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Algebras, Projective Geometry, Mathematical Logic, and Constructing the World: Intersections in the Philosophy of Mathematics of A. N. Whitehead

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TO THE MEMORY OF W. V. O. QUINE (1908–2000)

A. N. Whitehead (1861–1947) contributed notably to the foundations of pure and applied mathematics, especially from the late 1890s to the mid 1920s. An algebraist by mathematical tendency, he surveyed several algebras in his book *Universal Algebra* (1898). Then in the 1900s he joined Bertrand Russell in an attempt to ground many parts of mathematics in the newly developing mathematical logic. In this connection he published in 1906 a long paper on geometry, space and time, and matter. The main outcome of the collaboration was a three-volume work, *Principia Mathematica* (1910–1913): he was supposed to write a fourth volume on parts of geometries, but he abandoned it after much of it was done. By then his interests had switched to educational issues, and especially to space and time and relativity theory, where his earlier dependence upon logic was extended to an ontology of events and to a general notion of "process," especially in human experience. These innovations led to somewhat revised conceptions of logic and of the philosophy of mathematics.

A. N. Whitehead (1861–1947) contribuiu de forma marcante para os Fundamentos da Matemática Pura e Aplicada, especialmente entre o fim da década de 1890 e meados da década de 1920. Sendo um algebrista na sua vertente matemática, fez um levantamento de diversas álgebras no seu livro *Universal Algebra* (1898). Pouco depois de 1900 juntou-se a Bertrand Russell numa tentativa para basear várias partes da matemática sobre a lógica matemática, que se começava então a desenvolver. Nesse âmbito publicou em 1906 um longo artigo sobre geometria, espaço e tempo, e matéria. O principal resultado da colaboração foi um trabalho em três volumes, *Principia Mathematica* (1910–1913): estava previsto que Whitehead escrevesse um quarto volume sobre aspectos das geometrias, mas abandonou-o depois de uma boa parte já estar escrita. Por essa altura os seus interesses tinham-se voltado para questões educacionais; especialmente para o espaço e o tempo e para a teoria da relatividade, onde a sua anterior dependência da lógica se estendeu a uma ontologia de acontecimentos e a uma noção geral de "processo" especialmente na experiência humana. Estas inovações levaram a concepções um pouco revistas da lógica e da filosofia da matemática.

MSC 1991 subject classifications: 00A30; 01A60; 03-03; 03A05.
Key Words: Whitehead; foundations of mathematics; *Principia Mathematica*; process philosophy.

Poetry allies itself to metre, philosophy to mathematic pattern.

—Whitehead, end of *Modes of thought* (1938)

1. INTRODUCTION

Between 1910 and 1913 there appeared three volumes of *Principia Mathematica*, a detailed account of the claim that (quite a lot of) mathematics could be obtained from logic

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alone, specifically the propositional and predicate calculi (including the logic of relations) with set theory closely involved. The authors were A. N. Whitehead (1861–1947) and Bertrand Russell (1872–1970); but soon after its appearance, and ever since, the literature on that work routinely referred only to Russell. This practice infuriated him, for he insisted that the preparation had been a truly collaborative affair; he even indicated parts of the work for which Whitehead had been initially responsible [see especially Russell 1948].

A main theme here is the role of Whitehead in the preparation of *Principia Mathematica*, including his failure to complete the fourth volume (Sect. 4). The paper begins with his *entrée* into foundational studies in mathematics in the late 1890s (Sect. 2), and then considers his extended foray into a logical treatment of space, time, and matter in a paper of 1906 (Sect. 3). Finally, the roles of mathematics and logic in his later philosophy and cosmology is reviewed (Sect. 5); it included a new approach to the physical sciences, an attempt of 1934 to modify the logic of *Principia Mathematica*, and a philosophy of mathematics based upon patterns and symbolism.

The general context for most of this work is provided by [Grattan-Guinness 2000, esp. Chaps. 6–8] which, rather unusually amidst the massive literature on Russell's philosophy, focuses upon the mathematical background and content of *Principia Mathematica*. The literature on Whitehead's views on logic is otherwise scanty: even the survey [Quine 1941] deals as much with Russell. Elsewhere, [Smith 1953] argues that Whitehead's grasp of logic was inferior to Aristotle's, and [Henry 1970] rather confuses things, such as algebraic logic with mathematical logic. Some further historical details are to be found in [Grattan-Guinness 1977] and [Moore (G.H.) 1982].

Among earlier general writings on the influence of Whitehead's mathematics upon his philosophy, [Palter 1960, 1963] is especially strong on relativity theory and cosmology, [Code 1985] takes up several aspects concerning order, and [Fitzgerald 1979] reports much but appraises rather little. Whitehead's contributions to geometry and physics have been ignored in pertinent specialist histories, most recently in [Boi and others 1989] and [Gray 1999].

The other literature on Whitehead largely deals with his (later) philosophy: [Schilpp 1941], a general commentary on all of his work, is a source itself now of historical value, over and above Quine's contribution just cited. Further worthy writings include [Lowe 1962], based upon his own Schilpp piece, which the author was elaborating in turn into a full biography until his death in 1988 [Lowe 1985, 1990]; it includes a complete Whitehead bibliography. Of relevance also are [Mays 1959, 1977] and [Saint-Sernin 2000]. Among obituaries, [Whittaker 1948] on his former teacher (Sect. 4.1) excels.

This paper amplifies several of the connections noted by these authors and adds some new ones. It also draws upon manuscript sources which have not been much used hitherto. Very little survives directly from Whitehead, because after his death his widow followed his instructions to destroy all manuscripts; but, in one of the many contrasts between the two men (Sect. 5.2), Russell's massive *Nachlass*, conserved in the Bertrand Russell Archives in McMaster University, Canada, includes dozens of letters and a few manuscripts from Whitehead, which help especially in determining his role in *Principia Mathematica.*

They did not give any specific name to the program presented in that work. From 1904 the name "logistic" was used to refer both to it and also to the logical treatment of mathematics pursued by Giuseppe Peano (1858–1932) (Sect. 2.2). In order to remove this ambiguity, in the late 1920s Rudolf Carnap (1891–1970) proposed "logicism" as an alternative name for the Whitehead/Russell thesis: gradually it became standard, and I shall use it here. I shall also follow the usual practise of the time, especially in *Principia Mathematica*, of using "class" as the technical term where today we usually prefer "set" (and as did Whitehead himself in some later contexts); but I shall use "set theory" to refer to the subject in general. Whitehead and Russell sometimes used square brackets in their letters and manuscripts; so when quoting from them I enclose my editorial interferences in angled brackets "⟨ ⟩."

2. WHITEHEAD'S MATHEMATICS: FROM ALGEBRAS TO LOGIC

2.1. Whitehead's Algebraic Phase

As a student at Trinity College, Cambridge, in the early 1880s, Whitehead was exposed to the tedium of the undergraduate mathematics syllabi but also to the strength of the University in applied mathematics. It was therefore typical of the time that he wrote on electrodynamics to gain a college fellowship; sadly, this work is now lost. A couple of papers on fluid mechanics soon followed, and one on geometry [Whitehead 1898a]; but his main product was a large book started in 1891 and published seven years later by Cambridge University Press [Whitehead 1898b].

The title, *A Treatise on Universal Algebra, with Applications*, was inspired by a paper by the algebraist J. J. Sylvester (1814–1897) on "universal algebra" [Sylvester 1884]. But in neither case was the title was happily chosen, since no all-embracing algebra was presented [Novy 1976]. Whitehead's book included an impressive selection of the new algebra*s* of recent decades [Grattan-Guinness 1999]. The one due to Hermann Grassmann (1809–1877) was his principal source; of impressive though not universal generality, it focused upon the algebraic representation of geometrical/spatial magnitudes of various kinds [Schubring 1996]. Whitehead rehearsed some of the main features in his first Book, along with the theory of manifolds (also known in English sometimes as "multiplicities") due to Bernhard Riemann (1826–1866) [Riemann 1867]. An interesting feature is Whitehead's discussion of the relationship between identity and (mathematical) equality; he regarded the latter as "equivalence of validity" to cope with the "paradox" and "truism" both evident in "$a = b$" [Mays 1977, Chap. 3].

The second Book treated "The Algebra of Symbolic Logic," which largely drew upon the algebra that had been proposed in the mid-19th century by George Boole (1815–1864) and various later developments, including some algebraization of syllogistic logic. While Whitehead drew on some aspects of the massive treatment given by Ernst Schröder (1841–1902), he took little note of the pioneering contributions made by C. S. Peirce (1839–1914), and ignored their most significant contribution, the logic of relatives; and he seems not to have known that Robert Grassmann (1815–1901) had adapted his brother's algebra to produce an algebraic logic similar to Boole's.

The third Book handled "positional manifolds" without reference to measurement. It amounted to a survey of many features of projective geometry, in which metrical properties were not assumed and all lines intersected, if necessary as a presumed point or figure at infinity. The fourth Book delved into many details of Grassmann's algebra, and incorporated parts of metrical geometry; some properties were expressed in matrix theory. Not yet a widely used branch of mathematics, matrices also appeared in the next Book, where parts of mechanics were handled. The sixth Book, on "metrics," by far the longest, worked through non-Euclidean geometries in some detail, with quadratic forms prominent; it related to a contemporary paper [1898a] on geodesics on surfaces in elliptic and hyperbolic geometries.

Further applications to mechanics were made, especially a treatment of space in terms of transformations which had been developed by Felix Klein (1849–1925). In the final Book Whitehead applied Grassmann's algebra to geometry by outlining much vector algebra and vector analysis. Finally, a bibliography of Grassmann's writings was given.

The book was unusually well received. In Britain G. B. Mathews (1861–1922), Senior Wrangler in the Mathematical Tripos at Cambridge University in 1883 when Whitehead had been joint fourth Wrangler, welcomed it very wittily in *Nature*, contrasting the "good old days" when "two and two were four" with the panorama exposed in this book, "which appears to set every rule and principle of algebra and geometry at defiance. Sometimes *ba* is the same thing as *ab*, sometimes it isn't" [Mathews 1898]. Abroad the French philosopher Louis Couturat (1868–1914) wrote 40 pages on the book and the German Paul Natorp (1854–1924) 25 pages [Couturat 1900, Natorp 1901], while the Scottish-born mathematician and historian Alexander MacFarlane (1851–1913] greeted it warmly for the American journal *Science*, although he bemoaned the small attention given to Peirce and Schröder [MacFarlane 1899]. The reviewing *Jahrbuch über die Fortschritte der Mathematik* noticed the book in an extremely unusual way: no copy had been sent to them, but co-editor Emil Lampe composed a review based upon those written by Mathews and MacFarlane [Lampe 1900].

The book was subtitled "Volume 1," and Whitehead continued with other algebras, including the quaternions of W. R. Hamilton (1805–1865). During 1899 he submitted to the Royal Society a long paper on finite groups and their interpretation as classes with imposed operations; the paper was accepted and a summary published [Whitehead 1899], but then he withdrew it when he found (or felt) that a significant proportion of his results had recently been proved by the German mathematician Georg Frobenius (1847–1917).[1] Further work on the volume was interrupted by an international event, and a growing relationship with a former student who had already read the proofs of his book.

2.2. *Russell on the Foundations for Mathematics and Philosophy*

Whitehead's duties at Trinity College included the instruction of undergraduates, and in the early 1890s the cohort had included Russell. After taking the Mathematical Tripos (seventh Wrangler), Russell had decided that that was enough mathematics for a long time, and so he took Part 2 Tripos in philosophy. Winning a Trinity College Prize Fellowship for six years in 1895, he united his two trainings in an effort to present epistemological foundations for mathematics. A version of his dissertation appeared from Cambridge University Press as *An Essay on the Foundations of Geometry* [Russell 1897], soon to be followed by forays into arithmetic and parts of mathematical analysis and mechanics, of which Whitehead read several. But no satisfactory system emerged, even when he drew upon the set theory of Georg Cantor (1845–1918) and upon Whitehead's book [Russell, *Papers*, Vol. 2].

