# Embeddability of multiple cones

D. Repovš [a,*], W. Rosicki [b], A. Zastrow [b], M. Željko [a]

[a] *Institute of Mathematics, Physics and Mechanics, University of Ljubljana, PO Box 2964, Ljubljana 1001, Slovenia*
[b] *Institute of Mathematics, Gdańsk University, Ul. Wita Stwosza 57, 80-952 Gdańsk, Poland*

**Abstract**

The main result of this paper is that if $X$ is a Peano continuum such that its $n$th cone $C^n(X)$ embeds into $\mathbb{R}^{n+2}$ then $X$ embeds into $S^2$. This solves a problem proposed by W. Rosicki.
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## 1. Introduction

The classical Lefschetz–Nöbeling–Pontryagin Embedding Theorem [10] asserts that every compact metric space $X$ of dimension $n$ embeds into $\mathbb{R}^{2n+1}$. We are interested in the relationship between the embeddability of $X$ and embeddability of its Cartesian product $X \times I^n$ with a cube $I^n$ (respectively its cone $C(X)$, iterated cone $C^n(X) = C(\ldots(C(X))\ldots)$, suspension $\Sigma(X)$). Clearly, if $X$ embeds in $\mathbb{R}^m$, then $X \times I^n$ and $C^n(X)$ embed into $\mathbb{R}^{n+m}$. However, sometimes they embed into lower-dimensional Euclidean space. Such is the case for the spheres $S^n$, where $S^n$, $C(S^n) \cong B^{n+1}$ and $S^n \times I$ all embed into $\mathbb{R}^{n+1}$.

Let $X$ be a Peano continuum. It was proved in [14] that if the cone $C(X)$ of $X$ embeds into $\mathbb{R}^3$, then $X$ embeds into $S^2$. As a consequence, if the suspension $\Sigma(X)$ of $X$ embeds into $\mathbb{R}^3$, then $X$ is planar. Note that for each $n \geqslant 3$, there exists a Peano continuum $X_n$ such that $X_n$ is not embeddable in $S^n$, whereas the cone $C(X_n)$ of $X_n$ is embeddable in $\mathbb{R}^{n+1}$ (see [14]).

The main result of this paper is Theorem 1.1 which solves a problem from [14]. Our proof is based on the methods of [4] and [14].

**Theorem 1.1.** *Let $X$ be a Peano continuum. Suppose that for some $n \in \mathbb{N}$, $C^n(X)$ is embeddable in $\mathbb{R}^{n+2}$. Then $X$ is embeddable in $S^2$.*

* Corresponding author.
  *E-mail addresses:* dusan.repovs@guest.arnes.si (D. Repovš), wrosicki@math.univ.gda.pl (W. Rosicki), zastrow@math.univ.gda.pl (A. Zastrow), matjaz.zeljko@fmf.uni-lj.si (M. Željko).
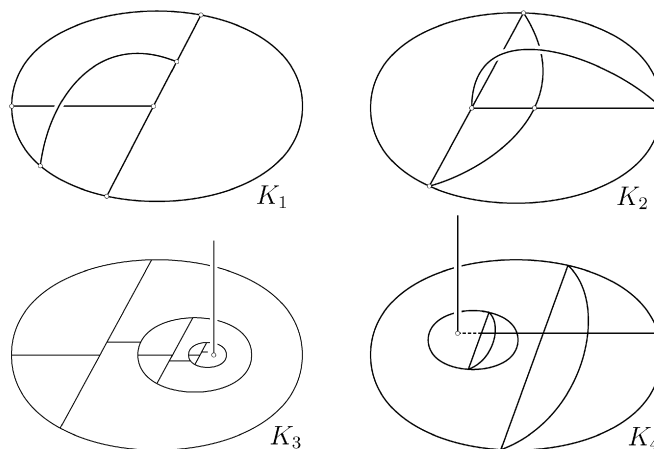
Fig. 1. Kuratowski curves $K_1$, $K_2$, $K_3$, $K_4$.

Let $X$ be a Peano continuum. Claytor [7] proved that $X$ is embeddable in $S^2$ if and only if $X$ does not contain any of the Kuratowski curves $K_1$, $K_2$, $K_3$, $K_4$ (see Fig. 1).

