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Polar functions—II: completion classes of archimedean f-algebras vs. covers of compact spaces

Jorge Martinez*

Department of Mathematics, University of Florida, P.O. Box 118105, Gainesville, FL 32611-8105, USA

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Abstract

There is an inclusion preserving bijection between the class of completion classes of uniformly complete real f-algebras with identity and the partially ordered class of covering classes of compact Hausdorff spaces. In this setting a completion class $\mathbf{A}$ is a hull class of uniformly complete f-algebras, with the additional feature that $G \in \mathbf{A}$ if and only if $G^* \in \mathbf{A}$. Using an idempotent invariant polar function $\mathscr{X}$ and the covering function $\mathfrak{K}$ derived from it, the main theorem of this article states that the covering class associated with the uniformly complete f-algebras having no proper $\mathscr{X}$-splitting extensions is the class of compact spaces X which equal their $\mathfrak{K}$-cover.

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This article is concerned, in the first instance, with an effort to abstract the type of hull classes of uniformly complete archimedean f-algebras with identity, which correspond in a natural way to covering classes of topological spaces. This study is intimately related to [22]. That article initiates the study of polar functions, and they are of considerable relevance here. Polar functions lead to a discussion of the hull classes which can be defined in terms of the requirement that a predetermined class of polars become summands. The link here with the constructions in [22] is Theorem 5.8. This paper also devotes some attention, in particular, to the extension here called

* Tel.: +1-352-392-0281-245; fax: +1-352-392-8357.
E-mail address: martine@math.ufl.edu (J. Martinez).

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the ω_1-splitting completion, which is the counterpart to the cloz cover of Henriksen, Vermeer and Woods in [17]. The ambient category for the discussion below is **W**, the category of all archimedean ℓ-groups with designated weak unit, together with all the ℓ-homomorphisms which preserve the designated weak unit.

1. Invariant polar functions

The discussion begins with a brief review of the basic constructions and ideas of [22]. In the following remarks some concepts and terms from the theory of lattice-ordered groups are also reviewed. As to unexplained terminology about lattice-ordered groups the reader is referred to [5]. The common designation "ℓ-group" will be used to abbreviate "lattice-ordered group". Throughout, the symbol $\leqslant$ will be used to denote inclusion of subgroups, subalgebras and the like.

To start, let us review the notion of a hull class, from [13]. The definition employs the concept of an essential extension in **W**; we review that in 1.2.

Definition and Remarks 1.1. (a) Consider a class **H** of **W**-objects closed under formation of ℓ-isomorphic copies. An **H**-*hull* on **W** is a function assigning to each **W**-object A an extension hA, such that

(i) $A \leqslant hA$ is an essential extension, with $hA \in \mathbf{H}$, and
(ii) $A \leqslant B \leqslant A^e$, and $B \in \mathbf{H}$ imply that there exists a **W**-embedding $g: hA \to B$, extending the identity on A.

If there is such an h associated with the class **H**, we call the latter a *hull class* with *hull operator h*. One also uses the phrase "each **W**-object A has an **H**-hull" when such a hull operator exists.

It is shown in Proposition 2.4 of [13] that each **W**-object A has an **H**-hull if and only if **H** is *essentially intersective*; that is, for each essentially closed **W**-object E, and each collection $\mathfrak{A}$ of subobjects of E, such that $\mathfrak{A} \subseteq \mathbf{H}$ and $\bigcap \mathfrak{A}$ is essential in E, then $\bigcap \mathfrak{A} \in \mathbf{H}$. Note that any hull class contains all the essentially closed **W**-objects. In particular, every hull class is nontrivial.

At the outset, one should point out one of the most elementary hulls in algebra, the divisible hull; the extension is here denoted $G \leqslant dG$.

(b) Let **H** be a hull class. Denote by $\mathbf{H}^{\#}$ the class of all compact spaces X for which there is a **W**-object $A \in \mathbf{H}$ such that $YA \cong X$.

Definition and Remarks 1.2. (a) Suppose that G is an arbitrary **W**-object. A *polar* is a subgroup of the form

$$S^{\perp} \equiv \{\, g \in G \,:\, |g| \wedge |s| = 0, \quad \forall s \in S \,\},$$

for a suitable subset S of G. The set $\mathscr{P}(G)$ of all polars is a boolean algebra under inclusion; this is well known, and was first established by Šik for an arbitrary ℓ-group. The infimum in $\mathscr{P}(G)$ is intersection. Whenever there is a risk of confusion the subscript $\perp_G$ will be used to indicate that the polar in question is being considered in the group G.

A polar P for which $G = P \oplus P^\perp$ is called a *summand* of G; $\mathscr{S}(G)$ stands for the subalgebra of all summands of G.

(b) An extension $G \leqslant H$ is said to be *essential* if for each $h > 0$ in H there is a $g > 0$ in G and a positive integer n such that $g \leqslant nh$. Equivalently, H is an essential extension of G if and only if the trace map $P \mapsto P \cap G$ is a boolean isomorphism of $\mathscr{P}(H)$ onto $\mathscr{P}(G)$. The sufficiency in this equivalence uses the archimedean feature of the ℓ-group; this equivalence is part of Theorem 11.1.15 in [3], and is generally attributed to Conrad. Conrad first showed (in [4]) that each archimedean ℓ-group can be embedded in an *essential closure*: that is, there is an archimedean ℓ-group G^e which extends G essentially, such that $G \leqslant H$ is essential if and only if the inclusion of G in G^e extends to an ℓ-embedding of H in G^e. G^e is essentially closed, in the sense that it has no proper essential extensions of its own in **W**.

(c) Suppose that $\mathscr{X}$ denotes a subalgebra of $\mathscr{P}(G)$. An essential extension H of G is *$\mathscr{X}$-splitting* if, for each $K \in \mathscr{X}(G)$, $K^{\perp_H \perp_H}$ is a summand of H. For any subalgebra $\mathscr{X}$, G^e is an $\mathscr{X}$-splitting extension; for each $K \in \mathscr{X}(G)$ and each $a \in G^e$, $a[K]$ denotes the projection of a on the component $K^{\perp_{G^e} \perp_{G^e}}$. It is shown in [22, Lemma 2.2], that H is an $\mathscr{X}$-splitting extension if and only if, for each $K \in \mathscr{X}(G)$ and each $a \in G$, $a[K] \in H$.

Here is the main result from Section 2 of [22].

Theorem 1.3. *Let G be a* **W***-object and $\mathscr{X}$ be a subalgebra of $\mathscr{P}(G)$. Then there is a least $\mathscr{X}$-splitting extension $G[\mathscr{X}]$ in G^e. The elements of $G[\mathscr{X}]$ can be expressed in the following way: if $x \in G[\mathscr{X}]$ then there exist $a_1, \ldots, a_n \in G$ and $K_1, \ldots, K_n \in \mathscr{X}(G)$, the latter pairwise disjoint, such that*

$$x = \sum_{i=1}^{n} a_i[K_i].$$

Let α be an infinite cardinal. Recall that G is *α-projectable* if for each subset $S \subseteq G$, with $|S| < \alpha$, $G = S^{\perp\perp} \oplus S^\perp$. As in [11] the class of all α-projectable **W**-objects is denoted by $\mathbf{P}(\alpha)$.

Definition and Remarks 1.4. Now suppose that $\mathscr{X}$ denotes a function which assigns to a **W**-object G a subalgebra $\mathscr{X}(G)$ of $\mathscr{P}(G)$ which contains all the summands of G; following [22], we call $\mathscr{X}$ a *polar function*. It is assumed here that $\mathscr{X}$ is *invariant* in the sense that if $G \leqslant H$ is essential, then the assignment $K \mapsto K^{\perp_H \perp_H}$ carries $\mathscr{X}(G)$ into $\mathscr{X}(H)$. The abbreviation *ipf*, standing for "invariant polar function", is also carried over from [22].

With regard to the notation introduced in Theorem 1.3, please note that we abbreviate $G[\mathscr{X}(G)]$ to $G[\mathscr{X}]$ when an ipf is involved in the stipulation of the *least $\mathscr{X}$-splitting extension* (as opposed to "least $\mathscr{X}(G)$-splitting extension").

Call an ipf *idempotent* if $(G[\mathscr{X}])[\mathscr{X}] = G[\mathscr{X}]$, for each **W**-object G. Equivalently, $\mathscr{X}$ is idempotent if and only if $\mathscr{X}(G[\mathscr{X}]) \leqslant \mathscr{S}(G[\mathscr{X}])$.

Apart from $\mathscr{P}$ itself, with its associated hull $G \leqslant G[\mathscr{P}]$ and hull class $\mathbf{P}(\infty)$, here are what might be considered the standard examples of ipfs, discussed amply in [22]; $\mathscr{P}$ and the first two in the following list are idempotent, whereas the third is not:

(i) $\mathscr{P}_\omega$, the ipf which selects the subalgebra of $\mathscr{P}(G)$ generated by all the principal polars; that is, the polars of the form $g^{\perp\perp}$. As is explained in [22], $G[\mathscr{P}_\omega]$ is the projectable hull of G. The associated hull class is $\mathbf{P}(\omega)$. The reader should refer to [5,22] for details.

(ii) Let α be an uncountable, regular cardinal. $\mathscr{P}^\alpha_\alpha$ is the ipf which selects the α-generated polars of G which have an α-generated complement. ($P \in \mathscr{P}(G)$ is α-*generated* if there is a generating set having fewer than α elements.) The idempotence of $\mathscr{P}^\alpha_\alpha$ is argued in [22, 5.5(a)]. The corresponding hull class will be discussed in Example 4.8.

(iii) Let α be as in (ii). $\mathscr{P}_\alpha$ denotes the ipf which picks out the subalgebra generated by the α-generated polars of $\mathscr{P}(G)$. $\mathscr{P}_\alpha$ is not idempotent; for more discussion of this example the reader is referred to [13, Section 3]. The associated hull class here is $\mathbf{P}(\alpha)$.