Part of the difficulty lay in the philosophical climate which Russell had inherited: the staple Cambridge fare of "neo-Hegelianism," a comprehensively idealist soup in which all was human conception. In particular, the clash of "thesis" with "anti-thesis" were to be resolved in a "synthesis" which itself awaited its own contrary and further synthesis [Griffin 1991]. Russell's slightly younger philosophical colleague G. E. Moore (1873–1958) also

[1] The manuscript is kept in the Archives of the Royal Society, AP, Vol. 74, No. 4, 82 fols.; for discussion see [Grattan-Guinness 1986].

subscribed to this faith; but around 1899 he crossed over to empirical realism, replacing wherever possible assumptions of abstract or mental objects with a strong reliance upon sensual data and constructs made from them. Russell accepted this change soon afterwards, relishing also its preference for pluralism over monism; but mathematics remained uncaptured. However, a further reform soon occurred.

One feature of European intellectual life in the late 19th century was internationalization: co-operation and contacts between the various European countries and empires and elsewhere, especially the emerging United States of America. Mathematicians exhibited this characteristic quite strongly [Stump 1997], including the launch of international congresses (where Cantor had played a prominent role). The first congress was held in 1897, and the successor in Paris in August 1900: many disciplines held such meetings that year in the city, and the philosophers organised theirs to precede immediately that of the mathematicians. Whitehead and Russell went to it, and Whitehead stayed on for the second one.

The great moment for both men occurred with the philosophers during the morning of Friday 3 August, when Peano and three of his main followers either presented papers on the Peano approach or had them read *in absentia.* The principal aim of the "Peanists" was to establish the rigor possible in mathematical knowledge.[2] To this end they both axiomatized and symbolized a mathematical theory as much as possible, and also treated in the same way the required logic—that is, "mathematical logic," Peano's own name (in the modern sense) covering both propositional and predicate calculi with quantification over domains of objects (integers, say, or points). Cantor's set theory played a major role as a means of handling collections of objects satisfying predicates (such as the class of integers or of points); it had suffered a somewhat difficult reception from Cantor's inauguration in the early 1870s, but Peano had adopted it quickly in the early 1880s, and from the mid 1890s it was gaining prestige rapidly among mathematicians and some philosophers. One significant feature was its distinction between membership of individuals to a class and (im)proper inclusion of subclasses in that class, which was not observed in the traditional part–whole theory of collections. More controversial was Cantor's revolutionary theory of the actual infinite, where infinities came in different sizes and cardinal and ordinal properties of "transfinite" numbers were studied.[3]

From around 1897 Russell had become gradually more interested in Cantor's theory, initially for a third feature: his novel emphasis on "order-types," the different ways in which (infinite) collections of objects could be ordered. The interest for Russell was that these types could be expressed by various kinds of relation. Whitehead's *Universal Algebra* also exercised some sway on him from 1899, especially for the emphasis on manifolds. But the great change, perhaps the major one in Russell's philosophical career, was the discovery of Peano's enterprise and its attendant aspects such as the correct means of framing definitions (the subject of his Paris lecture). During the next 12 months or so Russell learnt the Peanists' system. Discovering to his understandable surprise that they had not formed a logic of relations (that is, propositional functions of more than one variable), he formulated it as an extension of mathematical logic and published two papers on it in a journal edited by Peano.

[2] On the Peanists, see especially [Borga and others 1985; Rodriguez-Consuegra 1991, Chap. 3; Grattan-Guinness 2000, Chap. 5].

[3] On these developments see [Dauben 1979, esp. Chaps. 6–9; Grattan-Guinness 2000, esp. Chaps. 3–4], and further references given there.

Like Whitehead before him (Sect. 2.1), he largely ignored the important contributions of Peirce and Schröder.

Russell also recognised the Peanists' practice of keeping logical and mathematical notions separate; but he must have noticed that sometimes notions from set theory appeared in each column, and occasionally both, and so he concluded that there was no difference between them. Thus he formulated the logicist thesis, that mathematical logic could supply not only the required modes of reasoning (as it should) but also the objects required in mathematics. He started out from his definition of cardinal integers: 0 as the unit class of the empty class, and the others as classes of "similar" classes (that is, classes whose members could be set up in one–one correspondence), and built up through rational to irrational numbers to continuous domains and thence to the calculus, geometries, and even some mechanics.

That was the good news. But while studying set theory, sometime in the spring or summer of 1901 Russell also found the bad news, a paradox which ever since has carried his name: the class of all classes that do not belong to themselves belongs to itself if and only if it does not. This news was as bad as it could be: misfortune falls upon any theory when a significant paradox surfaces, but when the theory embodies logic *itself* the malaise is maximal. But he carried on: in the middle of 1902 he sent to Cambridge University Press a book manuscript on "The Principles of Mathematics." While it was in press he added an appendix on the work by Gottlob Frege (1848–1925) with special reference to the anticipations of logicism for arithmetic, and another one attempting to solve his paradox. He also made many additions to the main text on proof (for example, most of the footnotes). The preface was written at the end of 1902, with fulsome thanks offered to Whitehead for help and to G. E. Moore for philosophical leads.[4] The book finally appeared in May 1903; it too was a "Volume 1" [Russell 1903, xviii].

2.3. *The Gestation of a Collaboration*

"I found that Whitehead has a great reputation," Russell told Moore soon after returning from Paris; "all the foreigners who knew Mathematics had read and admired his book, and were delighted to meet him" [Russell 1992, 202]. Whitehead also became greatly impressed by Peano at Paris, but he took a somewhat different research direction. As well as carrying on with his own Volume 2, he studied the Peanists' writings carefully, lecturing upon them in 1904 to the annual meeting of the British Association for the Advancement of Science.[5] But he was drawn more to set theory and transfinite arithmetic, especially algebraic aspects such as factorization, duality, possible invariants under transformation, and group-theoretic properties; he also studied certain kinds of classes of classes, and used Russell's new theory of relations. Three papers and a short note were produced [Whitehead 1901, 1902, 1903, 1904]; they appeared in the *American Journal of Mathematics*, edited by Frank Morley (1860–1937), who had been a fellow student with Whitehead at Trinity College in the early 1880s but had emigrated to the U.S.A. and was then a professor at Johns Hopkins University [Archibald 1938, 194–201].

[4] Readers aware of Russell's claim in his autobiography that the whole book was basically written in 1900 [1967, 145] will be surprised at the summary given here. In fact the writing was still more convoluted; for details see [Grattan-Guinness 1997a, or 2000, Chap. 6 *passim*].

These connections between Russell, Whitehead, and G. E. Moore may show some interaction via the undergraduate debating society The Apostles, of which all three were members [Levy 1979].

[5] *Report of the British Association for the Advancement of Science* (1904), p. 440.

To the second paper Russell contributed a section in which he published for the first time his logicist definition of cardinal numbers. This is a sign of close working, and sometime during 1902 it seems to have evolved into a formal collaboration with Russell's Volume 2 as the target [Russell 1903, xvi] and Whitehead's own Volume 2 postponed. Whitehead may have realized that, while his own successor would add nicely to his catalog of algebras, it did not have the mathematical or philosophical weight carried by Russell's ambition, whose fulfilment would require too much effort for even that young man. Furthermore, when defining mathematics "in its widest signification" early in his own Volume 1 as "the development of all types of formal, necessary, deductive reasoning," he had advocated the importance of logic: "the sole concern of mathematics is the inference of proposition from proposition" [1898b, vi]. However, he had not worked out the details of this claim (which surely characterizes logic more than mathematics[6]); now he saw a possibility in Russell's project.

So the two men were together intellectually, but not often geographically. Russell's fellowship had ended in 1901, and during the rest of the decade he seems to have lived largely on his own resources, and mostly away from Cambridge; in particular, from the spring of 1905 he lived for much of the next six years in a lovely house built for him and his wife Alys at Bagley Wood near Oxford. During this period the Whiteheads lived at Grantchester near Cambridge until 1907, when they moved back into the city. So, fortunately for historians, they corresponded extensively. As was mentioned in Sect. 1, only the Russell collection has survived (with one letter published in [Russell 1992, 285–286]), but from it one can gain some impression of the division of role and of labor.[7]

By and large, Whitehead left philosophical decisions and plans to Russell. The main issue was solving (or rather, philosophically Solving) the paradoxes, not only Russell's but also others such as of the greatest Cantorian transfinite cardinal and ordinal numbers. Other questions included the relationship between propositional functions and mathematical functions, which was resolved as a special case of his 1905 theory of definite descriptions, in which negation of the definite article was handled like negation with quantifiers [Russell 1905].[8]

But Whitehead was no passive admirer. In particular, he outlined the main properties that are required of a "reducible" propositional function, that is, one whose associated class is a legitimate object: the paradoxes exhibit exceptions, such as "not belonging to itself" [Russell, *Papers*, Vol. 2, xxiv–xxvi]. Figure 1 shows an extract from a long exegesis which Russell received from him and added a line. One of the main criteria centered around Cantor's power-class theorem, that the number of members of a class is strictly less than that of the members of the class of all its subclasses, for Russell had found his paradox by developing a variant of the proof [Grattan-Guinness 1978, Coffa 1979]; however, despite many efforts, no Solution was emerging.

[6] Whitehead seems to have been influenced by Benjamin Peirce; he cited Peirce in his [1911d, 200–201]. On Peirce's own cryptic formulations, see [Grattan-Guinness 1997b].

[7] The authoritative source for the period 1900–1905 is [Russell *Papers*, Vol. 3–4]; unfortunately the succeeding volume, covering 1906–1908, is still not ready.

[8] Or at least this is my reading of Russell's theory, though I do not find it noted in the endless literature. Just as the true proposition "Not all numbers are even" is distinguished from the false "All numbers are not even," so the false (by Russell's criteria for existence) "The present king of France is not bald" is to be distinguished from the true "Not (the present king of France is bald)"; their respective negations are the true "Not (the present king of France is not bald)" and the false "The present king of France is bald." That is, four propositions need to be considered, not two.

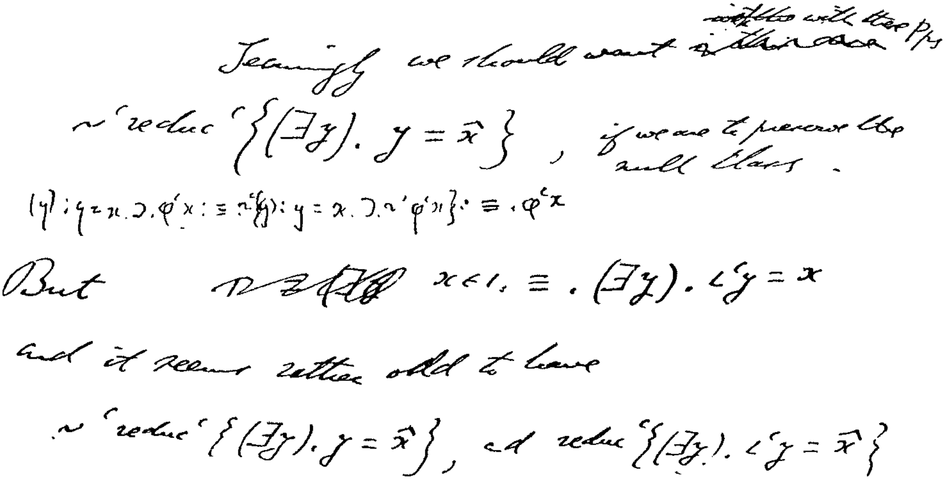

FIG. 1. A fragment from the foundry: Whitehead on reducibility, with an addition by Russell.