## 2. Preliminaries

A space $X$ is said to be *planar* if $X$ is embeddable in $\mathbb{R}^2$. We say that $X$ is *locally planar* if for every point $x \in X$ there exists a neighbourhood $U_x$ of $x$ in $X$ such that $U_x$ is embeddable in $\mathbb{R}^2$. Rosicki [13, Theorem 1.1] proved that if a Peano continuum $X$ is embeddable in $\mathbb{R}^3$ and $X$ is a nontrivial Cartesian product $X = Y \times Z$ then one of the factors is either an arc or a simple closed curve.

Rosicki [13] also proved that if a Peano continuum $X$ is embeddable in $\mathbb{R}^3$ and is homeomorphic to the product $Y \times S^1$ then the factor $Y$ must be planar. Alternatively, if $X = Y \times [0, 1]$ is embeddable in $\mathbb{R}^3$ and $\check{H}^1(X) = \check{H}^2(X) = 0$ then $Y$ must be planar. Cauty [4], generalizing Rosicki [13], proved that for every $n > 3$ and every Peano continuum $X$ such that $X \times I^{n-2}$ is embeddable into an $n$-manifold, it follows that $X$ must be locally planar. This theorem was stated earlier by Stubblefield [15]. However, Burgess [2] found a mistake in his proof.

Borsuk [1] constructed an example of a locally connected, locally planar continuum $X$ which is not embeddable into any surface. This continuum contains a sequence $(X_n)$ of subsets homeomorphic to Kuratowski curve $K_1$ which converge to an arc. Cauty [4] proved that $X \times I^{n-2}$ is not embeddable into any $n$-manifold so the converse to his theorem does not hold.

## 3. Local separation

We say that a subset $D \subset \mathbb{R}^n$ *locally separates* $\mathbb{R}^n$ at the point $x_0 \in D$ into $k \in \mathbb{N}$ components if there exists $\varepsilon > 0$ such that for all $0 < \delta < \varepsilon$, the set $B(x_0, \delta) \setminus D$ has exactly $k$ components $A_1, \ldots, A_k$ for which $x_0 \in \overline{A_i}$, for all $i \in \{1, \ldots, k\}$.

It is easy to prove the following lemma using similar methods as in the proof of Lemma 1 in [14].

**Lemma 3.1.** *A homeomorphic image of any $n$-disk locally separates $\mathbb{R}^{n+1}$ at any point of its interior into two components.*

Note that $C^n(X) = \sigma^{n-1} * X = \{xt + y(1-t); \ x \in \sigma^{n-1}, \ y \in X, \ t \in [0, 1]\}$, where $\sigma^{n-1}$ is an $(n-1)$-simplex. Then $\sigma^{n-1} * \{x\}$ is an $n$-ball and $\sigma^{n-1} * I$ is an $(n+1)$-ball. We consider $\sigma^{n-1}$ as a subset of $\sigma^{n-1} * X$.

**Lemma 3.2.** *Let $I_i$, $i \in \{1, \ldots, k\}$, $k > 1$ be arcs with common endpoints and pairwise disjoint interiors and $C_k = C^n(\bigcup_{i=1}^k I_i) = \sigma^{n-1} * (\bigcup_{i=1}^k I_i)$. Let $h: C_k \to \mathbb{R}^{n+2}$ be an embedding. Then $h(C_k)$ locally separates $\mathbb{R}^{n+2}$ at any point $h(x_0)$, where $x_0$ is an interior point of $\sigma^{n-1}$, into $k$ components (where $\sigma^{n-1}$ is considered as a subset of $C_k$).*

**Proof.** The proof is by induction on $k$. If $k = 2$, then $C_2 = \sigma^{n-1} * S^0 * S^0$ hence $h(C_2)$ locally separates $\mathbb{R}^{n+2}$ at $h(x_0)$ into two components, by Lemma 3.1.

Assume that Lemma 3.2 holds for $k - 1$. Choose $\varepsilon > 0$ smaller than the distance between $h(x_0)$ and the image of $\partial\sigma^{n-1} * (\bigcup_{i=1}^{k} I_i)$. Let $\delta > 0$ be so small that

$$D_k = h\big(C_k \cap B(x_0, \delta)\big) \subset B\big(h(x_0), \varepsilon\big).$$

There exists an open connected set $U_k \subset \mathbb{R}^{n+2}$ such that $D_k = U_k \cap h(C_k)$. Consider the exact sequence of the pair $(U_k, U_k \setminus D_k)$:

$$\to H_1(U_k) \to H_1(U_k, U_k \setminus D_k) \to H_0(U_k \setminus D_k) \to H_0(U_k) \to H_0(U_k, U_k \setminus D_k) \to 0.$$

Since $U_k$ is an open $(n + 2)$-manifold, $H_1(U_k) \cong \check{H}_c^{n+1}(U_k)$ by the Poincaré duality, where $\check{H}_c$ denotes the Čech cohomology with compact supports. Also $H_1(U_k, U_k \setminus D_k) \cong \check{H}_c^{n+1}(D_k)$ (see [9, VIII, 7.14], where $L = \emptyset$, $K = D_k$ and $X = U_k$).