For use in Section 5, we record the following remark. The proofs are immediate from Theorem 1.3. The first item in Proposition 1.5 is Corollary 2.5 of [22]. Recall that a lattice-ordered ring A is an f-*ring* if $a\wedge b=0$ and $c \geqslant 0$ imply that $ac\wedge b=0$. An abelian ℓ-group G which is also a real vector space such that $0 \leqslant r \in \mathbb{R}$ and $0 \leqslant g \in G$ imply that $rg \geqslant 0$ is a *vector lattice*.

Proposition 1.5. *Suppose that $\mathscr{X}$ is a subalgebra of polars of the* **W**-*object G. Then each of the following features of G is also a property of $G[\mathscr{X}]$:*

(a) *G is an f-ring.*
(b) *G is divisible.*
(c) *G is a vector lattice.*

Proof (Sketch). Apply to the three cases, in succession, the observation that G^{e} is an f-ring, divisible and a vector lattice. Then use Theorem 1.3. □

Definition and Remarks 1.6. In [22] is described a transfinite process which constructs, given the ipf $\mathscr{X}$, the least idempotent ipf $\mathscr{X}^\flat$ exceeding $\mathscr{X}$. A similar one occurs in Section 5. Here is a sketch of the construction, which simultaneously produces two transfinite towers. The reader should refer, in [22], to the remarks in 5.1 and to Proposition 5.2.

Suppose that $\mathscr{X}$ is an invariant polar function. Define, at the first step of the induction, $\mathscr{X}^1 \equiv \mathscr{X}$. Assume now that λ is an ordinal, and that for each ordinal $\gamma < \lambda$, $\mathscr{X}^\gamma$ is defined, such that for each **W**-object G and each $\gamma < \delta < \lambda$, we have that $\mathscr{X}^\gamma(G)$ is a subalgebra of $\mathscr{X}^\delta(G)$. If λ is a limit ordinal, then put

$$\mathscr{X}^\lambda(G) \equiv \bigcup_{\gamma<\lambda} \mathscr{X}^\gamma(G),$$

for each **W**-object G. On the other hand, if κ precedes λ, then set

$$\mathscr{X}^{\lambda}(G) \equiv \{\, G \cap P \;:\; P \in \mathscr{X}(G[\mathscr{X}^{\kappa}])\,\}.$$

It is easy to verify [22, 5.1] that $\mathscr{X}^{\kappa}(G) \leqslant \mathscr{X}^{\lambda}(G)$.

This procedure defines a transfinite sequence of ipfs

$$\mathscr{X} = \mathscr{X}^{1} \leqslant \cdots \leqslant \mathscr{X}^{\lambda} \leqslant \cdots \leqslant \mathscr{P}, \qquad (\dagger)$$

where it should be understood that, for polar functions $\mathscr{X}$ and $\mathscr{Y}$, $\mathscr{X} \leqslant \mathscr{Y}$ means that $\mathscr{X}(G) \leqslant \mathscr{Y}(G)$, for each **W**-object G.

The sequence in (†) must stabilize, owing to cardinality conditions; that is, for each **W**-object G there is an ordinal λ such that $\mathscr{X}^{\lambda}(G) = \mathscr{X}^{\mu}(G)$, for each $\mu > \lambda$. Finally, here is the definition we have been aiming for: for each **W**-object G, let

$$\mathscr{X}^{\flat}(G) \equiv \mathscr{X}^{\tau}(G),$$

where τ is the least ordinal such that $\mathscr{X}^{\tau'}(G) = \mathscr{X}^{\tau}(G)$, for each ordinal $\tau' > \tau$. This is the *idempotent closure* of the sequence $\mathscr{X}^{\lambda}$.

[22, Proposition 5.2] also tells us that, for each ordinal λ and each **W**-object G,

$$G[\mathscr{X}^{\lambda+1}] = G[\mathscr{X}^{\lambda}][\mathscr{X}]$$

and if λ is a limit ordinal then

$$G[\mathscr{X}^{\lambda}] = \bigcup_{\gamma<\lambda} G[\mathscr{X}^{\gamma}].$$

One also gets that $G[\mathscr{X}^{\flat}] = G^{\tau}[\mathscr{X}]$, for a suitable ordinal τ.

The above reviews but half of the machinery from [22]. In the next section the topological “half” of this machinery is reviewed. It is prefaced by a review of the Yosida Representation Theorem for **W**.

2. Invariant covering functions

In this section algebras of regular closed sets are considered. Denote by $\mathfrak{R}(X)$ the boolean algebra of all regular closed sets of the space X. The reader is reminded that all spaces are compact and Hausdorff. It is well known that, under inclusion, $\mathfrak{R}(X)$ is a complete boolean algebra, in which finite suprema coincide with set-theoretic unions. (Recall that a closed subset A of X is *regular* if A is the closure of its interior.)

We begin with a review of covering classes. Examples will be discussed in some detail in Section 4. For a more extensive background on covers, the reader is referred to [9] or Chapter 6 of [25].

Definition and Remarks 2.1. (a) Suppose that $f : Y \to X$ is a continuous surjection (of compact spaces). f is said to be *irreducible* if X is not the image of a proper closed subset of Y. It is well known that if f is irreducible then for each open set $U \subseteq Y$ there is an open set $V \subseteq X$ such that $f^{-1}(V)$ is dense in U. Also if f is irreducible then it induces a boolean isomorphism from $\mathfrak{R}(X)$, the algebra of regular closed sets of X, onto $\mathfrak{R}(Y)$, by the assignment $f^{\leftarrow}(A) \equiv \mathrm{cl}_Y f^{-1}(\mathrm{int}_X A)$. (The $f^{\leftarrow}$ designation is convenient; it is borrowed from [22].) It is worth noting that the inverse of the isomorphism $A \mapsto f^{\leftarrow}(A)$ is the direct image map $B \mapsto f(B)$.

Now fix the space X. Let $\mathrm{Cov}(X)$ denote the set of all irreducible surjections $f : Y \to X$, modulo the equivalence relation defined by $f \sim f'$ (where $f' : Y' \to X$ is an irreducible surjection) if there is a homeomorphism $h : Y \to Y'$ such that $f' \cdot h = f$. It is convenient, especially where the notation is concerned, to identify an irreducible surjection with the equivalence class in which it lies; no confusion should ensue from this identification.

One can partially order $\mathrm{Cov}(X)$: with $f : Y \to X$ and $g : Z \to X$ irreducible, let $f \leqslant g$ if there is a continuous surjection $g^* : Z \to Y$ (necessarily irreducible) such that $f \cdot g^* = g$. Indeed, $\mathrm{Cov}(X)$ is a complete lattice ([9]).

(b) Suppose that $\mathfrak{T}$ is a class of spaces which is closed under formation of homeomorphic copies. $\mathfrak{T}$ is called a *covering class* if, for each space X, $\mathfrak{T} \cap \mathrm{Cov}(X)$ has a minimum.

Recall that a space X is said to be *extremally disconnected* if the closure of every open set is open. It is well known—see [9]—that if $f : Y \to X$ is irreducible and X is extremally disconnected, then f is a homeomorphism. Thus it is seen that any covering class contains all extremally disconnected spaces.

Definition and Remarks 2.2. An *invariant covering function* $\mathfrak{K}$ is a function assigning to each compact space X a subalgebra $\mathfrak{K}(X)$ of $\mathfrak{R}(X)$ which contains every clopen set, such that for each irreducible surjection $g : Y \to X$, $g^{\leftarrow}(\mathfrak{K}(X)) \subseteq \mathfrak{K}(Y)$; as with polar functions, "invariant covering function" is abbreviated to *icf*. Martinez [22] introduced the $\mathfrak{K}$-*cover* of a space X: if $g : Y \to X$ is an irreducible surjection, Y is a $\mathfrak{K}$-cover if $g^{\leftarrow}(A)$ is clopen, for each $A \in \mathfrak{K}(X)$.

Aside from $\mathfrak{R}$, here are the "typical" icfs.

(i) $\mathfrak{R}_{\omega}$, which, for each space X, selects the subalgebra generated by the closures of all cozerosets of X.
(ii) Again, α stands for an uncountable, regular cardinal. $\mathfrak{R}_{\alpha}^{\alpha}$ is the icf which picks out the closures of α-cozerosets which are α-complemented. (Recall that an α-*cozeroset* is a union of fewer than α cozerosets. The α-cozeroset U is α-*complemented* if there is an α-cozeroset V disjoint from U, such that $U \cup V$ is dense in X.)
(iii) α is taken once more as in (ii). Let $\mathfrak{R}_{\alpha}(X)$ be the subalgebra of regular closed sets generated by all closures of α-cozerosets.

It is time to review the essential facts concerning the Yosida Representation Theorem. Our favorite reference for this material is [15].

For the record, βX denotes the Stone-Čech compactification of the Tychonoff space X. It is also convenient to introduce $\mathfrak{B}(X)$ for the algebra of clopen sets of X. $S(X)$ stands for the subalgebra of $C(X)$ consisting of all continuous functions of finite range.

Definition and Remarks 2.3. *YG* stands for the *Yosida space* of the **W**-object G; that is to say, the space of values of the designated unit, with the hull-kernel topology. It is well known that *YG* is a compact Hausdorff space.

Now for any compact Hausdorff space X, $D(X)$ shall denote the set of all continuous functions $f : X \to \mathbb{R} \cup \{\pm\infty\}$, where the range is the extended reals with the usual topology, such that $f^{-1}(\mathbb{R})$ is a dense subset of X. $D(X)$ is a lattice under pointwise operations, but not a group or ring under the obvious pointwise operations, unless some assumptions are made about X. One need not raise those issues here. What is needed is an understanding of the term "ℓ-group in $D(X)$": a subset $H \subseteq D(X)$ which is an ℓ-group such that under the lattice operations it is a sublattice of $D(X)$, and, for each $h, k \in H$ there is an $l \in H$ such that $l(x) = h(x) + k(x)$ for all x in a dense subset of X.