Whitehead also criticized more general features of Russell's presentation. While the collaboration was being formed, some time in 1902 he criticized Russell's first attempt to write the opening parts of *Principia Mathematica*: "Everything, even the object of the book, has been sacrificed to making the proofs look short and neat. It is *essential*, especially in the early parts, that the proofs be written out fully." Around that time he also criticized Russell's notations for two-place relations [Russell, *Papers*, Vol. 4, 38]: he often proposed notations and became responsible for most of them. However, he was rather profligate, especially in connection with relations, so that several theorems merely serve as dictionary entries [Quine 1941, 154].

3. WHITEHEAD'S ESSAY ON GEOMETRY, SPACE, AND MATTER, 1905

> Years ago, in a communication to the Royal Society in 1906, I pointed out that the simplicity of points was inconsistent with the relational theory of space. At that time, so far as I was aware, the two inconsistent ideas were contentedly adopted by the whole of the scientific and philosophic worlds.
>
> Whitehead 1920b [1947a, 337; 1947b, 241–244]

3.1. The Preparation of the Essay

While Russell was struggling with the logical system and the need to Solve the paradoxes, Whitehead seems to have decided around 1905 to work out some of the later parts of logicism in a technical way which could be followed or modified when the definitive system was adopted. He chose to try out some features of projective and Euclidean geometries and their applications to mechanics and mathematical physics. He kept closely in touch with Russell; according to letters he started to work out the paper in April 1905. He submitted it in September to the Royal Society, to which he had been elected in 1903 after proposal by his senior Cambridge colleague A. R. Forsyth (1858–1942); all papers were refereed there, and his went to Mathews and Sir David Niven (1842–1917). The latter also showed it to the algebraist William Burnside (1852–1927), who had added an appendix [Burnside 1898] to his earlier paper on geodesics (Sect. 2.1); he produced the main comments. Whitehead made suitable revisions to his paper and also stated reasons for preferring his presentation on occasion; [9] the final version was accepted by the Society in December and appeared late in the following year as the 61-page paper [Whitehead 1906b]. It was republished nearly 50 years later in an anthology of Whitehead's writings; below I give both page numbers, in chronological order.

No manuscript version of the paper has survived. There seems to have been a draft version; for Russell's *Nachlass* contains 12 folios called "Whitehead," in which he wrote out the main definitions and features of undoubtedly this paper but with a somewhat different sequence of notions. A further sequence of 18 folios entitled "[Final Recension] *Whitehead* Material World" recorded the sequence of definitions exactly as it appeared in print in 1906; presumably Russell took it from the final manuscript, although maybe he used the published version.[10] There does not seem to be any major difference of content or procedure between the two lists, and I shall not refer to it further; however, the referees' main comments are noted. On the philosophical aspects of the paper see [Mays 1961], summarized in his [1977, Chap. 4].

3.2. *Whitehead's Principal Sources*

The full title of Whitehead's paper was "On Mathematical Concepts of the Material World"; in it he applied some of the logical and set-theoretic notions available, especially the logic of relations. In a short summary written for the Royal Society's *Proceedings* [Whitehead 1906a] stated his task thus: given a many-place relation R and the members of its field, "the problem here discussed is to find various formulations of axioms concerning R, from which, with appropriate definitions, the Euclidean Geometry issues as expressing properties of the field of R," which could come from "the Material world."

Whitehead's sources included his own treatment of projective geometry from the third Book of his *Universal Algebra*; but he was also strongly influenced by two other authors. One was Russell, especially the appropriate parts of his *Essay* on geometry and *The Principles* [Russell 1897, esp. Chap. 3; 1903, esp. Chaps. 45–47]. The other was the American Oswald Veblen (1880–1960), who had recently published his doctoral thesis on the foundations of geometry, prepared under E. H. Moore (1862–1932) at the University of Chicago [Veblen 1904]; Whitehead wrote to Russell about it at some length in letters during April 1905.

A main source for Veblen was the study of the foundations of geometry made by David Hilbert (1862–1943) at the end of the century [Hilbert 1899]; he had stressed the need to find full sets of axioms for a mathematical theory and show their consistency and independence [Toepell 1986]. Veblen imitated this procedure with different primitive notions. Carefully laying out definitions and axioms, he constructed Euclidean geometry up to three dimensions, followed by an outline of projective and metrical geometry. Whitehead liked Veblen's decision to take order as a primitive,[11] for he could express ordered objects in

[9] This material is kept together in the Royal Society Archives, file RR, Vol. 15, fols. 399–403. Whitehead's letter of 24 December 1905 to the Society asking for the return of the manuscript for alterations is kept at MC. 06655.

[10] These two lists are held in the Russell Archives, file 230.030473.

[11] Veblen may have followed a line of influence from Cantor similar to Russell's (Sect. 2.2); for his supervisor Moore was very interested in set theory, including order-types, and was soon to launch an ambitious topic called "general analysis" [Moore (E.H.) 1910)], in imitation of Cantor's term "general set theory." On projective geometry at that time, see [Cassina 1940].

logicist terms as fields of relations of various kinds. Some other features of Veblen's paper also affected Whitehead, as we shall soon see.

3.3. Whitehead's First Three "Concepts"

> Everyone makes for himself a clear idea of the motion of a *point*, that is to say, of the motion of a corpuscle which one supposes ⟨to be⟩ infinitely small, and which one reduces by thought in some way to a mathematical point ⟨. . .⟩
>
> Louis Poinsot (1777–1859), opening of [1834]

In outlining his aims, Whitehead stressed that his view of space was relativistic, not absolutist [1906b, 467/14];[12] he used the adjective "Leibnizian" at several places to claim eminent parentage for his position, which was opposed by Russell among others. He began his main exegesis with a brisk survey of the basic notions and proposed notations for set theory and mathematical logic, especially the logic of relations [Whitehead 1906b, Part 1]. Ignoring his worries of 1898 (Sect. 2.1), identity meant simply that "*x is defined to stand for y*" [the enigmatic p. 471/19]. Class abstraction was marked by semi-colons, and existential quantification by raised periods; thus, to quote one of his examples for a three-place relation "R⁏(xyz),"

$$\text{“R}^{;}(;\, y\,\cdot)\,.\!=\!.\,\dot{x}\{(\exists z).\text{R}^{;}(xyz)\}\quad \text{Df, ”} \tag{1}$$

where "Df" abbreviated "definition" [pp. 474/22]. The superscript semicolon was always, though unnecessarily, attached to the relation symbol. To his opening flourishes he added a paragraph for Niven's benefit, amplifying the role and attached properties of R [p. 470/17, beginning "Thus in"].

Among the relations used, the main ones were "essential" and the others "extraneous." Whitehead laid out carefully his axioms and definitions, though rather oddly he presented axioms as "Df's" of the *names* of the relationships asserted; however, unlike Veblen he did not claim that his axiom system was independent [p. 476/24]. Among the defined "entities" were "points" (a primitive notion with Veblen), of which points in space, instants of time and particles of matter were examples. Similarly, lines could be geometrical lines, but also lines of physical force. He used the terms "objective reals" to refer to all nontemporal notions [p. 467/13]; the adjective manifested his empirical attitude to physical world, which presumably followed G. E. Moore and Russell (Sect. 2.2).

The exegesis was based upon a sequence of five "concepts," that is "Each complete set of axioms, together with the appropriate definitions and the resulting propositions." Those which required two classes of projective reals were "dualistic," and only one class were "monistic"; Whitehead's reductionist spirit preferred the former kind if possible [pp. 466–467/13].

The first concept was purely spatial; the essential relation "R⁏(abc) means *the points a*, *b*, *c are in the linear order* (*or the* R-*order*) *abc*"; in this case both "*points of space* and *particles of matter*." Lines were construed as the classes of its points, so that "R⁏$(a; b)$ is the *segment between a and b*," "R⁏$(ab;)$ is the *segmental prolongation of ab beyond b*," and so on [pp. 476/24–25). He proceeded to define rectilinear figures, and spaces defined by

[12] This use of "relativity" has no link with the theory being developed at exactly this time by Albert Einstein, who introduced the word (most regrettably) for his own theory only a decade later (compare Sect. 5.3 below).

the closed specifying boundaries (for example, of a triangle). A noticeably vectorial flavor comes over, redolent of the fifth and sixth Books of *Universal Algebra*. Twelve axioms were put forward, similar to the systems of Hilbert and Veblen; several claimed the existence of some points and properties of order among them, and another "secures the continuity (in Cantor's sense) of the points on a line." Extraneous relations were briefly offered to bring in temporal aspects, when particles of matter moved in time [pp. 478–480/27–29].

Whitehead pointed out that "none of the definitions contain⟨s⟩ any reference to length, distance, area, or volume" [p. 477/26]; and Burnside had taken exception when he refereed the paper, wondering at the absence of axioms on the congruence of figures. The traditional way from Euclid onwards had been to state (or assume) congruence in terms of sameness of shape and size; but during the previous 30 years or so, as part of the rise of projective geometry, an alternative approach had been pursued by mathematicians such as Arthur Cayley (1821–1895) and Klein, in which a metric was defined relative to a given conic placed at infinity (Sect. 3.6, (3)); congruence was them handled in terms of invariance of shape under (projective) transformations in the plane [Richards 1989]. Among various commentators, both Whitehead and Russell had discussed or used it in their books, and Veblen in his paper; yet Burnside seemed to have been completely unaware of it. "One of them has apparently never heard of the Projective Theory of Metrics," Whitehead wrote to Russell on 25 December 1905 when he received the referees' reports, and he added a sentence to his manuscript, pointing out that the theory was "well-known" and giving some references in a footnote [p. 478/26].

The second concept was a monistic variant of its predecessor in applying only to points in space; a two-place extraneous relation linked them to instants of time, an approach which he credited [p. 480/29] to Russell's *The principles* [1903, 468]. The next concept was also monistic, "obtained by abandoning the prejudice against points moving" and taking the "modern (and Cartesian) point of view of the ether, as filling all space. The particles of ether (or moving points) compose the whole class of objective reals" [p. 480/30], in a rather incoherent passage which seems to involve both the elementary particles of physical matter and the much smaller particles composing the punctiform aether which was then often favored [Buchwald 1980]. Later on he seemed to assume a nonpunctiform aether, "which 'lies between' the corpuscles of gross matter" [p. 483/34].

The essential relation, "R;(*abct*)," "may be read as stating *the objective reals a, b, c are in the* R-*order abc at the instant t*." Help came from an extraneous relation "S;($uvwt$)" asserting that "u, v, w are intersecting straight lines mutually at right angles" at time t, and so forming a usable system of co-ordinate "kinetic axes." Whitehead also distinguished between cases of the concept where the objective reals maintained "the same special type of motion" involved, such as in "Kelvin's vortex ring theory of matter"; and those where a motion continued but where the reals might change, for example through disintegration [pp. 480–482/30–31].

3.4. *The Concept of Interpoint*

The last two concepts were primarily concerned with "linear objective reals" of various kinds. Projective geometry now came further into play with a major concept: "*intersection-points* (shortened into *interpoints*)" [p. 484/35]. This was not just the usual business of lines intersecting in points, for it was related to the underlying duality in projective geometry that

"Any valid projective proposition remains valid if the words ⟨'⟩point⟨'⟩ and ⟨'⟩plane⟨'⟩ are interchanged in its complete statement," as Veblen had put it [1904, 376]. In his logicist quest Whitehead went further, to conceive not only of lines as classes of (all) points and solids as classes of planes but also conversely of lines as classes of intersection of (all) planes and points as classes of intersection of lines or of planes [pp. 482–483/32–33]. Thus "interpoints are, in general, only portions of points" since they would occur only in connection with *certain* lines, often some pair of them; and "a point may contain no interpoint or many interpoints" [p. 484/35, in a passage somewhat revised after refereeing].