We know that $H_0(U_k, U_k \setminus D_k) = 0$ because $U_k$ is arc-connected and $U_k \setminus D_k \neq \emptyset$. Therefore we can consider the exact sequence

$$\to \check{H}_c^{n+1}(U_k) \to \check{H}_c^{n+1}(D_k) \to H_0(U_k \setminus D_k) \to H_0(U_k) \to 0.$$

Next we show by induction that the map $\check{H}_c^{n+1}(U_k) \to \check{H}_c^{n+1}(D_k)$ is trivial. If $k = 2$ then $D_k$ is an open $(n + 1)$-ball. Then $H_0(U_k \setminus D_k) \cong \mathbb{Z}^2$, by Lemma 3.1. Since $\check{H}_c^{n+1}(D_k) \cong \mathbb{Z}$ and $H_0(U_k) \cong \mathbb{Z}$, we obtain the exact sequence

$$\check{H}_c^{n+1}(U_k) \to \mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z} \to 0.$$

Hence the map $\check{H}_c^{n+1}(U_k) \to \check{H}_c^{n+1}(D_k)$ is indeed trivial, as asserted.

Since $\check{H}_c^{n+1}(D_2) \cong \mathbb{Z}$, we obtain by induction that $\check{H}_c^{n+1}(D_k) \cong \check{H}_c^{n+1}(D_{k-1}) \oplus \check{H}_c^{n+1}(D_2') \cong \mathbb{Z}^{k-2} \oplus \mathbb{Z}$, where $D_2' = h(C^n(I_1 \cup I_k) \cap B(x_0, \delta))$.

The map $\check{H}_c^{n+1}(U_k) \to \check{H}_c^{n+1}(D_k) \cong \check{H}_c^{n+1}(h(D_{k-1})) \oplus \check{H}_c^{n+1}(D_2')$ is trivial because both of its coordinates are trivial, by inductive hypothesis.

Therefore the sequence

$$0 \to \check{H}_c^{n+1}(D_k) \to H_0(U_k \setminus D_k) \to H_0(U_k) \to 0$$

is exact. So the sequence

$$0 \to \mathbb{Z}^{k-1} \to H_0(U_k \setminus D_k) \to \mathbb{Z} \to 0$$

is also exact. Hence $H_0(U_k \setminus D_k) \cong \mathbb{Z}^k$ and $U_k \setminus D_k$ has $k$ components.

The point $h(x_0)$ belongs to the closure of each of them. Indeed, if $X_k$ is $D_k$ with a small open neighbourhood of $h(x_0)$ removed then $\check{H}_c^{n+1}(X_k) \cong 0$ and the sequence

$$0 \to H_0(U_k \setminus X_k) \to H_0(U_k) \to 0$$

is exact, therefore $H_0(U_k \setminus X_k) \cong \mathbb{Z}$. $\quad\square$

## 4. Proof of Theorem 1.1

We shall need two more lemmata:

**Lemma 4.1.** *Consider the Kuratowski curve $K_1$ and let $n \in \mathbb{N}$. Then $C^n(K_1)$ is not embeddable in $\mathbb{R}^{n+2}$.*

**Proof.** Suppose to the contrary, that there exists an embedding $h : C^n(K_1) \to \mathbb{R}^{n+2}$. Consider $K_1 \subset \mathbb{R}^3$ and denote (see Fig. 2)

$$I_1 = [c, a] \cup [a, b], \quad I_2 = [c, p] \cup [p, b], \quad \text{and} \quad I_3 = [c, d] \cup [d, b].$$