Suppose that $H \subseteq D(X)$ is an ℓ-group in $D(X)$. Recall that H is said to *separate the points of* X provided that, for each $x \neq y$ in X, there is an $h \in H$, such that $h(x) \neq h(y)$.

Here is a formulation of the Yosida Representation Theorem:

For each **W**-*object* G *with designated unit* $u > 0$, *there is an* ℓ-*isomorphism* ϕ *of* G *onto an* ℓ-*group* G' *in* $D(YG)$ *such that* $\phi(u) = 1$; *furthermore*, G' *separates the points of* YG.

The Yosida space *YG* is, up to equivalence, the only space satisfying the above theorem. More precisely:

Suppose that Z *is a compact Hausdorff space and that there is an* ℓ-*isomorphism* θ *of* G *onto an* ℓ-*group* H *in* $D(Z)$ *which separates the points of* Z *and such that* $\theta(u) = 1$, *then there is a homeomorphism* $t : Z \to YG$ *such that*, *for each* $g \in G$ *and* $z \in Z$,

$$\phi(g)(tz) = \theta(g)(z).$$

Finally, if the unit u of G is strong, the image of the Yosida Representation lies in $C(YG)$.

Central to the understanding of the relationship between the algebra and the topology in this context is the passage from polars to regular closed sets and back. Here are the pertinent facts.

Definition and Remarks 2.4. (a) Suppose that P is a polar of the **W**-object G. Associate a regular closed subset $\rho(P)$ of $Y = YG$, as follows:

$$\rho(P) \equiv \mathrm{cl}_Y \left(\bigcup_{g \in P} \mathrm{coz}(g) \right).$$

ρ defines an isomorphism of boolean algebras from $\mathscr{P}(G)$ onto $\mathfrak{R}(Y)$. The inverse map is

$$\rho^{-1}(A) = \{ g \in G : \mathrm{coz}(g) \subseteq A \}.$$

Observe that if $P \in \mathscr{S}(G)$ then $\rho(P)$ is clopen in YG, although the converse may fail.

(b) Now, if H is an extension of G in G^{e}, then there is an induced irreducible surjection $t : YH \to Y$. This happens because of the functorial properties of the Yosida Representation. A brief explanation is in order; the reader is referred to [15] for additional discussion.

Suppose that $\theta : G \to H$ is a $\mathbf{W}$-morphism. Let $\phi_G : G \to G'$ and $\phi_H : H \to H'$ denote the Yosida representations, over YG and YH, respectively. Then there is a unique continuous map $t : YH \to YG$ such that for each $p \in YH$ and $g \in G$,

$$\phi_G(g)(t(p)) = \phi_H(\theta(g))(p). \tag{$\odot$}$$

When θ is an essential embedding, t is an irreducible surjection, and conversely.

What all this is leading up to is an identity involving the map ρ for essential extensions. It follows from ($\odot$); for more details the reader is referred to 4.1 in [22]. The subscripts on ρ in the display which follows should speak for themselves:

Let $G \leqslant H$ be an essential extension; then for each polar P of G,

$$\rho_H(P^{\perp_H \perp_H}) = t^{\leftarrow}(\rho_G(P)).$$

(c) Suppose that $\mathscr{X}$ is an ipf and $\mathfrak{K}$ is an icf, and for each compact space X, $\mathfrak{K}(X) = \rho(\mathscr{X}(C(X)))$; then say that $\mathfrak{K}$ is *derived* from $\mathscr{X}$, or that $\mathfrak{K}$ is the *covering derivative* of $\mathscr{X}$. One writes, perhaps suggestively, that $\mathfrak{K} = \rho(\mathscr{X})$.

Here is the topological counterpart to Theorem 1.3; [22, Theorem 3.5] makes up the essential part of this.

Theorem 2.5. *Let $\mathfrak{K}$ be an icf. Then each space X has a least $\mathfrak{K}$-cover $X[\mathfrak{K}]$, which is obtained as $YC(X)[\rho^{-1}(\mathfrak{K}(X))]$.*

We conclude this review with the topological counterpart of the idempotence in 1.4.

Definition 2.6. Suppose that $\mathfrak{K}$ is an icf and that $(X[\mathfrak{K}])[\mathfrak{K}] = X[\mathfrak{K}]$, for each compact space. Then $\mathfrak{K}$ is said to be *idempotent*. This is equivalent to saying that, for each X, each member of $\mathfrak{K}(X[\mathfrak{K}])$ is clopen.

Of the examples in 2.2, $\mathfrak{R}_\omega$ is not idempotent, and $\mathfrak{R}_\alpha$ is not either, for any regular, uncountable cardinal α. $\mathfrak{R}_\alpha^\alpha$, on the other hand, is idempotent, for each regular, uncountable cardinal; although not expressly put in those terms, that is the content of Theorem 7.4, [14]; the reader will also find a more explicit account of this, below, as Proposition 4.9. $\mathfrak{R}$ itself is also idempotent.

As in 1.6, [22, 5.8] describes a transfinite construction that generates the least idempotent icf over a given icf. Here is an outline of that.

Definition and Remarks 2.7. Suppose that $\mathfrak{K}$ is an icf and X is a compact space. First, set $\mathfrak{K}^1 \equiv \mathfrak{K}$. Next, suppose that λ is an ordinal number, and that, for each $\gamma < \lambda$, icfs $\mathfrak{K}^\gamma$ are defined, such that for $\gamma < \delta < \lambda$, and each compact space X, $\mathfrak{K}^\gamma(G)$ is a subalgebra of $\mathfrak{K}^\delta(G)$.

If λ is a limit ordinal, let

$$\mathfrak{K}^\lambda(X) \equiv \bigcup_{\gamma<\lambda} \mathfrak{K}^\gamma(X).$$

On the other hand, if κ is the predecessor of λ, then define

$$\mathfrak{K}^\lambda(X) \equiv \{\, g^\kappa(A) \,:\, A \in \mathfrak{K}(X[\mathfrak{K}^\kappa])\,\},$$

where $g^\kappa : X[\mathfrak{K}^\kappa] \to X$ is the covering map associated with $X[\mathfrak{K}^\kappa]$. In either case $\mathfrak{K}^\lambda$ is an icf, and by induction we have a transfinite sequence of icfs

$$\mathfrak{K} = \mathfrak{K}^1 \leqslant \cdots \leqslant \mathfrak{K}^\lambda \leqslant \cdots \leqslant \mathfrak{R}.$$

(*Note*: For icfs $\mathfrak{K}_1$ and $\mathfrak{K}_2$, define $\mathfrak{K}_1 \leqslant \mathfrak{K}_2$ to mean that $\mathfrak{K}_1(X) \leqslant \mathfrak{K}_2(X)$, for each compact space X.)

For each compact space X there is an ordinal τ for which $\mathfrak{K}^\tau(X) = \mathfrak{K}^{\tau'}(X)$, for each $\tau' > \tau$. This enables one to define the *limit covering function* $\mathfrak{K}^\flat$ by

$$\mathfrak{K}^\flat(X) \equiv \mathfrak{K}^\tau(X).$$

It is shown in [22, Proposition 5.9], that each $\mathfrak{K}^\lambda$ and $\mathfrak{K}^\flat$ are invariant. By design, $\mathfrak{K}^\flat$ is idempotent, and the least idempotent icf exceeding $\mathfrak{K}$.

Also by [22, Proposition 5.9], one has, for each ordinal λ and each compact space X, that

$$X[\mathfrak{K}^{\lambda+1}] = (X[\mathfrak{K}^\lambda])[\mathfrak{K}],$$

and, if λ is a limit ordinal, that

$$X[\mathfrak{K}^\lambda] = \bigvee_{\gamma<\lambda} X[\mathfrak{K}^\gamma].$$

$\mathfrak{K}^\flat$ is called the *idempotent closure* of $\mathfrak{K}$.

3. Completions vs. covers

Hull classes in **W** feature a fairly extensive range of ℓ-group-theoretic properties. Now, for the heart of the matter in this article, it is time to limit the scope of the discussion. From this point onward we shall mainly be concerned with hull classes consisting of uniformly complete real f-algebras. (In the sequel the qualification "real" in front of "f-algebra" will be dropped, as all f-algebras considered here will be over the real field.) These are the objects which are f-rings and also real vector lattices, and uniformly complete **W**-objects into the bargain. All f-rings in these pages are commutative and have an identity, and, moreover, that the identity is the designated unit.

It is known that the class of archimedean f-rings itself is a hull class—and more; see Remark 5.10(b) for details. Thus, since every divisible uniformly complete **W**-object is a vector lattice, it should also be clear that the class of uniformly complete f-algebras is itself a hull class in **W**. To begin, here is a brief review of the basics regarding uniform completion.

Definition and Remarks 3.1. Let A be a **W**-object with designated unit $u>0$. A sequence $(s_n)_{n<\omega}$ in G is *uniformly Cauchy* if for each positive integer m there is a positive integer k such that $m|s_{n+k}-s_n| \leqslant u$, for each $n \in \mathbb{N}$. The given sequence is said to *converge uniformly* to $s \in A$ if for each $m \in \mathbb{N}$ there is a positive integer k such that $m|s_{n+k}-s| \leqslant u$, for each $n \in \mathbb{N}$. If every uniformly Cauchy sequence converges A is *uniformly complete*.

It is well known that each **W**-object A has a *uniform completion*, defined in the following sense: an extension $uA \leqslant A^{\mathrm{e}}$ of A which is uniformly complete, such that if H is an essential extension of A which is uniformly complete, then the identity function on A extends to an ℓ-embedding $B \to uA$. In fact, the assignment $A \mapsto uA$ is a reflection, although that will not be used here.