Whitehead seems to have missed an intriguing possibility here. In a long paper published by the Royal Society A. B. Kempe (1849–1922) had elaborated a theory of multisets (to use the modern name), where membership can occur more than once: for example, the repeated root 5 and 5 of a polynomial equation [Kempe 1886]. Then he had applied it to a study of the relationship between classes and "the geometrical theory of points," including projective properties [Kempe 1890]. Whitehead cited this second paper in an early footnote of his 1906 paper as one of his minor sources, but only for its use of a three-place relation to express projective properties, which he could handle by means of the logic of relations [1906c, 469/16]. The possibility of treating points as multisets of interpoints was not floated. Kempe's extraordinary foray made virtually no impact on him, or on any other compatriot.[13]

The essential relation was now the five-place "R‘(*abcdt*)," stating that "*the objective real a intersects the objective reals b, c, d in the order bcd at the instant t*" [p. 484/34]. The need for four lines was imposed by projective geometry: since no metric was imposed, three rather than two points were required to specify a line, and thus three lines b, c, and d to generate the points of intersection with the fourth line a (Sect. 3.6, 3)). Further, to ensure intersection a point (or line or circle) at infinity was added to the repertoire, as the location of the meet of lines which in Euclidean geometry were regarded as parallel.

A major derived projective relationship was that of similarity between entities x and y with respect to R, defined by the property that if R were valid for them in their positions among its three central places, then it remained valid if they were switched (for example, if they were neighbouring lines when intersecting with a). The ghastly "symbol R‘$(\frac{a???t}{x})$ denotes the class of entities with positions similar to that of x in the relation R, a being first term and t last term" [p. 485/35]. Many of the ensuing definitions and axioms involved R itself, and/or this and related classes: they played an important role in the rest of the theory, thus justifying their place here against Burnside's request that the section be moved forward in the paper.

The fourth concept dealt with particles; their motion was represented by linear objective reals portraying lines of force. It split into two cases, as with the previous concept. In the dualistic case enduring particles normally differed in location at different instants of time, so that they would be determined by an "indefinite number of extraneous relations," presumably expressible via the coordinate kinetic axes. In the second case involving possible disintegration of particles the concept was monistic, and those relations would be only two-place, between points and instants [pp. 491–493/42–44, including some cryptic remarks about electricity and magnetism].

[13] On Kempe's work and its amazing effect upon C. S. Peirce and then Josiah Royce, see [Grattan-Guinness 2002; briefly in 2000, 137–140, 154–156].

3.5. The Theory of Dimensions

The fifth and last concept was the dominant one; the account composed the second half of the paper. As part of the extensive preparation some predicate calculus was introduced: if "$\phi!x$ is some proposition involving the entity $x\langle\ldots\rangle$, then any entity z, for which $\phi!z$ is true, is said to *possess the property* ϕ." An extension to "ϕ-classes" u was also defined, along with the union of such classes and the "*common sub-region for u*" as the intersection of all ϕ-classes which contained u [pp. 492–493/44–45].

A major definition was that of the flatness of a class of straight lines: namely, either it contained only one line, or if and only if a third member line met two other lines at a point distinct from that of their own intersection. Further, two classes "have ϕ-equivalence" if they possessed the same subregion, and class u was "ϕ-prime" if it was not ϕ-equivalent to any of its subclasses. The latter property bore upon the dimensions of classes of (non-)concurrent lines; when the lines *were* concurrent, the class was "ϕ-axial." Then its dimension number was defined as the largest cardinality of the classes both ϕ-equivalent to it and ϕ-prime. He indicated that the required number 3 would satisfy this definition for properties such as "being a pair of nonintersecting lines" [pp. 492–495/45–47]. The arrival of a number in the theory may seem surprising; it came via Whitehead's adoption of Russell's logicist definition of cardinal integers as classes of similar classes [p. 473/21].

With this ingenious suite of definitions Whitehead built up an elaborate theory of related notions, such as points satisfying ϕ and points defined in terms of ϕ-axial classes of lines. The "*Proposition*" that all nonempty subclasses of a ϕ-prime class were themselves also ϕ-prime for any ϕ formed part of the "foundation of the whole theory"; for it followed that the cardinality and the dimension of a ϕ-class were equal under equivalence and axiality [pp. 497–498/50–51]. He also proposed five new axioms, allowing the formation of classes of coplanar and of noncoplanar but concurrent lines [pp. 498–499/52]. Elaboration of the relationships between ϕ-axiality, ϕ-maximality, and dimensions of classes of lines provided him with the machinery for the last concept, "linear and monistic" and once again "Leibnizian," where "points are classes of objective reals, and disintegrate from instant to instant." R‘(*abcdt*) obtained as in the previous concept, but "*copunctual* objective reals do *not* necessarily *intersect*, though two intersecting objective reals are necessarily copunctual" [the cryptic p. 505/60, drawing upon properties of ϕ-axiality]. A "*punctual line*" was defined as the class of points formed by (only) two distinct intersecting planes [p. 507/62].

A major new definition was of an objective real p being "doubly secant" relative to a class u of objective reals as the property that at least two members x and y of the class intersect p at instant t but do not both belong to any interpoint on p, so that there must be a segment of p in between those two reals. Then Whitehead could generalize the previous definition of flatness to that of "homaloty" of u at t: either it contained only one member, or if and only if each of its members was doubly secant with it [p. 506/60–61]. Thereby he could specify the order of points on a line in terms of their intersection with other lines, and thus give order a generality to emulate Veblen but without having to take it as a primitive.

Projective geometry required points at infinity, which Whitehead renamed "*cogredient points*." Two objective reals were cogredient if trios of interpoints on each of them could always be put into the *same* order by some interpoint relation. He also defined two figures (that is, any two classes of points) being in perspective in the usual ways in terms of a projective pencil from a point in the full sense [pp. 508–509/63–64]. This time 17 axioms were needed, concerning order of points on lines, relationships between interpoints and points, various existential requirements, and continuity; at the end he stressed that they applied not only to orthodox geometry but also to the relationships with objective reals and interpoints [pp. 511–513/67–69]. The lengthy further exegesis included many theorems on the validity here of the previous theories of interpoints, further properties of cogredient points, and orders and points on lines (or objective reals) and in figures in general [pp. 514–524/70–81]. Finally came the extraneous relation for velocity and acceleration and remarks on types of points and "corpuscles." He concluded the paper with a call for "some simple hypothesis concerning the motion of objective reals and correlating it with the motion of electric points and electrons," requiring "the assumption of only one class of entities as forming the universe" [pp. 524–525/81–82].

3.6. Some Limitations of Whitehead's Enterprise

Apart from the spectacular aspiration of the last quotation, some surprising silences and queries attend aspects of his ambitious study. Seven of them are mooted; the order runs from axiomatization through set theory and geometry to mechanics and physics.

(1) Although he was heavily influenced by Veblen's recent paper, Whitehead did not use a feature of it which soon was recognised as a highlight. Soon after publishing his study of geometry in 1899 Hilbert had realized the need for a further axiom, of "completeness." The word was not used in the sense to which his later work on the foundations of mathematics has accustomed us: it was a meta-axiom (as we would understand it), asserting that the other axioms captured *all* the objects required by the theory. It was included in the French and then the English translations of the book [Hilbert 1902, 25], whereupon Moore and Veblen were inspired to consider the relationship between two (or more) collections of objects satisfying an axiom system and to form the notion of "categoricity" between them when an isomorphism could be established across all their respective members [Veblen 1904, 346].[14] This new notion added considerably to the fledging subject of model theory [Scanlan 1991] and surely bears upon the interpretation of Whitehead's theory. He seems either to have ignored it, or else took it for granted: for example, that categoricity held between the points of space, particles of matter and instants of time as interpretations of objective reals.

(2) Cantor had proved that an isomorphism could be found between the points in a square and those of any one of its sides [Johnson 1979]. The theorem shocked even its prover, for it refuted the usual understanding of dimensions; finding an alternative definition of it turned out to be a daunting task, not satisfactorily completed until the 1920s [Johnson 1981]. Whitehead surely was aware of this aspect of set theory, but he never mentioned it; however, his own theory of dimensions must be susceptible to it, for he used coordinates as set by kinetic axes.

(3) In his exegesis of the concept of definitions Whitehead made some use of projective geometry; however, in contrast to Veblen and Kempe, he did not emphasize the quadrilateral construction. It can be built up from Fig. 2. In this geometry three points are needed to

[14] The name, in its adjectival form "categorical," had been suggested to Veblen by John Dewey, who was then Professor of Philosophy at Chicago University. For noncategoricity he proposed "disjunctive," which has not endured.

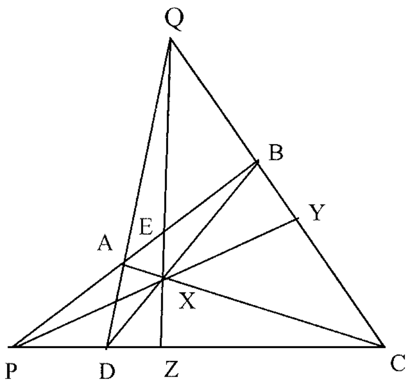

FIG. 2. The quadrilateral construction.

determine the line PDC since no metric is available. Assume another point, A, not on it; then P and A lie on a second line, and let B be chosen as its third point. Draw DA and CB to meet at Q; draw the diagonals AC and BD to meet at X; finally draw QX and PX to meet the original lines at Z and Y respectively.

A major theorem due to Karl von Staudt (1798–1867) stated that, given P, D, and C, then Z was *uniquely* determined by the construction; and similarly with Y on QBC [von Staudt 1847, 45]. Further, the cross-ratio (then often called the "anharmonic ratio") of P, Z, D, and C was invariant under projective transformation (for example, its value was taken also by P, E, A, and B) and a metric was defined from it by taking a logarithm. The metric used by Cayley and Klein (Sect. 3.3) took D and G to be the points of intersection of that line with the conic placed at infinity. Among other theorems, "A segment AB is congruent to a segment $A'B'$ if AB can be transformed into $A'B'$ by a finite number of reflections" (and vice versa), the latter notion being a projective transformation which maps, for example, P and Z of Fig. 2 onto each other [Veblen 1904, 381–383]. It seems odd that Whitehead, impressed by the power of projective geometry, did not use such theorems; but clearly Burnside was quite unfamiliar with all of them!

(4) In confining himself to Euclidean geometry, Whitehead surely showed a narrow conception of the applicability of mathematics to the material world, especially with electricity and magnetism included. A student of Riemann's theory of manifolds (Sect. 2.1), he must have been aware of the remarkable observation on the possible connections between geometry and the distribution of matter [end of Riemann 1867]; yet he did not take it up.

(5) In addition, Whitehead's lines and their objective cousins were always *straight*. Perhaps this restriction was needed because he did not assume or produce the differential calculus and so could not handle curved lines and their tangents. But this is a limitation which he did not seem fully to absorb. In connection with extraneous relations for the third and fifth concepts, he stated that "velocity and acceleration can now be defined" [p. 481/31; compare p. 524/81]; but he had not supplied the required mathematical means. The issue also bears upon logicism; Russell had included chapters on the calculus in *The Principles* [1903, Chaps. 39–41], but he and Whitehead were to omit it from *Principia Mathematica*, without discussion.

(6) Whitehead handled space and time separately; the notion of space–time was only to come around 1908, especially with Hermann Minkowski (1864–1909), in connection with the theory of relativity due to Albert Einstein (1879–1955) [Hentschel 1990, 22–25]. Whitehead made this link, apparently independently, but only in 1911; the circumstances are recorded in Sect. 5.3 below. Such a lacuna is quite understandable: much less so is the silence over the status of his points, instants, and particles *within logicism*, the program which had fathered this enterprise. The issue was to recur in *Principia Mathematica*, where Whitehead was to make a serious related mistake (Sect. 4.6).

