phi) \subset I_t(\phi)$ and hence E_P is not R_P-free. Therefore, by assumption and the above remark

$$\begin{aligned}\text{grade } I_t(\phi) = \text{ht}(P) &\geqslant v(E_P) - \text{rk}(E) + k = n - l - \text{rk}(E) + k \\ &\geqslant n - (t-1) - \text{rk}(E) + k = (n - \text{rk}(E)) - t + k + 1 \\ &= \text{rk}(\phi) - t + k + 1. \quad \blacksquare\end{aligned}$$

Remark. We can replace (b) by

(b′) $v(E_P) \leqslant \text{ht}(P) + \text{rk}(E) - k$, for each prime P with $\text{ht}(P) \geqslant k$, provided we are given that E is free in codimension $\leqslant k-1$. Since E is always assumed to be free in codimension 0, (b) $\Leftrightarrow$ (b′) for $k = 1$. We apply this in

COROLLARY 6.7. *Let R be a Cohen–Macaulay ring and let I be a Cohen–Macaulay ideal that is generically a complete intersection. Then the following conditions are equivalent for a presentation $\phi: R^m \to R^n$:*

(a) *grade* $I_t(\phi) \geqslant (n-1) - t + 3$, $1 \leqslant t \leqslant n - \text{ht}(I)$.

(b) $v(I_P) \leqslant \text{ht}(P) - 1$, *for every prime* $P \supset\ \supset I$ *with* $\text{ht}(P) \geqslant \text{ht}(I) + 1$.

Proof. Note that the assumptions include: If $I \neq (0)$, then $\text{ht}(I) > 0$ and I/I^2 has a rank.

It suffices to apply Lemma 6.6 along with the remark above, provided we show that

(1) Condition (a) is equivalent to

$$\text{grade } I_t(\phi^*) \geqslant \text{rk}(\phi^*) - t + 2, \qquad 1 \leqslant t \leqslant \text{rk}(\phi^*),$$

where

$$(R/I)^m \xrightarrow{\phi^*} (R/I)^n \to I/I^2 \to 0.$$

For this, note that $I \subset \sqrt{I_{n-1}(\phi)} \subset \sqrt{I_t(\phi)}$, $1 \leqslant t \leqslant n-1$ and that $\text{rk}(\phi^*) = n - \text{rk}(I/I^2) = n - \text{ht}(I)$. Since R/I is Cohen–Macaulay, we get

$$\text{grade } I_t(\phi) = \text{grade } I_t(\phi^*) + \text{ht}(I), \qquad 1 \leqslant t \leqslant n\text{-grade } I.$$

(2) Condition (b) is equivalent to

$$v((I/I^2)_P) \leqslant \text{ht}(P/I) + \text{rk}(I/I^2) - 1, \quad \text{for every prime } P \supset I$$

with $\text{ht}(P) \geqslant \text{ht}(I) + 1$.

But the equalities $v(I_P) = v((I/I^2)_P)$ and $\text{ht}(P/I) = \text{ht}(P) - \text{ht}(I)$ translate (2) into (b). ∎

The conditions of Proposition 6.5 are realized when $\text{Sym}(E)$ is Cohen–Macaulay. More generally

PROPOSITION 6.8. *Let R be a Cohen–Macaulay local ring and let E be a finitely generated module with a rank.*

(i) *If* $\mathrm{Sym}(E)$ *is unmixed (that is, the minimal primes of* $\mathrm{Sym}(E)$ *have the same dimension), then* $\dim \mathrm{Sym}(E) = \dim R + \mathrm{rk}(E)$.

(ii) *Moreover the following are equivalent:*

(a) $\mathrm{Sym}(E)$ *is R torsion-free.*

(b) $\mathrm{Sym}(E)$ *is unmixed and* $v(E_P) \leqslant \mathrm{ht}(P) + \mathrm{rk}(E) - 1$ *for every prime P with* $\mathrm{ht}(P) \geqslant 1$.

Proof. The proof of (i) is the same as [21, (2.4)], where R was taken to be a domain.

(ii) (a) $\Rightarrow$ (b) First, to show that J, in the presentation $\mathrm{Sym}(E) = R[T_1,..., T_n]/J$, is height unmixed, it suffices to go over to K, the total ring of fractions of R; but $E \otimes K$ is free by hypothesis.

Let P be a prime of R of height $\geqslant 1$. Since $\mathrm{Sym}(E)$ is R torsion-free, $P\mathrm{Sym}(E)_P$ is a prime of $\mathrm{Sym}(E_P)$ of ht > 0. Thus

$$v(E_P) = \dim(\mathrm{Sym}(E_P)/P\mathrm{Sym}(E_P)) \leqslant \dim \mathrm{Sym}(E_P) - \mathrm{ht}(P\mathrm{Sym}(E_P)).$$

Since $\mathrm{Sym}(E_P)$ is unmixed, by (i) we have

$$\dim \mathrm{Sym}(E_P) = \mathrm{ht}(P) + \mathrm{rk}(E)$$

and the assertion follows.