If $X = \bigcup_i I_i$, then $\sigma^{n-1} * X = \bigcup_i (\sigma^{n-1} * I_i)$ is a union of $(n + 1)$-disks. Let $x_0 \in \text{Int}\,\sigma^{n-1}$ and choose $\varepsilon > 0$ so that (see Fig. 3)
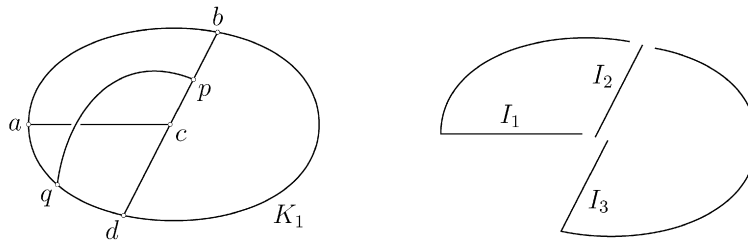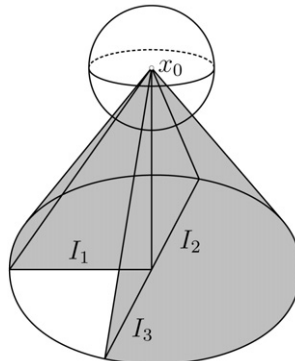
Fig. 2. Kuratowski curve $K_1$.



Fig. 3. Local separation at $h(x_0)$.

$$C_1 = h(\sigma^{n-1} * (I_1 \cup I_3)) \quad \text{locally separates } B(h(x_0), \varepsilon) \text{ into } B_1, A_1 \text{ at } h(x_0),$$

$$C_2 = h(\sigma^{n-1} * (I_1 \cup I_2)) \quad \text{locally separates } B(h(x_0), \varepsilon) \text{ into } B_2, A_2 \text{ at } h(x_0),$$

$$C_3 = h(\sigma^{n-1} * (I_2 \cup I_3)) \quad \text{locally separates } B(h(x_0), \varepsilon) \text{ into } B_3, A_3 \text{ at } h(x_0).$$

By Lemma 3.2 we have that $C = h(C^n(I_1 \cup I_2 \cup I_3)) = h(\sigma^{n-1} * \bigcup_{i=1}^{3} I_i) = h(\bigcup_{i=1}^{3} \sigma^{n-1} * I_i)$ locally separates $B(h(x_0), \varepsilon)$ into three components. We will show that we can adopt the notation for these three components to be $B_1$, $A_2$ and $A_3$.

We use abstract linear combinations for describing our joins, e.g.

$$\sigma^{n-1} * K_1 = \{xt + y(1 - t); \ x \in \sigma^{n-1}, \ y \in K_1, \ t \in [0, 1]\}.$$

For $\sigma^{n-1} \subset \sigma^{n-1} * K_1$, we have that $h(\sigma^{n-1})$ is a subset of $C_1$, but that $h|_{\sigma^{n-1} * I_2}$ maps all linear combinations with $t \neq 1$, but sufficiently close to 1, to a subset that is connected but disjoint from $C_1$. Hence this subset can only be contained either in $A_1$ or in $B_1$. We may assume that it is in $A_1$. Since the entire neighbourhood of $\sigma^{n-1}$ in $\sigma^{n-1} * I_2$ is mapped by $h$ into $A_1$, we have $h(\sigma^{n-1} * I_2) \cap B_1 = \emptyset$, provided $\varepsilon > 0$ is small enough. Then $B_1$ is not divided by $C$, so it is one of the three components.

Analogously, by considering $C_2$ (respectively $C_3$) we can make sure that $A_2$ and $A_3$ are the other two components and that $h(\sigma^{n-1} * I_3) \cap A_2 = \emptyset$ and $h(\sigma^{n-1} * I_1) \cap A_3 = \emptyset$. Since $C \cup B_1 \cup A_2 \cup A_3$ and $C \cup A_1 \cup B_1$ are both disjoint decompositions of a neighbourhood of $h(x_0)$, the set $h(\sigma^{n-1} * I_2) \cup C_1$ separates the component $A_1$ into components $A_2$ and $A_3$.

Note that

$$x_0 * K_1 = \{x_0 t + x(1 - t); \ x \in K_1, \ t \in [0, 1]\} \subset C^n(K_1).$$

Choose $t_0$ near 1 so that

$$h(\{x_0 t + x(1 - t); \ x \in K_1, \ t \geqslant t_0\}) \subset B(h(x_0), \varepsilon).$$

Let $p' = h(x_0 t_0 + p(1 - t_0)) \in A_1$. The arc $H = h(\{x_0 t_0 + x(1 - t_0); \ x \in (p, q)\})$ is contained in $B(h(x_0), \varepsilon) \setminus h(C)$. Therefore points $p'$ and $q' = h(x_0 t_0 + q(1 - t_0))$ are in the same component. Hence $q' \in A_2$ or $q' \in A_3$. So the arc

$I = h(\{x_0t_0 + x(1 - t_0); \ x \in (a, q] \cup [q, d)\})$ is contained either in $A_2$ or in $A_3$. But this yields a contradiction since $a' = h(x_0t_0 + a(1 - t_0)) \notin \overline{A_3}$ (so $I \not\subset A_3$) and $d' = h(x_0t_0 + a(1 - t_0)) \notin \overline{A_2}$ (so $I \not\subset A_2$). □

The proof of the next lemma can be obtained by changing the proof of [14, Lemma 4] in the same way as we did it for the proof of Lemma 2.3 using the proof of [14, Lemma 3].