4. TOWARD AND THROUGH *PRINCIPIA MATHEMATICA*

4.1. Whitehead's Two Tracts on Geometry, 1906–1907

Whitehead's interest in geometry was maintained by a new venture launched in 1903 by Cambridge University Press: a series of short "Tracts in Mathematics and Mathematical Physics," co-edited by his former student and now co-Fellow of Trinity College, E. T. Whittaker (1873–1950). He wrote two tracts for the series, on the axioms of projective and of descriptive geometry [Whitehead 1906c, 1907].

In the first one Whitehead recalled that in projective geometry "two coplanar lines necessarily intersect" [1906c, 4–5]. In his opening paragraphs he emphasized the importance of axiomatization and appraised geometry as "a department of what in a certain sense may be called the general science of classification" [p. 4]. He also characterized "rigid proofs of existence theorems" as "deductions from the premises of formal Logic" [p. 3]; the latter were not stated, but he gave a system of 19 axioms. It followed not Veblen but Peano's follower Mario Pieri (1860–1913), for whom the primitives were "point" and "line joining a point A with a point B" and beyond [pp. 7–8].[15] von Staudt was highly praised, and the quadrilateral construction and a version of the uniqueness theorem presented [p. 9]. He emphasized as the "fundamental theorem" the statement that "a projective correspondence between two lines is completely determined when the correspondents of three points on one line are determined on the other" [p. 17: compare Fig. 2 above], and throughout the book he pointed to the need in proofs of other theorems and their independence from certain collections of axioms. He also introduced coordinates and showed the invariance of cross-ratio [Chap. 7].

Another change concerned continuity, which was now defined on p. 30 by the method of cuts due to Richard Dedekind (1831–1916) rather than Cantor's approach, mentioned (though not seriously used) in the 1906 paper (Sect. 3.3). Presumably Whitehead made the change because Dedekind's assumption that a cut could always be made on a continuous line fitted better into Whitehead's concern with points than did Cantor's definition, which drew upon properties of classes themselves [for discussion see *PM*, *275–*276].

A similarity with the 1906 paper was an axiom restricting this geometry to three dimensions, needed (but necessary?) "when we come to the introduction of coordinates" in Chap. 7, [p. 15]. Although Whitehead cited several recent writings, [Veblen 1904] was omitted; however, on p. 18 he used Veblen's recent lithographed set of lecture notes [Veblen 1905]. The book included a detailed study of order [Chap. 3] and proofs of major theorems, such as Pappos's. Among reviews, the reviewing *Jahrbuch über die Fortschritte*

[15] Whitehead cited [Pieri 1898, 1905]; on Pieri's work see [Marchisotto 1995].

der Mathematik carried praise from the German mathematician Eugen Meyer (1871–1909), who also picked up a few slips on theorem and literature [Meyer 1909].

In the successor tract "Descriptive Geometry" was taken in a nonstandard way from [Russell 1903, Chap. 46], to refer to "any Geometry in which two straight lines do not necessarily intersect" [Whitehead 1907, 1], so that lines were open indefinitely at either end. This time [Veblen 1904] was used, for the emphasis upon order; but [Peano 1889] provided many of the axioms [Whitehead 1907, Chap. 1]. Continuity was again Dedekindian, although he remarked that "the dividing point" made by the cut "itself belongs to one of the two classes" into which the continuous line was cut [p. 9]—exactly the assumption which Dedekind did not make.

Much of the exegesis was concerned with topics which other authors would have construed as parts of projective geometry, including perspective, ideal points and transformations. The chapter on "Axioms of congruence" [Chap. 5] was rather unfortunately titled, because the bulk of it was taken up with the establishment of congruence by axioms alone, as in the recent paper; here he used other means due to Sophus Lie (1842–1899), whose theory of "infinitesimal rotations" was handled in the succeeding Chap. 6.

Whitehead concluded with material then more usually treated within projective geometry. After laying out coordinate systems in Chap. 7, he devoted the final Chap. 8 to "Metrical geometry," rehearsing the theory of transformations due to Cayley and Klein (Sect. 3.2) and deriving distance and angle in the usual way from cross-ratio; both Euclidean and non-Euclidean metrics were treated.

4.2. Whitehead's Lecture Course, 1907

As we shall see in Section 4.8, these tracts were prepared partly with *Principia Mathematica* in mind, although logic or logicism was not used or mentioned in them. But an opportunity for links arose in the Spring of 1907, while the second tract was being written; Whitehead gave a lecture course at Trinity College on "The Principles of Mathematics." As usual, no manuscripts survive; but extensive notes were taken by student H. W. Turnbull (1885–1961), who later followed a noted career as a mathematician and also a historian of mathematics.[16]

Although the final system for *Principia Mathematica* was still not settled, Whitehead explained the principal notions of mathematical logic, such as propositional functions, their "ranges of significance" divided into types (Sect. 4.3), and functional quantification; curiously, he seems not to have mentioned the paradoxes. He began with the construction of mathematics that was envisaged: from integers to rational and irrational numbers and up to transfinite arithmetic. On model-theoretic aspects of basic notions such as number, zero, and successor in the Peano axioms for arithmetic "might be fitted on to anything. B. Russell doesn't seem to object to this." He also summarized at some length both the two new tracts, and also parts of his 1906 paper, including classes (1) of interpoints and most of the notions in the final concept (Sect. 3.5) such as objective reals, homaloty, and punctual lines. Surely all this material was intended for *Principia Mathematica*, which at last was taking a definitive form.

[16] This manuscript is held in Cambridge University Library, Turnbull Papers, Box 3, file 3. Whitehead gave a similar course in the Lent term in 1910 (*Cambridge University Reporter* (1909–1910), 66), but no record of it seems to survive.

4.3. *The Forming of Principia Mathematica*

The name of their collaborative book seems to have been decided in 1906. Maybe it was intended to recall the work of their Trinity College predecessor Isaac Newton; another candidate is the *Principia Ethica* (1903) of their friend G. E. Moore.

For over a year around 1906 Russell thought that he had found the Solution to the paradoxes in a "substitutional theory," in which only individuals, propositions, and their truth-values were used: variables in the usual sense were banned, along with propositional functions, classes, and relations. But Russell seemed to find it hard to construct most of Cantor's transfinite numbers, to Whitehead's pleasure on 6 December 1905: "I sincerely hope that ω_1 ⟨the initial ordinal after the first class of transfinite ones⟩ is indefinable. The aleph-series of Cardinals has always bored me extremely, and I shall be glad to assist at its funeral." In later letters he suggested developments of this system (and objected on 21 February 1906 to "your excessive formalism"); but Russell was more sanguine about the loss of those transfinite numbers, and philosophically worried about the status of the referents of false propositions. Then by early 1907 at the latest he found that the theory admitted a paradox, and so he definitively abandoned it [Grattan-Guinness 2000, 360–364].

Back to variables and propositional functions and relations as primitive, then, with classes defined contextually from them; some of the devices of the substitutional theory were also used but then discarded. A complicated theory of orders and types of functions (to which Whitehead had alluded in his lecture course) was constructed to Solve the paradoxes. The objects of logic were divided into classes of individuals, classes of classes of individuals, classes of ordered pairs, triples, . . . of individuals, and so on into many complications; the structure was grounded epistemologically upon the corresponding classification of propositions and propositional functions, both in terms of orders of quantification. Membership to a class was restricted to the objects in the type immediately below: self-membership was forbidden, thus ruling out the paradoxes. Unfortunately some decent mathematics was also excluded; so for restitution they reluctantly assumed an "axiom of reducibility," which asserted that to any propositional function there existed a logically equivalent one free of quantifiers. Whitehead supported its use in a letter of 7 October 1906.

From around that time until the autumn of 1909 they worked out the story up to but excluding geometry, each reading the other's material. Whitehead sent several letters, mostly querying features of the theory of types and its constituents: propositions, propositional functions (of functions of . . .) and quantification, individuals, and possible alternatives to the unwelcome axiom.

Russell prepared the final manuscript for printing, but at several thousand sheets it was too expensive for Cambridge University Press to finance alone. So the authors successfully obtained a grant from the Royal Society, to which Russell had been elected in 1908 following proposal by Whitehead [Grattan-Guinness 1975b].

4.4. *Mathematics or Pure Mathematics in Logicism?*

The text of their application to the Royal Society[17] is very instructive about their aim, and also for differences between their respective conceptions of logicism; for three times

[17] The application is transcribed with extensive commentary in [Grattan-Guinness 1975a]; the text is also in [Grattan-Guinness 2000, 581–584].

Whitehead wrote in general terms about "mathematics," and three times Russell added "pure" to the sentences. The difference emerges also in a general article "Mathematics" which Whitehead wrote for the new 11th edition of the *Encyclopaedia Britannica*: after recounting features of axioms, set theory, Russell's paradox, arithmetic finite and transfinite, and the logic of relations, he offered as a "*Definition of Mathematics*" the "science concerned with the logical deduction of consequences from the general premises of all reasoning" [Whitehead 1911d, 210]—a version of logicism, but *not* Russell's definition of "pure" mathematics.

The role of this adjective relates to logicism itself. Russell's conception of it had been implicational in form: in the opening of *The Principles*, "Pure mathematics is the class of all propositions of the form 'p implies q,'" with a list of the credentials required of propositions p and q to be logical [1903, 3]. This was an eccentric use of "pure," quite different from the traditional claim of independence of the mathematics involved from experience, and Whitehead ignored it.

Principia Mathematica is much less clear. In the short preface to the work logicism was not even stated explicitly(!). In the opening material of *Principia Mathematica* only "mathematics" was mentioned, and the impression is given that logicism is *inferential*: given the axioms, then the theorems follow logically. But in the short introduction to the Section in which the propositional calculus was laid out we read that the ensuing "theory of deduction" would "set forth the first stage of the deduction of pure mathematics from its logical foundations," where deduction was "the principles by which conclusions are inferred from premises" and "depends upon the relation of *implication*" [*PM*, Vol. 1, 90]—that is, the implicational form again. Following their hero Peano, their logic was always severely incoherent on implication, inference, and entailment; all of them were called "implication."

4.5. The Organisation and Opening of Principia Mathematica

The material in the three volumes was divided into six Parts, each Part into Sections, and each Section into "Numbers" (of all names!) each with an identifying ordinal: in a system taken over from Peano, each theorem or definition then took a decimal number, such as "*41·341." Some features will be noted in the rest of this section; a much more extensive account is given in [Grattan-Guinness 2000, 380–410].

The first volume appeared in 1910. Russell was largely responsible for its contents, which comprised three introductory chapters and then Part I giving the axioms for the propositional and predicate calculi (including relations), set theory, and other techniques such as his theory of definite descriptions. According to [Russell 1948], Whitehead wrote much of the first chapter, where these basic tools were briefly explained; but most of them were Russell's inventions [Whitehead 1920a, 10]. Neither there nor in the detailed exegesis given in the Part did Whitehead lay much mark on the logic: for example, although an algebraist by inclination, he did not have duality emphasized in the presentation of the propositional calculus [Leblanc 1963], nor were model-theoretic aspects aired. However, a touch of structuralism was evident in *38, where he introduced a two-place connectival schematic letter "$\overset{\circ}{+}$," which took "values" such as "≈" (union of classes, and also of relations), "↔" (the same for intersection), and "|" (the compounding of two relations). Apparently he also wrote much of Section IB on "apparent variables" (Peano's name for quantified ones), which also contained identity and the theory of definite descriptions, and also a sketch of the theory of types and the axiom of reducibility; but again most of the basic notions were due to Russell.

Part II in the volume was a "Prolegomena to Cardinal Arithmetic." It started rather strangely with the logicist definitions of the first cardinals and ordinals as integers as classes of similar classes and of well-ordered series, respectively, and then went through a lot of set theory and of the logic of relations for later use in the systematic theory.