(b) $\Rightarrow$ (a) Let $Q \subset \mathrm{Sym}(E)$ be an associated prime of $\mathrm{Sym}(E)$ and denote by P its inverse image in R. We must show that $\mathrm{ht}(P) = 0$. Localizing at P we may assume that P is the unique maximal ideal of R without affecting the hypotheses.

If $\mathrm{ht}(P) > 0$, we have $n = v(E) \leqslant \mathrm{ht}(P) + \mathrm{rk}(E) - 1$. But $P\mathrm{Sym}(E)$ must be Q since $\mathrm{Sym}(E)$ is unmixed, and thus, again by (i), $\mathrm{ht}(P) + \mathrm{rk}(E) = \dim \mathrm{Sym}(E) = \dim(\mathrm{Sym}(E)/P\mathrm{Sym}(E)) \leqslant \mathrm{ht}(P) + \mathrm{rk}(E) - 1$, which is a contradiction. ∎

We single out the following consequence:

COROLLARY 6.9. *Let R be a Cohen–Macaulay ring and let I be an ideal of height* > 0.

(a) *If* $\mathrm{Sym}(I)$ *is Cohen–Macaulay, then for each prime ideal P,* $v(I_P) \leqslant \mathrm{ht}(P) + 1$.

(b) *If I is a Cohen–Macaulay ideal which is generically a complete intersection and* $\mathrm{Sym}(I/I^2)$ *is Cohen–Macaulay, then* $v(I_P) \leqslant \mathrm{ht}(P)$ *for each prime* $P \supset I$.

Proof of Theorem 6.4. (a) $\Rightarrow$ (b) We may assume R local. Since $\mathrm{Sym}(I) \cong \mathscr{R}(I)$ implies $\mathrm{Sym}(I/I^2) \cong \mathrm{gr}_I(R)$, it suffices to check that $\mathrm{gr}_I(R)$ is Cohen–Macaulay. For that it is enough to count depths relative to the maximal irrelevant ideal of $\mathscr{R}(I)$ along the two exact sequences

$$0 \to I\mathscr{R}(I) \to \mathscr{R}(I) \to \mathrm{gr}_I(R) \to 0$$

$$0 \to \mathscr{R}(I)_+ \to \mathscr{R}(I) \to R/I \to 0$$

and observing the isomorphism $I\mathscr{R}(I) \cong \mathscr{R}(I)_+$.

The remaining implications, (b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (a), as well as the second set of equivalences, all follow from Proposition 6.8. ∎

Remark. Note that the proof actually showed that the assumptions "Sym(I) is C–M" and "$v(I_P) \leqslant \mathrm{ht}(P)$, for all $P \supset I$" imply "$\mathrm{Sym}(I) \cong \mathscr{R}(I)$" and "$\mathrm{Sym}(I/I^2)$ is C–M" without the hypothesis that either R/I be C–M or I a generic complete intersection. Also, note that the last set of equivalences depended only on $\mathrm{Sym}(I/I^2)$ being C–M and not on actually Sym(I) being C–M, so that $\mathrm{Sym}(I/I^2) \not\cong \mathrm{gr}_I(R)$ might be the case there.

B *Criteria for Cohen–Macaulayness of* Sym(I)

Theorem 6.4 points out the interest of knowing when Sym(I) is Cohen–Macaulay. We give a criterion for this to happen, in terms of the acyclicity of the $\mathscr{Z}$ complex.

THEOREM 6.10. *Let R be a Cohen–Macaulay ring and let I be an ideal of height* > 0. *Assume*

(i) $v(I_P) \leqslant \mathrm{ht}(P) + 1$, *for every prime P.*

(ii) *depth*$(H_i)_P \geqslant \mathrm{ht}(P) - v(I_P) + i$, *for every prime P and* $0 \leqslant i \leqslant v(I_P) - \mathrm{ht}(I_P)$, *the H_i's denoting the Koszul homology modules on a set of generators of I.*

Then Sym(I) is Cohen–Macaulay.