What is noteworthy in this context is that $YuA = YA$ [15, Theorem 5.5]. To simplify the notation in the theorem ahead **U** will be used to denote the class of all uniformly complete **W**-objects.

Let us also adopt the following usage: for any class of **W**-objects **A** occurring in these pages, $\mathbf{A}^f$ will stand for the subclass of all objects in **A** which are uniformly complete f-algebras.

Definition 3.2. A hull class **H** consisting of uniformly complete f-algebras will be called a *completion class* if it also has the feature that, for any uniformly complete f-algebra A, $A \in \mathbf{H}$ if and only if $A^* \in \mathbf{H}$. (*Note*: A^* denotes the convex ℓ-subgroup of A generated by the designated unit.) Vacuously then, $\mathbf{U}^f$ itself is a completion class, since if A is a uniformly complete f-algebra, then so is A^*.

For later use, let us indicate that u_f will denote the hull operator for $\mathbf{U}^f$.

Thus, for a given completion class **H**, if $A \in \mathbf{H}$ and $A = A^*$, then, without loss of generality, we may apply the Stone-Weierstrass Theorem and assume that $A = C(X)$; evidently, $YA = X$. The point of this comment is that in computing $\mathbf{H}^\#$, without loss of generality, one may deal with bounded **W**-objects.

Before going any further, it may be dispiriting, but certainly worth recording the following comment.

Remark 3.3. One may wonder whether a hull class necessarily contains a largest completion class. In this regard it is first important to recall that any hull class contains all the essentially closed **W**-objects. Denote this class by **E**. Any completion class **H** contained in **E** must therefore coincide with **E**. But the reader will readily see that **E** is not a completion class.

These remarks should be contrasted with Proposition 5.5, which spells out a sufficient condition for such a largest "subcompletion class" to exist.

The first result should now be expected.

Proposition 3.4. *Suppose that* $\mathbf{H}$ *is a completion class in* $\mathbf{W}$. *Then* $\mathbf{H}^\#$ *is a covering class of compact spaces.*

Proof. Suppose that $\mathscr{Y} \equiv \{ Y_i : i \in I \}$ is the set of compact spaces in $\mathrm{Cov}(X)$, lying in $\mathbf{H}^\#$, and let $Y = \wedge_{i\in I} Y_i$ in $\mathrm{Cov}(X)$. We have that $C(Y_i) \in \mathbf{H}$, for each $i \in I$, and each of these is an essential extension of $C(X)$. Let A be the $\mathbf{H}$-hull of $C(X)$. Since $A \leqslant C(Y_i)$, for each $i \in I$, it follows that $A = A^*$, whence $A = C(YA)$. It should be evident that, in $\mathrm{Cov}(X)$, $X \leqslant YA \leqslant Y_i$, and hence, $YA \leqslant Y$. On the other hand, $YA \in \mathscr{Y}$, forcing the reverse inequality. This means that $Y \in \mathbf{H}^\#$. □

Conversely, let us suppose that $\mathfrak{T}$ is a covering class of compact spaces. Let $\mathfrak{T}^\#$ be the class of uniformly complete f-algebras A for which $YA \in \mathfrak{T}$.

Proposition 3.5. *Suppose that* $\mathfrak{T}$ *is a covering class of compact spaces. Then* $\mathfrak{T}^\#$ *is a completion class of* $\mathbf{W}$*-objects. For each* $\mathbf{W}$*-object* A, $Y(h_{\mathfrak{T}}A^*)$ *is the minimum cover of* YA *in* $\mathfrak{T}$, *where* $h_{\mathfrak{T}}A$ *is the* $\mathfrak{T}^\#$*-hull of* A.

Proof. As $YA = YA^*$, it is clear that $A \in \mathfrak{T}^\#$ if and only if $A^* \in \mathfrak{T}^\#$. And since $\mathfrak{T}^\#$ consists of uniformly complete f-algebras, by definition, it suffices to verify that $\mathfrak{T}^\#$ is a hull class.

To that end suppose $A \leqslant B_i \leqslant A^e$ $(i \in I)$, and each B_i belongs to $\mathfrak{T}^\#$; set $B = \bigcap_{i\in I} B_i$. Note that B is uniformly complete, since the class of all uniformly complete f-algebras is a hull class. Evidently, in $\mathrm{Cov}(YA)$, one has $YA \leqslant YB_i$, for each $i \in I$. Let $Y = \wedge_{i\in I} YB_i$. Then $Y \in \mathfrak{T}$, and it follows that

$$B^* = C(YB) \leqslant C(Y) \leqslant C(YB_i) = B_i^*,$$

while $B^* = \bigcap_{i\in I} B_i^*$. From this conclude that $C(Y) = C(YB)$, and that $Y = YB$. Hence $B \in \mathfrak{T}^\#$, and this proves that $\mathfrak{T}^\#$ is a completion class.

As to the final assertion, it should be clear that $Y(h_{\mathfrak{T}}A^*)$ is a cover of YA in $\mathfrak{T}$. If Z is a cover of YA in $\mathfrak{T}$, then A^* is essentially embedded in $C(Z)$, and the latter is a $\mathfrak{T}^\#$-object. Hence, we may take $h_{\mathfrak{T}}A^* \leqslant C(Z)$, which in turn induces a covering map $Z \to Y(h_{\mathfrak{T}}A^*)$. □

Here is a corollary, which now follows easily.

Corollary 3.6. *For any completion class* $\mathbf{H}$, $(\mathbf{H}^\#)^\# = \mathbf{H}$. *Conversely, for any covering class of compact spaces,* $\mathfrak{T}$, *we have* $(\mathfrak{T}^\#)^\# = \mathfrak{T}$.

And, separately, for emphasis, one has the following corollary:

Corollary 3.7. *Suppose that $\mathfrak{T}$ is a covering class of compact spaces and that X is compact. Then the $\mathfrak{T}^{\#}$-completion of $C(X)$ is $C(\mathfrak{T}X)$.*

Thus, the maps $\mathbf{H} \mapsto \mathbf{H}^{\#}$ and $\mathfrak{T} \mapsto \mathfrak{T}^{\#}$ define mutually inverse lattice isomorphisms between the lattice of all completion classes of **W***-objects and the lattice of covering classes of compact spaces.*

Wherefore our insistence on having f-algebras, the reader might reasonably ask? To contrast, here is a brief digression on **W**-objects with singular unit. For details the reader is referred to [12].

Definition and Remarks 3.8. The **W**-object G with designated unit $u > 0$ is said to be *singular*, if for each $0 \leqslant a \leqslant u$, $a \wedge (u-a) = 0$. Equivalently, G is singular if and only if, in its Yosida representation, each $g \in G$ takes on values in $\mathbb{Z} \cup \{\pm\infty\}$. It suffices here to simply note that any singular object is uniformly complete. This follows immediately from the definition of uniform completeness, as uniformly Cauchy sequences must in this context be eventually constant.

The point, for now, is that the comment in 3.2 on the use of the Stone-Weierstrass Theorem is far off the mark for singular **W**-objects. We return to this, briefly, in 5.14.

The next section consists of an ample discussion of examples. As will be conceded at the end, these are the only examples of completion classes we know.

4. Examples

The examples in this section may be reasonably classified according to one of two types. Roughly speaking, there are the covering classes which are defined by an icf, and, thus, membership in one depends on a class of open sets having open closure; then there are those which, clearly, cannot be so defined, such as the class in Example 4.4.

We adopt the convention, that α denotes either a regular, uncountable cardinal, or else the symbol ∞, which is to be imagined as a symbol exceeding all cardinals. In the results which follow here this convention will be encapsuled by the inequalities "$\omega_1 \leqslant \alpha \leqslant \infty$". Unless the contrary is mentioned all cardinals in the sequel are assumed to be regular.

To begin, recall two dual definitions.

Definition 4.1. (a) Recall that a space is called α-*disconnected* if the closure of every α-cozeroset is clopen. The class of compact α-disconnected spaces will be denoted by $\mathfrak{E}_\alpha$. $E_\alpha X$ designates the minimum α-disconnected cover of X. Note that for $\alpha = \infty$ one has the extremally disconnected spaces, whereas for $\alpha = \omega_1$, the basically disconnected spaces.

(b) A **W**-object G is said to be *conditionally α-complete* if every set of fewer than α elements in G with an upper bound in G has a supremum in G. Denote the class of conditionally α-complete **W**-objects by $\mathbf{C}(\alpha)$; it is well known that $\mathbf{C}(\alpha)$ is a hull class, and in [13] there is a fair amount of information about it.

Let us begin with a simple lemma which will set the tone for several of the examples that follow. First, recall that an f-ring A *satisfies the bounded inversion property* if each $a \geqslant 1$ in A is invertible. It is well known that every uniformly complete f-algebra satisfies the bounded inversion property ([19]).

Lemma 4.2. *Suppose that A is an f-ring satisfying the bounded inversion property, and that $\omega_1 \leqslant \alpha \leqslant \infty$. Then $A \in \mathbf{C}(\alpha)$ if and only if $A^* \in \mathbf{C}(\alpha)$. In particular, $\mathbf{C}(\alpha)^f$ is a completion class.*

Proof. The necessity is trivial. As to the sufficiency, suppose that $A^* \in \mathbf{C}(\alpha)$, and that $\{ a_i : i \in I \}$ is a set of positive elements of A bounded above by $b \in A$, with $|I| < \alpha$. Without loss of generality, $b \geqslant 1$ and hence invertible. Now consider the set $\{ b^{-1}a_i : i \in I \}$; the members are bounded above by 1. Thus, $x = \vee_i b^{-1}a_i$ exists. It is then easy to verify that $bx = \vee_i a_i$, using the notion that multiplication by an invertible positive element induces a lattice isomorphism. □

Example 4.3. *Conditional α-completeness vs. α-disconnectivity.*

Lemma 4.2 shows that $\mathbf{C}(\alpha)$ is a completion class. It is well known that $\mathbf{C}(\alpha)^{\#}$ is the class of α-disconnected spaces; this is essentially the Stone-Nakano Theorem.