4.6. How Many Individuals in Logicism?

The second volume came out two years after the first one. Of its two main Parts, the second concerned Cantor's theory of finite and transfinite ordinals; but Cantor's use of the "well-order-type" was replaced by logicist definitions of ordinals. This Part IV contained a beautiful generalization into "relation-arithmetic," where order-similarity between series was extended to *any* order-type. Russell had envisaged the theory in 1902 and was largely responsible for this Part; in the interim a similar theory had been formulated independently by the German mathematician Felix Hausdorff (1868–1942) in two articles [Hausdorff 1906, 1907] which they judged to be "brilliant" [*PM*, Vol. 2, 391; see also Vol. 3, 171].

This volume had been delayed in publication because Whitehead had made a mistake in its opening Part III, which treated Cantor's theory of cardinal arithmetic, finite and transfinite. It had been caused by the need to repeat every mathematical theory within every available type, together with the empirical realism which they had adopted from G. E. Moore (Sect. 2.2).

As part of his desire to assume as little as possible, over the years Russell had tried various means to construct an infinitude of objects, usually by using mathematical induction. However, begging the question was in the air, as Whitehead wondered in a letter of 23 May 1905, though on the following 9 May he hoped that an axiom would not be needed. But his optimism was forlorn; not long afterwards Russell had to assume an "axiom of infinity"—and type theory insisted that an infinitude *of individuals* serve as the bottom type.

But what sort of logicistic object was an individual? It was hardly a piece of logic, surely, like Whitehead's points, particles and instants of 1906 (Sect. 3.3). Whitehead expressed his doubts to Russell on 6 January 1908: "I feel so hopelessly at sea with the 'simple' entity, that I don't want our work to be irrevocably tied up to it," in approval of a seemingly lost manuscript from Russell where a "non-committed" view of individuals was taken. But again his fears were well founded. Individuals had to be *physical* objects [Russell 1911, 23]; thus the axiom of infinity was an empirical assertion and mathematical logic became *a posteriori*.

To minimize the effect of this axiom upon logicism, Russell stipulated that whenever an infinitude was not needed, only one individual should be assumed; but then it followed that only 0 and 1 could be defined in the type of classes of classes of individuals, 0, 1, 2, and 3 in the next type, 0, 1, ..., 7 in the one after that, and so on. But Whitehead had forgotten to obey this restriction, and used the axiom freely; and Russell had not noticed when checking his text. It only came to light in January 1911, when Whitehead was reading proofs of the Part. He suspended printing and worked for about five months to effect a remedy, reporting progress to Russell in letters.[18] Parts of several Numbers were rewritten (certainly ^{*100}, ^{*117}–^{*120}, and ^{*126}); but the main outcome was a new preface to the volume entitled

[18] This sequence includes a letter dated 'Jan 27th' 1911, but misassigned to 1908 in the Whitehead file in the Russell Archives.

"Prefatory Statement of Symbolic Conventions." In this long and very difficult essay he outlined a variety of conventions intended to prevent mathematics from being effected in types that were too low down the hierarchy to admit the numbers required; for example, $2 + 2 = 4$ only from the type of classes of classes of individuals upwards.

Another arithmetical feature arose from these corrections: additional printing costs. Russell seems to have borne them alone.

4.7. Logicising Measurement

The third volume appeared, seemingly without further mishap, in 1913.[19] It was much smaller than its predecessors, for it was only the first half of the volume as conceived; the rest was to treat geometry, which Whitehead would prepare on his own as Volume 4.

The first portion of this volume finished off from the previous one Part V on series and continuity. Authorship here could have been joint, as both authors were well familiar with the material; [Russell 1948] explicitly mentioned Whitehead's responsibility for Section VC.

The treatment was based upon "serial" relations, which were transitive and connected: that is, if x, y, and z belonged to the field of relation R, then either xRy or yRx, and if xRy and yRz, then xRz [*202–*204]. They were used in Sections VB and VC in a detailed treatment of upper and lower limits of sequences, continuity in both Cantor's and Dedekind's senses, the continuity of mathematical functions, and several parts of point set topology (derived classes, dense classes, and so on). Then they constructed Cantor's theory of transfinite cardinal and ordinal numbers, the latter as a special case of relation-arithmetic, making much use of ancestral relations [*90.01] which they had learnt from Frege; the successive application of this kind of relation located all the members of its field (for example, "successor of" the initial integer in Peano's axioms for arithmetic). However, the finite number of types and orders restricted the construction to those numbers starting from *finitely* indexed initial ordinals, and the corresponding alephs; thus the paradoxes of the greatest ordinal and cardinal numbers were solved in an unwelcome manner, in that they could not get anywhere near them anyway [*PM*, Vol. 3, 170–173].

The rest of this volume was Part VI on "Quantity." It was due mainly to Whitehead, and letters to Russell suggest that much of it was prepared during the later summer and autumn of 1909 as the full manuscript was being readied for the Press. It fell into two "subdivisions": ratios, including rational numbers; and the "Quantitative Relations (or quantity proper)," where the real line was laid out and used. One neglected feature is the theory of positive *and negative* integers, defined as relations rather than classes of similar classes and dependent in effect upon the notions of counting forward and counting backward, with the identity relation supplying 0 [*300]. Ratios were defined by a means due to Russell, in terms of powers of serial relations; and then real numbers and their arithmetic were laid out in Section VIA.

Most of the rest of the Part was concerned with "vectors," which were conceived in a very general way as uniquely specified sensed magnitudes "i.e., as an operation i.e., as a descriptive function" in the Russell sense (Sect. 2.3) [*PM*, Vol. 3, 339]; thus $+1$ and -1 could be construed as vectors. Among the kinds of "vector-families" [Section VIB], Whitehead defined "cyclic" ones [Section VID], where members might have several multiples; an

[19] Then the authors destroyed the manuscript; only one folio survives, in a letter sent by Russell to his mistress Ottoline Morrell.

important example was angles [*370], where, say, $30^\circ = 390^\circ = 750^\circ$. Other applications included a Section VIC on "Measurement," where a rectangular system of co-ordinates was specified, surely offered as preparation for the metrical geometry to come [*353–*355]. The algebraist came out again, when he noted an example of a group [*354 · 14–·17]; however, in the preface of the volume, presumably written by Russell, this finding, "which has generally been made prominent in such investigations, sinks with us into a very subordinate position, being sometimes not verified at all, and at other times a consequence of other more fruitful hypotheses" [Vol. 3, v].

In several places in both of these last Parts the authors emphasized the need for the axiom of infinity.[20] Maybe Whitehead added some of these remarks on proof, to make sure that no more panics would arise!

This Part gives a somewhat scrappy impression, topics loosely rather than logically put together: even at one point, "we have given proofs rather shortly in this Section," since many were "perfectly straightforward, but tedious if written out at length" [*PM*, Vol. 3, 461], which sounds like logicism betrayed. For some unstated reason the differential and integral calculus was not treated at all, even though all the relevant machinery seems to be in place or could easily be supplied. Instead the Part seems largely intended to prepare for the next Part on geometry to come. What did Whitehead intend to include there, and how was he going to handle it?

4.8. Preparing for Geometry

The application letter to the Royal Society provides information, for the text was followed by a complete summary of *Principia Mathematica* by Sections. Geometry was included, as Part VII, divided into the four Sections "Projective Geometry," "Descriptive Geometry," "Metrical Geometry," and "Constructions of Space" [Grattan-Guinness 2000, 584]. Presumably Whitehead would have broadly followed the content and maybe some of the order of material of the two tracts and then the 1906 paper, all to be expressed in the formal manner of logicism, with many-place relations much to the fore [Harrell 1988]. Presumably some of the difficulties surrounding the paper would arise again, for example, the specification of dimension despite Cantor's isomorphism of line and plane (Sect. 3.6, (2)).

In an article on "Axioms of Geometry," published at that time in the *Encyclopaedia Britannica* though maybe written around 1906,[21] Whitehead had also followed his tracts, almost quoting from them in places [Whitehead 1911c]. He reported some recent groundwork to Russell in a letter of 10 December 1908: "I cannot move a step in Metrical Geometry, until I have clearly settled in my mind the fundamental nature of Geometrical entities," with a reference to some enclosed sheets which unfortunately are now lost.

By September 1910 Whitehead had several of the axioms for projective geometry numbered and named, starting from *500 (presumably after a prolegomena to the Part, starting at *400): on the "Associated Symmetrical and Permutative Triadic Functions," with others on "Associated Relation," "Permutation and Diversity" and "Connection"; arguably they

[20] See *PM*, Vol. 3, especially p. 234; see also, for example, pp. 143, 169, 218 and *263·122. However, *216 was largely concerned with proving theorems about real numbers that did not need the axiom of infinity.

[21] Whitehead mentioned the commission of an article in his letter of 6 December 1905, which was mentioned for other purposes in Sect. 4.3.

are versions of axioms III, V, VI and VIII–XI of [1906c, 7].[22] In a book review of 1911 on "the philosophy of mathematics," to illustrate the role of axioms in general he stated axioms VIII, XI, and XII [Whitehead 1911b, 237–238].

A few further hints are provided in the *Encyclopaedia Britannica*, in a revision for which Whitehead was largely responsible of an article on "Non-Euclidean Geometry" which Russell had prepared for the previous edition. He reduced much of the philosophy and added in material on metrics, which presumably would have been handled in the volume [Whitehead and Russell 1911].

An important later source is a paper on "The Relationist Theory of Space," delivered at a congress on the philosophy of mathematics held in Paris in 1914 and published in a French translation as [Whitehead 1916b]: the original text is lost. Here he took relations as fundamental, and saw space as only an "expression" of properties of interaction of bodies in physical space [p. 430]. He then used set theory to set up a collection of properties and objects [pp. 431–434], such as the unions of the domains of a class σ of relations, especially the property of seriality. Imitating the procedure in projective geometry, he then specified "Material σ-points" and "σ-segments" as the physical objects in space toward which the regions of space might converge [pp. 448–451]. He cited the 1906 paper but noted that the part–whole relation had not been used there when defining geometrical entities [p. 441]. Once again all lines were straight, physical space had three dimensions, and space–time as such was not considered [p. 452].

From these ancillary sources we can gain some idea of the fourth volume; quite a range of geometrical concerns, but also with limitations in the coverage of mathematics consequent upon those of the previous volumes. In particular, since the differential and integral calculus had been omitted, the differential geometry of curves and surfaces could not have been developed. No concluding Number was planned for the work: it would "just stop with whatever formula came last" [Russell in 1912, in *Papers*, Vol. 6, xxix].

5. MOVING INTO PHILOSOPHY FROM THE 1910s

5.1. London, and Mathematics Education

While *Principia Mathematica* was going through the press, Whitehead resigned his fellowship at Trinity College in 1910, his 50th year, over the treatment of Forsyth after a love affair. He moved to London without a post, but then obtained a Readership in Geometry at University College London in 1912, followed by a chair in applied mathematics at Imperial College two years later, soon after the appointment of Forsyth there.[23]

For a time in 1910 Whitehead was without a post, and he wrote a short "introduction" to mathematics. His topics included arithmetic, elementary geometry and calculus, and complex numbers; logicism and its concerns were too elevated for his purpose, but in an early chapter he discussed "variables," ending with the claim that "the idea of *variable* is fundamental, both in the applications as in the theory of mathematics" [Whitehead 1911a,

[22] For discussion of these axioms and the background, see [Harrell 1988, 146–149]. The axioms are mentioned in [Lowe 1990, 93]; but the later letter quoted there should be dated 1 October 1910, and its "illegible" mathematical symbol is clearly "ν," denoting a finite ordinal.

[23] On this period, see [Lowe 1990, Chaps. 1–3]. Somewhat to my surprise, I was told by the archives at Imperial College that they have no significant material pertaining to Whitehead's activities there.