**Lemma 4.2.** *Consider the Kuratowski curve $K_2$ and let $n \in \mathbb{N}$. Then $C^n(K_2)$ is not embeddable in $\mathbb{R}^{n+2}$.*

**Proof of Theorem 1.1.** By Claytor's theorem (see [6,7]), it suffices to show that $C^n(K_i)$ is not embeddable into $\mathbb{R}^{n+2}$ for any $i \in \{1, 2, 3, 4\}$. Now, Cauty [4] proved that $K_i \times I^n$ is not embeddable into $\mathbb{R}^{n+2}$ for any $i \in \{3, 4\}$. Therefore also $C^n(K_i)$ is not embeddable into $\mathbb{R}^{n+2}$ for any $i \in \{3, 4\}$. Hence we only have to consider the cases $i = 1$ and $i = 2$. The proof is now completed by application of Lemmata 4.1 and 4.2. □

## 5. Epilogue

Repovš, Skopenkov and Ščepin [12] proved that if $X \times I$ PL embeds into $\mathbb{R}^{n+1}$, where $X$ is either an acyclic polyhedron and $\dim X \leqslant \frac{2n}{3} - 1$ or a homologically $(2 \dim X - n - 1)$-connected manifold and $\dim X \leqslant \frac{2n}{3} - 1$ or a collapsible polyhedron, then $X$ PL embeds into $\mathbb{R}^n$.

**Question 5.1.** What can one say about embeddability of $X$ into Euclidean spaces if one considers $C(X)$ or $C^n(X)$ or $\Sigma(X)$ or $\Sigma^n(X)$ instead of $X \times I$ for $X$ in [12]?

It follows by [12] that if $X$ is a contractible polyhedron such that $X \times I$ embeds into $\mathbb{R}^{n+1}$ then $X$ embeds into $\mathbb{R}^n$. So if $X$ is contractible and $C(X) \subset \mathbb{R}^{n+1}$ then $X$ embeds into $\mathbb{R}^n$.

Note that there exists a polyhedron $P_n$ such that $P_n$ is not embeddable into $\mathbb{R}^n$ but $C^2(P_n)$ is embeddable in $\mathbb{R}^{n+2}$. Namely, Cannon [3] proved that if $H^n$ is a homology $n$-sphere then its double suspension $\Sigma^2(H^n)$ is the $(n + 2)$-sphere (see [8] and [11] for a far reaching generalization of this result by Edwards). So if $P_n = H^n \setminus B^n$ where $B^n$ is an $n$-ball then the double cone $C^2(P_n)$ embeds in $\mathbb{R}^{n+2}$. The polyhedron $P_n$ is acyclic but not contractible.

**Question 5.2.** Does there exist a contractible $n$-dimensional polyhedron $X^n$ such that $C^k(X^n)$ embeds into $\mathbb{R}^{n+k}$, but $X^n$ does not embed into $\mathbb{R}^n$?

In [14, Theorem 2] contractible continua $X_n$ were constructed, such that $X_n$ is not embeddable in $\mathbb{R}^n$, $C(X_n)$ is embeddable in $\mathbb{R}^{n+1}$, and $X_n$ is not a polyhedron. By [12], if $X$ is an $n$-polyhedron then $X \times I$ embeds into $\mathbb{R}^{2n+1}$. If $X$ is an $n$-polyhedron then $C(X)$ need not embed into $\mathbb{R}^{2n+1}$. For example, the Kuratowski curves $K_1$ and $K_2$ are 1-polyhedra but the cones $C(K_1)$ and $C(K_2)$ do not embed into $\mathbb{R}^3$.

**Question 5.3.** Suppose that $X$ is a compact contractible $n$-dimensional polyhedron. Does the cone $C(X)$ embed into $\mathbb{R}^{2n+1}$? Does the same hold if $X$ is only acyclic?

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