That the class of extremally disconnected spaces is a covering class was first proved by Gleason. The minimum extremally disconnected cover of X is called the *absolute* of X. It is also the only extremally disconnected cover of X. In [27] Vermeer constructs the minimum basically disconnected cover of a space. By appealing to criteria developed by Vermeer ([9] or [26]), it can be shown directly that $\mathfrak{E}_\alpha$ is a covering class.

It is well known (see [9]) that the class $\mathfrak{E}_\infty$ is the least covering class of compact spaces. Thus, according to Corollary 3.6, $\mathbf{C}(\infty)$, consisting of all conditionally complete f-algebras is the least completion class.

This example should not be left behind without commenting on the role of α-projectability. Recall that $\mathbf{P}(\alpha)$ stands for the class of all α-projectable $\mathbf{W}$-objects. Now, by appealing to the preceding lemma and Theorem 5.2, I, of [11], one can easily establish that a uniformly complete f-algebra is α-projectable if and only if it is conditionally α-complete; that is, $\mathbf{C}(\alpha)^f = \mathbf{P}(\alpha)^f$.

Example 4.4. *o-Completeness vs. quasi F-spaces.*

Recall that in a $\mathbf{W}$-object A a sequence $(s_n)_{n\in\mathbb{N}}$ is said to be *o-Cauchy* if there is a decreasing sequence $(v_n)_{n\in\mathbb{N}}$ in A, such that $\wedge_n v_n = 0$ and $|s_n - s_{n+p}| \leqslant v_n$, for all $n, p \in \mathbb{N}$. The sequence $(s_n)_{n\in\mathbb{N}}$ *o-converges* if there is an $s \in A$ and $(v_n)_{n\in\mathbb{N}}$ in A, such that $\wedge_n v_n = 0$ and $|s_n - s| \leqslant v_n$, for all $n \in \mathbb{N}$. A is *o-complete* if every o-Cauchy sequence o-converges. Here are the facts concerning o-completeness, pertinent to our discussion.

(i) *Every o-complete divisible* $\mathbf{W}$-*object is uniformly complete* [21, Theorem 16.2(i)].
(ii) *A is o-complete if and only if for each pair of sequences*

$$a_1 \leqslant a_2 \leqslant \cdots \leqslant \cdots \leqslant b_2 \leqslant b_1,$$

such that $\wedge_n b_n - a_n = 0$, *there is a* $c \in A$ *such that* $\vee_n a_n = c = \wedge_n b_n$ [24]. Evidently, when such a c exists it is unique.

(iii) *Letting* **O** *denote the class of o-complete* **W**-*objects*, $\mathbf{O}^\#$ *is the class of quasi F spaces*. Recall that X is *quasi F* if each dense cozeroset of X is C^*-embedded.

This is the place to observe that $\mathbf{O}^\#$ cannot be defined by an icf. Note that the class of quasi F spaces contains connected spaces, such as $\beta\mathbb{R}^+ \setminus \mathbb{R}^+$, where $\mathbb{R}^+$ stands for the set of nonnegative real numbers; (see [7, 6.10(b)], where it is shown that this space is connected, and [7, 14.27], which tells us that it is an F-space.)

(iv) $\mathbf{O}^f$ *is a completion class*. As in Lemma 4.2, the only matter that requires some checking is this: if A is a uniformly complete f-algebra, then $A \in \mathbf{O}^f$ if and only if $A^* \in \mathbf{O}^f$. The nontrivial part of this, the sufficiency, is a consequence of [23, Theorem 11.7] or (ii) in this list.

(A substantial amount of information on quasi F spaces as a covering class may be found in the following references: [6,16,20,23].) We summarize: $\mathbf{O}^\# = \mathfrak{q}\mathfrak{F}$ and $\mathfrak{q}\mathfrak{F}^\# = \mathbf{O}^f$.

Example 4.5. *A pair of* "*non-examples*".

(a) Consider the class **L** of laterally complete, uniformly complete f-algebras. It is not a completion class: for example $C(\mathbb{N}) \in \mathbf{L}$, but $C^*(\mathbb{N})$ is not laterally complete.

(b) F-spaces correspond to the class of **W**-objects A which satisfy the so-called *σ-interpolation property*: if

$$a_1 < a_2 < \cdots < \cdots < b_2 < b_1,$$

then there is a $c \in A$ such that $a_n < c < b_n$, for each $n \in \mathbb{N}$. Observe that $C(X)$ has the σ-interpolation property if and only if X is an F-space; which is to say, every cozeroset is C^*-embedded (see [18, Theorem 10.2]). Note that if A has the σ-interpolation property it is uniformly complete [18, Theorem 9.10], but the class of these objects is not a hull class. Alternatively, it is well known that the class of F-spaces is not a covering class; see [26], or else [9, Proposition 9.4(c)].

Having brought up laterally complete **W**-objects in 4.5(a), that should be contrasted with the following example.

Remark 4.6. Let the stipulation on the cardinal α be as before. Recall that a **W**-object A is *boundedly laterally α-complete* if every set of pairwise disjoint elements S, with $|S| < \alpha$ and which is bounded above, has a supremum. Let $\mathbf{bL}(\alpha)$ denote the class of all boundedly laterally α-complete **W**-objects. It is not hard to see that $\mathbf{bL}(\alpha)$ is a hull class. The proof given for Lemma 4.2 works to show that $\mathbf{bL}(\alpha)^f$ is a completion class.

Of course, this completion class is not new. It is shown in [11] that each $A \in \mathbf{bL}(\alpha)$ is also α-projectable, and that YA is α-disconnected; thus, $\mathbf{bL}(\alpha)^f \subseteq \mathfrak{C}^\#_\alpha$. Conversely, if X is α-disconnected, then $C(X)$ is conditionally α-complete and, in particular, boundedly laterally α-complete. This means that $\mathbf{bL}(\alpha)^f = \mathbf{C}(\alpha)^f$.

Example 4.7. *qF_α and $q\bar{F}_\alpha$.*

Let X be a space. Recall that an open subset U of X is called an α-cozeroset if it a union of fewer than α cozerosets. An α-*Lindelöf* space is one for which every open cover has a subcover consisting of fewer than α subsets. Clearly, every α-cozeroset is α-Lindelöf. Now X is a *quasi F_α space* if every dense α-Lindelöf subspace of X is C^*-embedded. The reader is referred to [2] for additional discussion of quasi F_α spaces. There it is shown that this class is a covering class.

Let us, briefly, give a more precise account.

(a) Suppose that $f: Y \to X$ is an irreducible surjection. Say that f is α-*irreducible* if for each α-cozeroset U in Y there is an α-cozeroset V in X such that $f^{-1}(V)$ is dense in U. Theorems 4.9 and 4.11 of [2] assert the following:

For each space X there is an α-irreducible surjection $q_\alpha : qF_\alpha X \to X$, with $qF_\alpha X$ a quasi F_α space, least among $g: Z \to X$ in $\mathrm{Cov}(X)$ with Z quasi F_α. Thus, the class $\mathfrak{q}\mathfrak{F}_\alpha$ of quasi F_α spaces is a covering class. $qF_\alpha X$ is the minimum quasi F_α cover of X.

When $\alpha = \omega_1$ one recovers the quasi F spaces. $\mathfrak{q}\mathfrak{F}_\infty = \mathfrak{E}_\infty$, the class of extremally disconnected spaces.

(b) To describe $\mathfrak{q}\mathfrak{F}_\alpha^\#$ one needs the concept of an α-cut, borrowed (most recently) from the discussion in [10], which deals with α-cut completion of a boolean algebra. In [2] the notion of α-cut completeness appears as α-*jamd completeness*. All spaces are compact, as before.

Suppose that A is a **W**-object. The pair of subsets of A, (F,H) is called an α-*cut* if

(i) $|F|, |H| < \alpha$;
(ii) $F \leqslant H$, meaning that $f \leqslant h$ for each $f \in F$ and $h \in H$;
(iii) $\wedge\{ h - f : f \in F, h \in H \} = 0$.

A is α-*cut complete* if for each α-cut (F,H) of A, there is an $a \in A$ such that $F \leqslant a \leqslant H$. Note that if this occurs, then $a = \bigvee F = \bigwedge H$. Alternatively, if $\bigvee F = \bigwedge H$ exists for each α-cut (F,H), then A is α-cut complete. It should be evident that conditional α-completeness implies α-cut completeness. Furthermore, for uniformly complete f-algebras, ω_1-cut completeness "is" o-completeness, by the observation in 4.4(ii). Now, if A is α-cut complete, then it is α'-cut complete, for each $\alpha' \leqslant \alpha$, and consequently o-complete, whence it is uniformly complete, provided it is also divisible.

Finally, in this progression of remarks, if A is a uniformly complete f-algebra, then A is α-cut complete if and only if A^* is α-cut complete. This is proved using (ii) above, and calculating as in the proof of Lemma 4.2. Upshot?

Let $\mathbf{H}(\alpha)$ be the class of α-cut complete **W**-objects. This is, apart from other considerations, a hull class. Now the discussion in the preceding paragraph says that $\mathbf{H}(\alpha)^f$ is a completion class. Furthermore, recall Theorem 4.6 of [2]: $C(X)$ is α-cut complete if and only if X is a quasi F_α space. Thus, $\mathfrak{q}\mathfrak{F}_\alpha^\# = \mathbf{H}(\alpha)^f$.