Chap. 2]. This claim extended to logicism; he ended the book review on the philosophy of mathematics [1911b] in that vein, presumably also with quantification in mind. In a later work he explained the source of the importance of the variable: that "by the employment of this notion that general conditions are investigated without any specification of particular entities" [Whitehead 1925, 33].

Whitehead's interest in mathematics education and heuristics increased during the 1910s; he saw them as a natural extension of the philosophy of mathematics beyond logicism. He became President of the Mathematical Association (the British organization for school-teachers of mathematics) and produced some reflective essays on the needs of "boys." For example, in his Presidential Address to the Mathematical Association he emphasized the merit of teaching congruence, in a metrical manner [Whitehead 1913, 129–130].

Sometimes Whitehead invoked his experience with foundations, such as this contrast in the book review [1911b, 238–239]:

> Elementary mathematics and the elements of mathematics are widely different subjects, though a confusion still exists and has survived from bygone times when the psychology of education was not understood so well as it is at present. It seems natural that the learner should begin at the beginning of the subject. The truth is entirely the opposite. The learner's natural beginning is the study of the complex facts of some application of the subject. The simple abstract ideas are very difficult to grasp, because in practical life we never consider them directly; for beginners they require some particular embodiment.

In a lecture to the British Association for the Advancement of Science on "the organisation of thought" Whitehead appraised logic in a manner not stated in *Principia Mathematica* [1917a, 165–168]. He divided it into four "departments," starting with the "arithmetic" when dealt with connectives between propositions. Then came the "algebraic" department, where "letters are sometimes variables and sometimes parameters"; this is an important distinction often evident between mathematical analysis/set theory and an algebra. He included propositional functions and relations in the former category, and again emphasized the importance of the variable. The next department was "general-function theory," where mathematical functions were introduced via Russell's theory of definite descriptions; finally came the "analytic," where he summarized the mathematical contents of *Principia Mathematica.*

5.2. *The End of a Collaboration*

Heuristic aspects of mathematics did not enter into Russell's philosophy. His collaboration with Whitehead finished, though Whitehead reported progress on the fourth volume in letters. Even contacts diminished, although Russell was often living in Cambridge after receiving a lectureship at Trinity College in 1910 in effect to replace Whitehead on foundational topics in mathematics [Russell 1967, 198]. The Great War strained their relationship further, due to their opposing attitudes to its legitimacy; when the College dismissed Russell from his lectureship Whitehead wrote a very equivocal leaflet [1916a] as part of the controversy that ensued among the fellows [Delany 1986]. Perhaps for this reason, Whitehead terminated their professional relationship in 1917 when he hinted in a letter his view that Russell had plagiarized some of his recent work in his own philosophy, despite Russell's ample acknowledgement in the text in question [Russell 1968, 78].

Another difference between the two men lay in their philosophies. Both adopted empiricism, and used *Principia Mathematica* both as a model for reductionism and as a source of techniques, especially the logic of relations; but while Russell tried to avoid metaphysical issues and assumptions if possible [Russell 1959, Chaps. 9–11], Whitehead conspicuously became absorbed in them. The gaps started to emerge around 1912, when Whitehead sent several undated pages of rather skeptical comments in response to Russell's recently published *The Problems of Philosophy* of that year (in the same book series as his own recent short introduction to mathematics), which restricted problems to those of its author's liking [Russell 1912]. In addition, for several years from 1911 Russell was quite strongly influenced by the work of his student Ludwig Wittgenstein (1889–1951), to whom Whitehead was not sympathetic either philosophically or personally [Lowe 1990, 278]. By contrast, to some extent Whitehead was in tune with the philosophy of Henri Bergson (1859–1941) and with evolutionary epistemology, which Russell despised; and from the mid-1910s he soon focused upon general relativity theory, which Einstein had recently published. Whitehead's main aim was to find a new philosophical basis for our sense experiences to replace the classical mind–body dualism.

It is remarkable that Whitehead and Russell had collaborated at all, for their differences in both personality and philosophical inclination are very marked. Whitehead put it beautifully to Russell once: "You think the world is what it looks like in fine weather at noon-day; I think it is what it seems like in the early morning when one first wakes from deep sleep" [Russell 1956, 41].

5.3. Space and Time, with Relativity Theory

During the 1910s and into the Great War Whitehead continued with the fourth volume of *Principia Mathematica*; much work was done in 1913 and 1914, probably stimulated by a lecture course on "mathematical logic with applications to geometry" given that winter at Imperial College, and also the forthcoming Paris lecture on space. Apparently one decisive factor in its abandonment was the death in action of his son Eric in January 1918.

But geometry remained prominent in Whitehead's growing concern with philosophical questions. After some incomplete forays in [1917a] and other papers of the mid 1910s [Kultgen 1966], the main outcome was three somewhat overlapping books: a general philosophical "enquiry" [Whitehead 1919a], rehearsed in a rather less technical manner in a study [1920a] of "the concept of nature," and then a detailed account [1922] of his own version of relativity theory.[24] A few points are made here, with selected references; they deal mainly with parentage of notions in the 1906 paper and *Principia Mathematica*, and some significant differences.[25] One of the latter was the use of the differential and integral calculus, especially in the excursions into relativity theory.

We recall that time and space were treated separately in Whitehead's 1906 paper (Sect. 3.6, 6), and that only space was intended for the last Part of *Principia Mathematica.* However, while preparing it Whitehead had reported to Russell on 3 September 1911: "But last night ⟨...⟩ the idea suddenly flashed on me that <u>time</u> can be treated in exactly the same way that I

[24] Whitehead's short account [1920b] of relativity theory, quoted at the head of Sect. 3.1, was written for the unusual venue of *The Times Educational Supplement.* It followed discussions by H. Wildon Carr (p. 47) and F. A. Lindemann (p. 59), the trio written in response to the "verification" of Einstein's theory by the observations of A. S. Eddington in 1919.

[25] Some inconsistencies are not pursued here. For more detailed analyses see especially [Palter 1960], and also [Mays 1977, Chaps. 5–7; Lowe 1990, Chaps. 5–6]. Some rather generalized connections between Whitehead's mathematics/logic and his cosmology are made in [Harrah 1959].

have now got space [which is a picture of beauty by the bye]." However, it is not likely that the Part would have been modified; for (instants of) time would have to be logicist or empirical, and so the difficulties of the second volume, experienced only a few months earlier, might have recurred. Further, in the later philosophical writings time was a major concern, but he admitted similarity of it with space only in that both categories exemplified extension; otherwise he emphasized differences, especially concerning the ways of passage [1922, 67–68].

As [Taylor 1921] suggested in a review, Whitehead's philosophy of science resembles a kind of *Naturphilosophie*. The prime notion was the event, a "spatio-temporal happening" [1922, 21]. With this change of primitive a major difference concerned elements of both space and time. In 1906 he had taken the temporal elements to be instants: now, possibly under the influence of Bergson, "A moment is a limit to which we approach as we confine attention to durations of minimum extension" [1920a, 57]—approach *but not reach*, since "a moment of time is nothing else than an instantaneous spread of nature" [1922, 54], and the minimum "value" is indeterminate. Similarly, the definition of velocity depended upon a neighborhood of instants [Whitehead 1919b, 45] and thereby upon (the existence of) limits of sequences. The notion of points was generalized to "event-particles," specified as "ideal minimum limits of events" [1920a, 86].[26]

5.4. The Mereology of "Abstractive Sets"

Whitehead explained the means of understanding events in most detail in connection with time. He considered sequences of its duration, nesting downwards but not to a zero duration. Such a sequence was called an "abstractive class" (or "set"); the nesting was expressed by the transitive relation of whole to part, itself conceived of within set theory rather than in the traditional part–whole theory: "an abstractive set p covers an abstractive set q when every member of p contains as its parts some members of q" [1920a, 83]. The procedure and even the name strongly recall Cantor's theory of "covering" ("Belegung") a class α by constructing the class of all its subclasses and proving that it had more members than α itself [Cantor 1892]. This method was well known; for example, Russell had found his paradox from a variant of it (Sect. 2.2). The relationship between equality and identity arose again: two such sets were equal if "the intrinsic characters of the two sets are identical" [1920a, 84].

An important consequent property was that an abstractive set was "σ-prime" relative to any "condition" σ (such as being an event, or a duration), when "it has the two properties (i) that it satisfies the condition σ and (ii) that it is covered by every abstractive set which both is covered by it and satisfied the condition σ" [1920a, 87–88; compare 1919a, 106]. Whitehead had used the name in the 1906 paper for classes under a "*property* ϕ" (Sect. 3.5), and the defining conditions echo the geometrically projective definition of points by the concurrence of lines (Sect. 3.4); but now that limiting situation was ideal, and the part–whole relation led finally to an indeterminate nonempty minimum.

Upon this basis Whitehead constructed a "method of extensive abstraction," in which instants or points or particles were taken to be the common part of abstractive sets [1919a, Pt. 3;

[26] Since ancient times there had been disputes over interpreting matter and physics in terms of points or corpuscles and related questions such as the (un)interpretability of lines/planes/solids as ensembles of points/lines/planes; the issues were often aired in connection with a variety of atomistic philosophies [North 2000]. Whitehead knew some of this story, though [Smith 1953] queries his familiarity with Aristotle.

1920a, Chap. 4]. He used continuity in Dedekind's sense [1920a, 102]; equality, including its preservation under Lie and other transformations [1922, Chap. 2]; and congruence of regions, which he expected to obtain in space and time.[27] This last requirement has its parentage in flatness in the 1906 paper (Sect. 3.5) and perhaps shows an algebraist extolling invariants; however, requiring space to be Euclidean must have surprised other students of relativity theory.[28] It was a major component of his new philosophy of science[29]—and also the subject of alleged plagiarism by Russell (Sect. 5.2).

Whitehead was now also deeply concerned with human perception of nature; indeed, he had replaced the objective reals of 1906 by the ambiguous "instantaneity" of human sense-experience. The elaboration of this concern forms one of the most important but also obscure parts of his philosophy [Broad 1920, esp. p. 228]. In this connection he used another word from the 1906 paper, though with a different sense (Sect. 3.5): the relation of "cogredience" between a perceived finite event and any part of its duration [1919b, 50; 1920a, 188–189].[30]

5.5. *Harvard and Professional Philosophy*

Two events took place in 1923. First, *Principia Mathematica* was out of print, and so Cambridge University Press had proposed a new edition to Russell. Whitehead initially responded positively to Russell on 24 May, promising to send notes on various aspects of the work for an "appendix": he had in mind clarifying the various meanings of "function," and possible modifications to the theory of types. However, in the end he took no part in the revisions; indeed, when the first volume of the second edition appeared in 1925, he published a rather frosty letter [Whitehead 1926] in *Mind* emphasising his noninvolvement.[31] He may well have taken exception to the changes, which gave the system a much more extensional character than before (compare Sect. 5.7).

The second circumstance involved Whitehead's situation in the University of London. From 1918 he had been the Dean of the Faculty of Science, and from 1921 also a Faculty representative on the University Senate;[32] now in 1923 Forsyth retired as head of the Department of Mathematics at Imperial College, and Whitehead succeeded. So he now had still more administration to handle, for which he was not really suitable. Nevertheless, he seems to have been a candidate for Vice-Chancellor of the University [Lowe 1990, 84–85];

[27] See especially [Whitehead 1919a, Chap. 4; 1920a, Chap. 6]; his presentation was formalized in [Leśniewski 1928, 258–263], accompanied by complaints over his careless characterization of events. Another example is his use of the principle of least action, which steals in, none too clearly, in his [1922, Chap. 4, at Eqs. (10) and (11)]. He may have been influenced by his senior Cambridge colleague J. J. Larmor (1857–1942), who gave it very high status in his mathematical physics [see especially Larmor 1900].