Proof. The conditions imply, in a manner analogous to Theorem 6.1, that the $\mathscr{Z}$ complex is acyclic. The same depth chasing then yields (locally) depth $\mathrm{Sym}(I) \geqslant \dim R + 1$. Finally, condition (i) is equivalent, by Corollary 6.9, to having $\dim \mathrm{Sym}(I) \leqslant \dim R + 1$. ∎

COROLLARY 6.11. *Let R be a Cohen–Macaulay ring and let I be an ideal of finite projective dimension. Assume either*:

(i) $\mathrm{ht}(I) = 2$ *and R/I is Cohen–Macaulay, or*

(ii) $\mathrm{ht}(I) = 3$ *and R/I (and R) is Gorenstein.*

Then Sym(I) *is Cohen–Macaulay if and only if* $v(I_p) \leqslant \mathrm{ht}(P) + 1$ *for every prime P.*

Proof. For such ideals' the Koszul homology modules on a generating set are Cohen–Macaulay (cf. [2, 14]). The equivalence thus results from Theorem 6.10 and Corollary 6.9. ∎

The following extends a result of [19].

COROLLARY 6.12. *Let* $(R, \mathfrak{m})$ *be a Cohen–Macaulay local ring of dimension* > 0 *and let be an* $\mathfrak{m}$*-primary ideal. The following conditions are equivalent*:

(a) *I is an almost complete intersection* (*meaning*: *I can be generated by* $\mathrm{ht}(I) + 1$ *elements*).

(b) Sym(I) *is Cohen–Macaulay.*

Proof. (a) ⇒ (b) follows from Theorem 6.10, while the converse is implied by Corollary 6.9. ∎

PROPOSITION 6.13. *Let* $(R, \mathfrak{m})$ *be a Cohen–Macaulay local ring and let I be an ideal of height* > 0. *Assume*:

(a) *I is generated by a proper sequence* $\{x_1, \ldots, x_n\}$.

(b) *depth* $R/(x_1, \ldots, x_j) \geqslant \mathrm{depth}\, R - j$, *for* $1 \leqslant j < n$ *and depth* $R/I \geqslant \dim R/I - 1$.

(c) $v(I_P) \leqslant \mathrm{ht}(P) + 1$ *for every P.*

Then Sym(I) *is Cohen–Macaulay.*

Proof. (a) implies that the $\mathscr{Z}$ complex is acyclic. Further, (a) and (b) combine to imply that depth $H_j(x_1, \ldots, x_n) \geqslant \dim R + j - n$, (cf. argument of [11, (5.8)]. Now by depth-chasing, as in Theorem 6.1, we get $\mathrm{depth}(Z_i \otimes S) \geqslant \dim R + i + 1$, and thus, by Proposition 6.2, depth Sym(I) $\geqslant \dim R + 1$. Finally, (c) provides for dim Sym(I) $= \dim R + 1$, s before. ∎

COROLLARY 6.14, *Let R be a Cohen–Macaulay local ring and let I be an ideal of height* > 0. *Assume*:

(a) *I is an almost complete intersection.*

(b) *depth* $R/I \geqslant \dim R/I - 1$.

Then Sym(I) *is Cohen–Macaulay.*

Proof. The proof follows from Proposition 6.13, as I is then generated by $\{x_1, \ldots, x_{n-1}, x_n\}$, where the first $n - 1$ elements may be chosen to form a regular sequence. ∎

C *Specialization Behaviour of* Sym(I)

In this section we show, in the vein of [7, (1.1)], under what conditions the Cohen–Macaulay property of the symmetric algebra is preserved under specialization.

Precisely, let R be a Cohen–Macaulay ring and let IR be an ideal of height > 0. We assume an ideal N is given such that (i) N is perfect, and (ii) I is not contained in any associated prime of N. We then ask when Sym(I) Cohen–Macaulay implies Sym(I/NI) Cohen–Macaulay. Note that (ii) implies that I/NI is of rank one as an R/N module.

PROPOSITION 6.15. *Let I and N be given as above. If* Sym(I) *is Cohen–Macaulay, then the following conditions are equivalent*:

(a) Sym(I/NI) *is Cohen–Macaulay.*

(b) $v(I_P) \leqslant \mathrm{ht}(P/N) + 1$, *for every prime* $P \supset I + N$.

Proof. (a) ⇒ (b) Since $v(I_P) = v((I/NI)_P)$ holds, this follows from Proposition 6.8.

(b) ⇒ (a) By the acyclicity lemma of [18], it suffices to show that $\mathrm{ht}(N\,\mathrm{Sym}(I)) \geqslant \mathrm{ht}(N)$. But this follows dorectly from Propositions 6.5 and 6.8. ∎

Remark. Note that the conditions imply $\mathrm{Tor}_1^R(I, R/N) = 0$, which, however, may not suffice to make I/NI isomorphic to an ideal of R/I. For this we need to stronger requirement: $v(I_P) \leqslant \mathrm{ht}(P/N)$ as in Proposition 6.8.

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