(c) One might opt for the following definition of a perfectly reasonable class of spaces: every dense α-cozeroset of X is C^*-embedded. Call these the *quasi* $\bar{F}_\alpha$ *spaces* and denote the class by $\mathfrak{q}\bar{\mathfrak{F}}_\alpha$. Obviously, every quasi F_α space is quasi $\bar{F}_\alpha$, and nothing is known about the converse except in the extreme cases:

(i) $\alpha = \infty$: the matter is easily resolved: $\mathfrak{q}\bar{\mathfrak{F}}_\infty = \mathfrak{q}\mathfrak{F}_\infty = \mathfrak{E}_\infty$.
(ii) $\alpha = \omega_1$: by [8, Theorem 3.6], if every dense cozeroset is C^*-embedded then it follows that every dense Lindelöf subspace is also C^*-embedded. That is to say, $\mathfrak{q}\bar{\mathfrak{F}}_{\omega_1} = \mathfrak{q}\mathfrak{F}_{\omega_1} = \mathfrak{q}\mathfrak{F}$.

By appealing to criteria developed by Vermeer ([9] or [26]) one can show that $\mathfrak{q}\bar{\mathfrak{F}}_\alpha$ is also a covering class. Little is known about $\mathfrak{q}\bar{\mathfrak{F}}_\alpha^\#$. Looking ahead to the examples in 4.8, note, citing [10, 3.5], that for zero-dimensional spaces,

$$\mathfrak{q}\mathfrak{F}_\alpha \Leftrightarrow \mathfrak{q}\bar{\mathfrak{F}}_\alpha \Leftrightarrow \mathfrak{C}\mathfrak{z}_\alpha.$$

Example 4.8. Let X be a space. Recall that an α-cozeroset V of X is α-complemented if there is an α-cozeroset W of X such that $V \cap W = \emptyset$ and $V \cup W$ is dense in X. If every α-complemented α-cozeroset of X has clopen closure X is called an α-*cloz space*. For $\alpha = \omega_1$ these are called *cloz spaces* introduced in [17]. These authors show that the class of cloz spaces is a covering class; indeed, by appealing to Vermeer's techniques of [9] once again, one can show that, for each α, the class $\mathfrak{C}\mathfrak{z}_\alpha$ of all α-cloz spaces is a covering class.

The reader will easily see that $\mathfrak{C}\mathfrak{z}_\infty = \mathfrak{E}_\infty$.

Now, Hager shows in [10, Section 3] that a zero-dimensional compact space is α-cloz precisely when its Stone dual is an α-cut complete boolean algebra. Theorem 7.4 of [14] shows that, for an archimedean f-ring A, YA is α-cloz if and only if A has the so-called α-splitting property. Let us explain; A stands for a **W**-object with designated unit u.

$S, T \subseteq A^+$ are α-*complemented* if $s \wedge t = 0$, for each $s \in S$ and $t \in T$ and $S \cup T$ generates a dense convex ℓ-subgroup and $|S|, |T| < \alpha$. Suppose that for every α-complemented pair (S, T) of subsets of A^+ there is a component v of u, such that $S^{\perp\perp} = v^{\perp\perp}$ (it then follows that $T^{\perp\perp} = v^{\perp}$). A is said to have the α-*splitting property*. Let $\mathbf{Spl}(\alpha)$ denote the class of all α-splitting **W**-objects. This is the hull class corresponding to the polar function $\mathscr{P}_\alpha^\alpha$ introduced in 1.4(ii).

Carrying over the proof of [14, Theorem 7.1], *mutatis mutandis* one has the following result.

Proposition 4.9. *Suppose that A is a* **W**-*object. Then A has the α-splitting property if and only if YA is an α-cloz space. Thus,* $\mathfrak{C}\mathfrak{z}_\alpha^\# = \mathbf{Spl}(\alpha)^f$ *is the completion class of all α-splitting uniformly complete f-algebras.*

For the ordinal ω_1 here is a description of the ω_1-splitting objects, in terms of o-convergence.

Proposition 4.10. *Let X be a compact space. Then X is a cloz space if and only if $C(X)$ has the following property*: *each increasing o-Cauchy sequence $(a_n)_{n\in\mathbb{N}}$ which* *satisfies, for each* $k \in \mathbb{N}$ *and* $m \geqslant 0$,

$$\{ (ka_n - m)^+ \ : \ n \in \mathbb{N} \}^{\perp} = f_{k,m}^{\perp\perp} \qquad (*)$$

for a suitable $f_{k,m} \in C(X)$, *o-converges.*

Proof. Suppose that $C(X)$ has the stated property. Let $\mathrm{coz}(f)$ and $\mathrm{coz}(g)$ be mutually complementary cozerosets; without loss of generality, $f, g \geqslant 0$. Let $f_n = 1 \wedge nf$; then, for every natural number k,

$$f_{n+k} - f_n \leqslant (1 - n(f \vee g))^+ \quad \text{and} \quad \wedge_n (1 - n(f \vee g))^+ = 0,$$

which says that $(f_n)_{n \in \mathbb{N}}$ is an increasing o-Cauchy sequence. As for condition $(*)$, if $m/k \geqslant 1$, $(kf_n - m)^+ = 0$, for each $n \in \mathbb{N}$, and $(*)$ certainly is satisfied, whereas if $m/k < 1$, then as long as $f(x) > 0$, $f_n(x) > m/k$, for large enough n; this means that

$$\{ (kf_n - m)^+ \ : \ n \in \mathbb{N} \}^{\perp} = g^{\perp\perp}.$$

Therefore, the f_n o-converge to h, which is a characteristic function, proving that $\mathrm{cl}_X \mathrm{coz}(f)$ is open and that X is a cloz space.

Conversely, suppose $(f_n)_{n \in \mathbb{N}}$ is an increasing o-Cauchy sequence satisfying condition $(*)$ for all nonnegative integers k and m, with $k > 0$. Then

$$U^{mk} \equiv \mathrm{cl}_X \bigcup \{ x \in X \ : \ f_n(x) > m/k, \, n \in \mathbb{N} \}$$

is open, for each rational number m/k. It is well known that the function f defined by

$$f(x) = \sup \{ m/k \ : \ x \in U^{mk} \}$$

is a continuous function. Evidently $f = \vee_n f_n$, and since $(f_n)_{n \in \mathbb{N}}$ is increasing it o-converges to f. □

To conclude this section, we recap the relationship between the various properties linked to a cardinal constraint, as introduced above.

Remark 4.11. What is obvious is that

$$\text{quasi } F_\alpha \ \Rightarrow \ \text{quasi } \bar{F}_\alpha \ \Rightarrow \ \alpha\text{-cloz}.$$

It is not known, in general, whether either arrow reverses, except as noted here:

(i) For $\alpha = \infty$ the three conditions are equivalent to "extremally disconnected". For ω_1, it has already been remarked that "quasi F_{ω_1}" is equivalent to "quasi $\bar{F}_{\omega_1}$". In [6] it is shown that every zero-dimensional cloz space is quasi F. Then an example is given there and also in [17] showing that this is not so in general.
(ii) As pointed out in [10, 3.5], every zero-dimensional α-cloz space is quasi $\bar{F}_\alpha$. This may also be deduced by extending [17, Theorem 3.4(b)] in the obvious way.
(iii) Finally, recall that X is α-*cozero complemented* if every α-cozeroset of X is α-complemented. It is not too hard to see that if X is α-cloz and α-cozero complemented, then it is α-disconnected.

5. Completions of Ipf's

This section reconnects the correspondence for completion classes with the preliminaries of the first two sections. The theorem quoted below is a paraphrase of [22, Theorem 5.13], which is, in effect, the culmination of that paper.

Theorem 5.1. (a) *For any ipf* $\mathscr{X}$, *the* **W**-*objects of the form* $G[\mathscr{X}^\flat]$ *form a hull class, denoted* $\mathbb{H}(\mathscr{X})$. *Indeed,* $G[\mathscr{X}^\flat]$ *is the hull of* G *in* $\mathbb{H}(\mathscr{X})$, *and* $\mathbb{H}(\mathscr{X}) = \mathbb{H}(\mathscr{X}^\flat)$.

(b) *For any icf* $\mathfrak{K}$, *the compact spaces which satisfy* $X = X[\mathfrak{K}]$ *form a covering class* $\mathbb{T}(\mathfrak{K})$. *The minimum cover of* X *in* $\mathbb{T}(\mathfrak{K})$ *is* $X[\mathfrak{K}^\flat]$, *and* $\mathbb{T}(\mathfrak{K}) = \mathbb{T}(\mathfrak{K}^\flat)$.

Remark 5.2. (a) Example 5.14 in [22] shows that the invariance provision of Theorem 5.1(b) cannot be dropped.

(b) One should also be careful in this regard: if $\mathscr{X}$ is an idempotent ipf, then the covering derivative $\rho(\mathscr{X})$ may fail to be idempotent. This happens with $\mathscr{P}_\omega$, for example; see [22, Example 5.11]. On the other hand, in the identity $\mathfrak{R}^\alpha_\alpha = \rho(\mathscr{P}^\alpha_\alpha)$, both polar and covering function are idempotent, for each α, including ∞, which takes $\mathscr{P}$ itself into account, along with its covering derivative $\mathfrak{R}$.

Consider an ipf $\mathscr{X}$. The objective ahead shall be to describe the pairing $\mathbb{H}(\mathscr{X})^f$ with $(\mathbb{H}(\mathscr{X})^f)^\#$, when the former is a completion class. Proposition 5.5 describes a reasonable sufficient condition on $\mathscr{X}$ such that $\mathbb{H}(\mathscr{X})^f$ is a completion class. Prior to that, however, here is an observation, the proof of which again shows that the product in the underlying ring is involved here in a significant way. This lemma is doubtless folklore, but we have been unable to find it explicitly stated anywhere.

Notice that uniform completeness is not assumed.

Lemma 5.3. *Suppose that* A *is an archimedean* f-*ring. Then the trace map* $P \mapsto P \cap A^*$ *on* $\mathscr{P}(A)$ *induces a boolean isomorphism of* $\mathscr{S}(A)$ *onto* $\mathscr{S}(A^*)$.

Proof. That the trace of a summand of A is a summand of A^* is evident; it is the converse that is at issue. So suppose that $P \in \mathscr{P}(A)$, and $P \cap A^*$ is a summand of A^*. Let $f \in A^+$; write $1 = a + b$, with $a \in P \cap A^*$ and $b \in P^\perp \cap A^*$. Then, since each polar in an f-ring is an ideal, we have that $f = fa + fb$, and $fa \in P$, while $fb \in P^\perp$. This proves that P is a summand of A. □

Remark 5.4. The proof of Lemma 5.3 as well as the proof of Lemma 4.2 ought to convince that our correspondence between completion classes of **W**-objects and covering classes of compact spaces ought to involve f-rings. The example in 3.8 shows that, at least, divisibility is required. And since this discussion involves rings of continuous functions intimately, uniform completeness is, apparently, also quite useful.

Proposition 5.5. *Let* $\mathscr{X}$ *be an ipf. Suppose that for each* **W**-*object* G,

$$\mathscr{X}(G^*) = \{\, K \cap G^* \,:\, K \in \mathscr{X}(G) \,\}.$$

Then $\mathbb{H}(\mathscr{X})^f$ *is a completion class. It is the largest completion class contained in* $\mathbb{H}(\mathscr{X})$.

Proof. We make use of Lemma 5.3. Thus, suppose that A is a uniformly complete f-algebra. Now, if $A \in \mathbb{H}(\mathscr{X})^f$, then for each $P \in \mathscr{X}(A^*)$, write $P = K \cap A^*$, with $K \in \mathscr{X}(A)$. But as K is a summand of A, P must be a summand of A^*. Conversely, suppose that $A^* \in \mathbb{H}(\mathscr{X})^f$; then (with the same notation) $P = K \cap A^* \in \mathscr{S}(A^*)$, whence $K \in \mathscr{S}(A)$. This shows that $A \in \mathbb{H}(\mathscr{X})^f$.

Since it is clear that $\mathbb{H}(\mathscr{X})^f$ is a hull class the above establishes that it is a completion class. The final claim of the proposition is obvious. □

Definition 5.6. If the ipf $\mathscr{X}$ satisfies the condition of Proposition 5.5 it is called *exact*. The reader will easily verify that, given an ipf $\mathscr{X}$, the rule

$$\mathscr{X}'(G) \equiv \{ P \in \mathscr{P}(G) : P \cap G^* \in \mathscr{X}(G^*) \},$$

for each **W**-object G, defines the largest exact ipf beneath $\mathscr{X}$.

It is straightforward to check that the ipfs $\mathscr{P}$, $\mathscr{P}_\omega$, $\mathscr{P}_\alpha^\alpha$ and $\mathscr{P}_\alpha$ all are exact. Also note that [22, Proposition 4.5] implies that, if $\mathscr{X}$ is exact, then $(G[\mathscr{X}])^* = G^*[\mathscr{X}]$, for each **W**-object G.

It is time now to introduce a transfinite construction which modifies the one in 1.6, in that it closes under $\mathscr{X}$-splitting, uniform convergence and "ringifies" in one construction. Recall that u_f is the hull operator associated with $\mathbf{U}^f$.

Definition and Remarks 5.7. Let $\mathscr{X}$ denote an ipf, and suppose that G is an arbitrary **W**-object. Set $u_f(G[\mathscr{X}]) \equiv G[[\mathscr{X}]] \equiv G^1[[\mathscr{X}]]$. Now for a given ordinal λ, suppose the sequence of extensions $\{ G^\gamma[[\mathscr{X}]] : \gamma < \lambda \}$ has been defined so that, for $\gamma < \delta < \lambda$, we have $G^\gamma[[\mathscr{X}]] \leqslant G^\delta[[\mathscr{X}]]$. Next, put

$$G^\lambda[[\mathscr{X}]] \equiv \left(\bigcup_{\gamma < \lambda} G^\gamma[[\mathscr{X}]] \right) [[\mathscr{X}]].$$

(Observe that this definition applies whether λ is a limit ordinal or not.) Note that $G^\lambda[[\mathscr{X}]] \leqslant G^{\mathrm{e}}$, for each λ, and consequently this sequence must stabilize at some ordinal τ. Now define

$$G^{[\mathscr{X}]} \equiv G^\tau[[\mathscr{X}]].$$

It should be evident that $G^{[\mathscr{X}]}$ is a uniformly complete f-algebra and that $G^{[\mathscr{X}]}[\mathscr{X}] = G^{[\mathscr{X}]}$; that is, $G^{[\mathscr{X}]} \in \mathbb{H}(\mathscr{X})^f$.

Next we have one of the central results of the section, which should not unduly surprise the reader. Recall the comments in 1.6, concerning the existence of a least idempotent ipf $\mathscr{X}^\flat$ exceeding $\mathscr{X}$.

Theorem 5.8. *Let $\mathscr{X}$ be an exact ipf. Then*

(a) $\mathscr{X}^\flat$ *is exact*.
(b) *For each* **W**-*object* G, $G^{[\mathscr{X}]}$ *is the hull of* G *in* $\mathbb{H}(\mathscr{X})^f$.

Proof. (a) Suppose that $\mathscr{X}$ is an exact ipf. We proceed by induction, proving each $\mathscr{X}^\lambda$ is exact. The object G under discussion in this argument is an arbitrary **W**-object.

Suppose now that λ is an ordinal, and for each $\gamma < \lambda$, $\mathscr{X}^\gamma$ is exact. If λ is a limit ordinal then we have, for a polar K of G^*, that $K \in \mathscr{X}^\lambda(G^*)$ if and only if there is a $\gamma < \lambda$ such that $K \in \mathscr{X}^\gamma(G^*)$, which is so precisely when $K = G^* \cap P$ for a suitable $P \in \mathscr{X}^\gamma(G)$, since $\mathscr{X}^\gamma$ is exact. Thus, $\mathscr{X}^\lambda$ is exact in this case.

Next, suppose that λ has predecessor κ. Note that, $K \in \mathscr{P}(G^*)$ lies in $\mathscr{X}^\lambda(G^*)$ if and only if $K = K' \cap G^*$, for some $K' \in \mathscr{X}(G^*[\mathscr{X}^\kappa]) = \mathscr{X}((G[\mathscr{X}^\kappa])^*)$, owing to the exactness of $\mathscr{X}^\kappa$. Since $\mathscr{X}$ is exact, this occurs precisely when $K' = K'' \cap (G[\mathscr{X}^\kappa])^*$, for a suitable $K'' \in \mathscr{X}(G[\mathscr{X}^\kappa])$. Put $P \equiv K'' \cap G$, and observe straightaway that $P \cap G^* = K$. Finally, by definition $P \in \mathscr{X}^\lambda(G)$. The reader will now realize that this shows that $\mathscr{X}^\lambda$ is an exact ipf.

It should now be easy to see that the completion $\mathscr{X}^\flat$ is exact as well.

(b) If $A \in \mathbf{U}^f \cap \mathbb{H}(\mathscr{X})$ and $G \leqslant A$ is an essential extension, then by transfinite induction each $G^\lambda[[\mathscr{X}]] \leqslant A$, whence $G^{[\mathscr{X}]} \leqslant A$. (The details of the induction are left to the reader.) □

Here is an immediate corollary of Theorem 5.8.

Corollary 5.9. *For any ipf* $\mathscr{X}$ *and any* **W**-*object* G, *one has*

(a) $G^{[\mathscr{X}]} = G^{[\mathscr{X}^\flat]}$.
(b) $(u_f G)^{[\mathscr{X}]} = G^{[\mathscr{X}]}$.

Theorem 5.8 is unsatisfactory, in the sense that one has no idea, for an arbitrary exact ipf $\mathscr{X}$, how drastically the Yosida space of the **W**-object G is changed in passing to $G^{[\mathscr{X}]}$. To better understand this problem one should look more closely at how the hull operator u_f changes the Yosida space.

Remark 5.10. (a) Recall that a divisible uniformly complete **W**-object is a vector lattice.

In brief we shall review the "ringification" reflection—item (b) below. Loosely speaking, the extension $G \leqslant u_f G$ is achieved by first taking the divisible hull, and then transfinitely iterating the composition $G \leqslant u(rG)$—or else the composition $G \leqslant r(uG)$, where r is the ringification functor of [15, Section 6]. It seems useful to point out some of the drawbacks in involving applications of the ringification. First, let us comment on what it is.

(b) Suppose that G is a **W**-object with designated unit $u > 0$. Then there is an f-ring rG with identity and an essential ℓ-embedding $r_G : G \to rG$ such that $r_G(u) = 1$ with the universal property that if $\theta : G \to B$ is an ℓ-homomorphism into the f-ring with identity B, satisfying $\theta(u)=1$, then there a unique ℓ-homomorphism of rings $\theta^r: rG \to B$, preserving the identities, such that $\theta^r \cdot r_G = \theta$. In particular, if θ is one-to-one then (owing to the essential containment of $r_G(G)$ in rG) θ^r is also one-to-one.

Now the applications of u and r do not commute. In fact, if A is an f-ring then uA might "lose" the ring structure; an example may be found in [15, 4.6], and, more dramatically, in [1]. And, similarly, if one ringifies a uniformly complete **W**-object, the result may fail to be uniformly complete. Thus, apart from taking the divisible hull, this explains why u and r must be iterated transfinitely, in general, in order to form u_f. This issue will not trouble us in the sequel, however.

(c) Unlike the situation with uniform completion or with the divisible hull, rA need not have the same Yosida space as A. The following example ought to suffice to point out what can go wrong; it is due to Tony Hager.

As in [7], X^* denotes the one-point compactification of the space X.

Suppose that A is the vector lattice consisting of all sequences which are eventually of the form $r + sf + tg$, with $r, s, t \in \mathbb{R}$, where

$$f(n) = \frac{1 + c(n)}{n}, \quad g(n) = n, \quad \forall n \in \mathbb{N},$$

and c denotes the characteristic function of the even integers. Note that $YA = \mathbb{N}^*$, upon which fg is not definable. Observe as well that A is projectable.

(d) By contrast, if A is a **W**-object and YA is already a quasi F-space, then, each $f^{-1}(\mathbb{R})$ (with $f \in A$) is C^*-embedded, and products in rA can be defined over the same Yosida space. Thus, $YrA = YA$.

There is one other important context in which the ringification does preserve the Yosida space. This happens when the **W**-object G being enlarged has a strong designated unit. This is so because the Yosida representation inserts G into $C(YG)$, and rG is then the ℓ-subring generated by G in $C(YG)$.

Our goal is to improve Theorem 5.8, by giving a reasonable description of the Yosida space of the hull $G \leqslant G^{[\mathscr{X}]}$. We have been able to do this for **W**-objects with a strong unit. Some preliminaries follow, and then the theorem which gives such a description.

Remark 5.11. Fix an ipf $\mathscr{X}$. Suppose that G has a strong designated unit. Then so does $G[\mathscr{X}]$. By the remarks in 5.10(d), $YrG[\mathscr{X}] = YG[\mathscr{X}]$.

Recall that by Proposition 1.5 the extension $G \leqslant G[\mathscr{X}]$ of a divisible object (resp. vector lattice, resp. f-ring) produces a divisible object (resp. vector lattice, resp. f-ring). Finally, although we cannot predict, in general, whether $\mathbb{H}(\mathscr{X}^\flat)$ is closed under uniform completion, $YuG[\mathscr{X}] = YG[\mathscr{X}]$, and, thus, $Yu_fG[\mathscr{X}] = YG[\mathscr{X}]$. This means that if $\mathscr{X}$ is exact, we have, for each ordinal λ (referring to the notation introduced in 5.7)

$$YG^\lambda[[\mathscr{X}]] = Y\left[\left(\bigcup_{\gamma<\lambda} G^\gamma[[\mathscr{X}]]\right)[\mathscr{X}]\right] = Y\left(\bigcup_{\gamma<\lambda} G^\gamma[[\mathscr{X}]][\mathscr{X}]\right)$$

and therefore,

$$YG^{\lambda}[[\mathscr{X}]] = \begin{cases} Y\left(\bigcup_{\gamma<\lambda} G^{\gamma}[[\mathscr{X}]]\right) & \text{if } \lambda \text{ is a limit ordinal,} \\ YG^{\kappa}[[\mathscr{X}]][\mathscr{X}] & \text{if } \lambda \text{ has predecessor } \kappa. \end{cases} \qquad (\dagger\dagger)$$

Theorem 5.12. *Suppose that the ipf $\mathscr{X}$ is exact. Suppose that $\mathfrak{K}$ is the covering derivative of $\mathscr{X}$. Then, for each ordinal λ, and each* **W**-*object G with a strong designated unit,*

$$YG^{\lambda}[[\mathscr{X}]] = (YG)[\mathfrak{K}^{\lambda}].$$

Consequently, $YG^{[\mathscr{X}]} = (YG)[\mathfrak{K}^{\flat}]$.

Proof. Suppose G has a strong designated unit. Without loss of generality, assume that $\mathscr{X}$ is idempotent. Throughout the proof $Y = YG$. To make labels match in the proof, set $Y[\mathfrak{K}^0] \equiv Y$.

We proceed by transfinite induction to prove the first claim. For $\lambda = 0$, it simply says that $Yu_f G = Y$. This is clear, in view of Remark 5.11. So suppose that $\lambda > 0$, and that, for each ordinal $\gamma < \lambda$, it has been established that

$$YG^{\gamma}[[\mathscr{X}]] = Y[\mathfrak{K}^{\gamma}].$$

Both instances of transfinite induction are applications of (††) in Remark 5.11: if λ is a limit ordinal then,

$$YG^{\lambda}[[\mathscr{X}]] = Y\left(\bigcup_{\gamma<\lambda} G^{\gamma}[[\mathscr{X}]]\right) = \bigvee_{\gamma<\lambda} YG^{\gamma}[[\mathscr{X}]] = \bigvee_{\gamma<\lambda} Y[\mathfrak{K}^{\gamma}] = Y[\mathfrak{K}^{\lambda}].$$

If κ is the predecessor of λ, then, invoking [22, Proposition 4.5] as well as Theorem 2.5, we have that

$$\begin{aligned} YG^{\lambda}[[\mathscr{X}]] &= Y(G^{\kappa}[[\mathscr{X}]][\mathscr{X}]) = Y(G^{\kappa}[[\mathscr{X}]][\mathscr{X}])^{*} = Y(G^{\kappa}[[\mathscr{X}]])^{*}[\mathscr{X}] \\ &= (Y(G^{\kappa}[[\mathscr{X}]])^{*})[\mathfrak{K}] \;=\; Y(G^{\kappa}[[\mathscr{X}]])[\mathfrak{K}] = (Y[\mathfrak{K}^{\kappa}])[\mathfrak{K}] = X[\mathfrak{K}^{\lambda}]. \end{aligned}$$

The first claim of the theorem is established, and then the second should be clear. □

The following corollary is immediate. The details are left to the reader.

Corollary 5.13. *For any exact ipf $\mathscr{X}$, with covering derivative $\mathfrak{K}$,*

$$(\mathbb{H}(\mathscr{X})^{f})^{\#} = \mathbb{T}(\mathfrak{K}^{\flat}).$$

Let us conclude this section by returning to the singular objects, by way of contrast, to indicate what happens if divisibility is dropped from consideration in hull formation.

Example 5.14. We take up the ipf $\mathscr{P}_{\omega}$ again. Let X be a compact, zero-dimensional space, and $S(X,\mathbb{Z})$ stand for the ℓ-group of all continuous integer valued functions with finite range. $S = S(X, \mathbb{Z})$ is an f-ring, projectable and, as has already been noted (in 3.8), uniformly complete. Thus, the hull of S in the class of uniformly complete f-rings is S itself, while $X[\mathfrak{R}^{\flat}_{\omega}]$ is again the minimum basically disconnected cover of X.

6. Work points

We leave the reader with a list of open questions, nontrivial ones, we believe. With regard to the first four, one might have thought that the correspondence between completion classes and covering classes highlighted in Section 3 would be useful in answering them; after some thought, there seems to be no evidence to encourage one that this is so.

Question 6.1. *If a covering class $\mathfrak{T}$ of compact spaces consists of zero-dimensional spaces, then must all the spaces in $\mathfrak{T}$ also be basically disconnected?*

Note, on the algebraic side, that if a completion class $\mathscr{H}$ consists of projectable **W**-objects then $\mathscr{H}^{\#}$ consists of basically disconnected spaces, and each $G \in \mathscr{H}$ is conditionally ω_1-complete.

To contrast, recall that the class of quasi F-spaces contains connected spaces.

Question 6.2. *If a covering class $\mathfrak{T}$ of compact spaces consists of cozero-complemented spaces, then must all the spaces in $\mathfrak{T}$ be basically disconnected?*

Recall that a space X is *cozero-complemented* if for each cozeroset U of X, there is a cozeroset V of X such that $U \cap V = \emptyset$ and $U \cup V$ is dense in X.

Question 6.3. *If a covering class $\mathfrak{T}$ of compact spaces consists of F-spaces then must all the spaces in $\mathfrak{T}$ be basically disconnected?*

Recall that the class of compact F-spaces itself is not a covering class. Note as well that any cozero-complemented F-space is basically disconnected, so that if $\mathfrak{T}$ is a covering class having both this property and the one in Question 6.2 then it does consist of basically disconnected spaces.

Question 6.4. *Suppose that $\mathfrak{T}$ is a covering class of compact spaces with the feature that if $X \in \mathfrak{T}$ and A is the closure of a cozeroset of X, then $A \in \mathfrak{T}$. Must such a class consist of basically disconnected spaces?*

If $\mathfrak{T}$ does consist of basically disconnected spaces it has this property—trivially so, since any closure of a cozeroset in a basically disconnected space is clopen.

Observe as well, as illustrations, that the class of quasi F-spaces does not have this feature: consider a discrete uncountable space D, and D^*, the one-point compactification of D. It is well known that D^* is a quasi F space. Any countable subset of D is a cozeroset of D^*, but its closure is homeomorphic to $\mathbb{N}^*$, which is not quasi F. Moreover, with regard to Question 6.2, D^* is also not cozero-complemented.

We close with a question which annoys because, intuitively, the answer is surely yes. It seems to be less tractable than one might think at first glance.

Question 6.5. *Is a hull class in* **W** *necessarily closed under taking products*?

There is, potentially, a second question: if **H** is a hull class of **W**-objects with hull operator h, then is

$$h\left(\prod_{i\in I} A_i\right) \cong \prod_{i\in I} hA_i\,?$$

If h satisfies the above then **H** is clearly product-closed. It is easy to find examples showing that the product-closure does not imply the above identity for the hull operator. Note that the class of divisible **W**-objects is closed under forming products. However,

$$d\,\mathbb{Z}^{\omega} = \{\, f \in \mathbb{Q}^{\omega} \;:\; \exists, k \in \mathbb{N} \text{ such that } kf \in \mathbb{Z}^{\omega}\,\},$$

which is not the product of countably many copies of $\mathbb{Q}$.

The reader familiar with reflections will easily see that if the hull class is epireflective then it is closed under taking products. In any event, it is either well known or else easy to check that all the hull classes discussed in Section 4 are product-closed.

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