[28] The historical context for relativity theory is described in detail in [Hentschel 1990] with Whitehead on pp. 481–488. [Russell 1923] briefly expressed his own reservation.

[29] See especially [Wind 1932; Palter 1960, Chap. 5].

[30] Whitehead's new use of "cogredience" does not link to its current use in the transformation of variables in tensor calculus in, for example, [Weyl 1918, Chap. 1]. On that work see [Hentschel 1990, 269–275].

[31] On Whitehead's (non-)role in the second edition see [Lowe 1990, app. 2]; Russell's revisions and their context are described in [Grattan-Guinness 2000, esp. pp. 434–440].

[32] On this part of Whitehead's life, see [Lowe 1990, Chap. 4], supplemented by files held in the University of London Archives, University Library: "Bound Minute Book. Faculty of Science," 1900–1928 (file AC 5/3/1/10); and "Attendance and Minute Book. Board of Studies in Mathematics," 1900–1914 and 1915–1927 (files AC 8/36/1/1–2), where Whitehead was not very active.

but everyone was saved from this fate by Harvard University, which invited him to take a professorship in the Department of Philosophy there, despite his age of 63 years. He accepted with alacrity[33] and passed the rest of his career at that institution.

Although Whitehead did not continue to work on logic, he gave some supervision to a graduate student at Radcliffe College, Susanne K. Langer (1895–1985). Her labors led to an interesting paper in *Mind*, in which she acutely spotted, with Whitehead's evident approval, the confusion of symbols with their referents in *Principia Mathematica*, especially concerning the theory of types [Langer 1926]. While she did not fully grasp the distinction of theory from metatheory, she made a notable contribution to American sensitivity on this major issue, where *Principia Mathematica* had been so wanting.[34]

5.6. The Importance of Process

At Harvard University Whitehead's philosophy entered a new phase when the method of extensive abstraction became a "philosophy of organism," a general metaphysics about process through durations, spreading from the past through the present into the future, with "prehensions" of processes following suit. The main outcome was his most important philosophical book, delivered as the Gifford Lectures at the University of Edinburgh in 1927 and 1928 and published, with some additional material, as *Process and Reality. An Essay in Cosmology* [Whitehead 1929a].[35]

The book and related works were much studied and led to a tradition of "process philosophy," especially at first in the USA.[36] While time and order remain prominent in it, rather little mathematics as such occurred; in particular, the scheme of "Categoreal Obligations" central to the book [pp. 35–38] shows only one influence, though interesting; the second obligation stipulated that "There can be no duplication of any element in the projective datum of the 'satisfaction' of an entity, so far as concerns the function of that element in the 'satisfaction.'" In other words, Whitehead retained Cantor's assumption that members belong to a class only once, and again ignored Kempe's theory of multiple membership (Sect. 3.6).

Elsewhere in Whitehead's process philosophy other themes from earlier work were rehearsed, especially within his "theory of extension," where he elaborated his cosmological principles. He gave a version of the method of extensive abstraction as improved by a suggestion from the American philosopher Theodore de Laguna in which separation was replaced by (non-)connection of regions, including contact of boundaries [de Laguna 1922]. Then homaloty of 1906 (Sect. 3.5) appeared as an account of "flat loci" [1929a, Pt. 4, Chaps. 2–3].

[33] Whitehead's departure from the University was noted in [University of London 1924, 25 June 1924, p. 22]. Sydney Chapman (1888–1970) was Whitehead's successor at Imperial College. He told me once that he went to see Whitehead in his room and found him sitting in an armchair and drafting a memorandum on the improvement of administrative efficiency, next to a table covered with unopened envelopes . . . .

[34] On the role of the Americans see [Grattan-Guinness 2000, esp. pp. 371–375, 421–430, 523–535]. Whitehead seems to have good relations with his Harvard colleague logician H. M. Sheffer (1882–1964), on whom see [Scanlan 2000].

[35] I cite here the original printing, which was poorly proofread: a list of main corrections was published in [Kline 1963, 200–207] and served as a basis for a corrected edition of the book, which appeared in 1978.

[36] For an introduction, see, for example, [Lowe 1990, Chaps. 11–12]. This philosophy did not excited much British interest, although Susan Stebbing (1885–1943), largely drawing upon [Whitehead 1920a, Chap. 3], included a treatment of extensive abstraction in her textbook on logic [Stebbing 1930, 446–455].

Specific influence from his mathematical background occurs on occasion; for example, a consideration of "any class of eternal objects" led him to consider, in paradox-like fashion, additional ones "which presuppose that class but do not belong to it," and the following discussion included the words "multiplicity" and "type" from the past [p. 63]. In addition, one section of the book was to lead him to a surprising coda to his logicist career.

5.7. Indication as a New Primitive, 1934

As part of his consideration of propositions in *Process and Reality*, Whitehead mused upon the manner of "indicating" them by registering their contents [1929a, 274–279]; and this and maybe other thoughts of the period led him five years later to sketch out a revision of logicism using the new primitive of "indication." He wrote a difficult paper which appeared after poor proofreading in *Mind* as [Whitehead 1934].

"Indication is the unique determination of a thing by the specification of some of its relationships to the functionings of some human body, and of some human intellect," Whitehead explained in his opening sentence. An event in the sense of his process philosophy, it seems to have been intended to replace as a primitive category the worryingly empirical individuals of *Principia Mathematica* (Sect. 4.6); indeed, he remarked that the definition of cardinals as classes of similar classes depended upon "the shifting accidents of factual existence," where "a new litter of pigs alters the meaning of every number, and of every extension of number, employed in mathematics" [p. 288]. At the end of this paper [p. 297] he asserted that

> *Principia Mathematica* does not solve the problem of basing upon constructions which are purely logical, abstracted from the metaphysical notion of types, and from the particularities of history. This memoir aims at supplying a logical doctrine of classes, defined with disengagement from all considerations other than those which are purely logical.

His "all considerations" implies that he regarded the notion of indication as logical.

"Ec!x is a proposition about x," Whitehead symbolized, "Ec" abbreviating "Ecce" ("behold"). Presumably the variable x took some particular value: it was "the object, not the symbol—which ascribes to x no intension other than the intension derived from purely logical notions" [pp. 282–283]. The rest of the theory closely followed *Principia Mathematica* in a sequence of similar definitions and theorems: both propositions and predicate calculi with attendant quantifications, and definitions of cardinal and ordinal numbers and operations upon them, even up to the start of Cantor's transfinites. Near the end he objected to Russell's decision in the second edition of *Principia Mathematica* to quantify all variables [p. 296].

However, Whitehead showed no awareness of the recent paper by Kurt Gödel (1906–1978) on the incompletability of any such exercise [Gödel 1931]. He was in common with his recent doctoral student in logic, W. V. O. Quine (1909–2000), whom he mentioned warmly in the paper. The law of commutativity, however, did not hold; as a student duty [Quine 1934a] reported his Master's studies in a lecture to the American Mathematical Society. He had been ploughing his own furrow in the reconstruction of *Principia Mathematica*, with new sets of primitives, first in a doctoral thesis [Quine 1932], and then in a revised formulation in his first book [Quine 1934b] (but also with no mention of Gödel), for which Whitehead wrote a warm preface. From then on Quine put forward further logicomathematical systems which have become much better known and indeed part of the folklore of modern mathematical logic [Ferreirós 1997]. Whitehead's farewell to the subject and the public debut of Quine in 1934 are a fine example of a change of generations in a discipline.

5.8. Mathematics as Symbolism and as Patterns

Early on in his paper Whitehead claimed that "A symbol, for a person who understands it, is a factor to establish a situation of indication" [1934, 282]. He had stressed symbolism since the mid-1920s, conceiving of it very generally: "The human mind is functioning symbolically when some components of its experience elicit consciousness, emotions, and usages, respecting other components of its experience" [Whitehead 1927, 9]. He acknowledged the importance of the philosophy of John Locke (1632–1704), whom he saw as a father of the philosophy of organism [p. v]. Locke had emphasized the importance of signs, of which words were only an (important) example; and Whitehead will have known that he had used the words "semiotike" and "logic" as synonyms [Locke 1690, Bk. 4, Chap. 21]. Likewise, Whitehead saw language as only one major kind of symbolism, with mathematics and logic also involved with other kinds; and his process philosophy included a concern with "symbolic reference" [1929a, Pt. 2, Chap. 8]. He claimed that "all fundamental physical quantities are *vector* and not *scalar*" [p. 249]: the algebraist had surfaced once again, though excessively so in forgetting about mass.

Especially from *Science and the Modern World* (1925) Whitehead began to recognise the importance of the related notion of pattern. He linked it to his earlier sort-of *Naturphilosophie* thus: "An event is the grasping into unity of a pattern of aspects" [1925, 149]. One of his first examples involved the differences between space and time (Sect. 5.3): namely, that patterns were present in the former but endured in the latter [p. 150; compare 1933, 192–193]. At this stage process was a contiguous notion [1925, 89–92], but soon it became the dominant one in *Process and Reality.* Pattern was now subsidiary, although it was emphasized in some nonmathematical contexts when it was abstracted from physical objects of various kinds [Whitehead 1929a, esp. pp. 160–162, 386–389].

Later, as was quoted at the head of this article, Whitehead allied pattern with a *general* link between mathematics and philosophy [1938, 238]. Around that time he explored this link in more detail in a brief review of the history of mathematics prepared for the Schilpp volume on his own philosophy (Sect. 1). He pointed to the importance of patterns in mathematics education and as a feature of geometry and especially algebra [Whitehead 1941, Arts. 3–4]. Again the algebraist came to the fore: "the fundamental notion at the base of Algebra" was "*Any* example of a given sort, in abstraction from some particular exemplification of the example or of the sort," and its history was "the story of the growth of a technique for the representation of finite patterns" [Arts. 6, 9]; common algebra at first but "an immense extension" during the 19th century, with mathematics as "the most powerful technique for the understanding of pattern, and for the analysis of the relationships of patterns" [Art. 10]. Modern logic had played a role in this advance by bringing in "any," that is, quantification over variables [Art. 11]. Once again Whitehead wove together mathematics, logic, education, and philosophy.

6. CONCLUSIONS: LINES OF INFLUENCE

Whitehead's late thoughts suitably prompt this final summary. The sextet of points below does not attempt to fulfil the cliché of locating a "unity of thought" in Whitehead; but there are some common themes running through his work from the algebras of 1898 to the process philosophy and symbolism 30 years later.

(1) At first, influenced by Grassmann and also by Boole, Whitehead handled collections in the traditional part–whole way. But after 1900 he became captured by Cantorian set theory and subsumed part–wholeness under (im)proper inclusion. He came to know very well all aspects of Cantor's set theory and its development by others; they turn up fairly regularly in his later philosophy. He followed Cantor to the extent of ignoring entirely Kempe's theory of multisets.

(2) Whitehead was concerned with equality in *Universal Algebra*, though he felt comfortable about identity in the 1906 paper. But the relationship between the two relations became quite significant in *Principia Mathematica*, where Russell adopted the Leibnizian definition of identity in terms of coextensive predicates [*PM*, *13]. It/they arose in Whitehead's later philosophy, especially concerning congruence but also with abstractive sets.

(3) Like Russell, Whitehead appreciated the importance of order-types in Cantor; and his interest increased still further in 1905 when he found that Veblen had taken it as a primitive notion in geometry. The attraction was its reducibility to relations of various kinds, a feature of logicism which was to be a staple of *Principia Mathematica.* It gave him a powerful means of expressing desired properties of space (an unfulfilled promise for the fourth volume of *Principia Mathematica*) and time.

(4) An associated common factor was vectoriality. Handled technically in *Universal Algebra*, evident in notions of the 1906 paper such as the "segmental prolongation" of one side of a given segment, and introduced near the end of *Principia Mathematica* for use in the fourth volume, it was present on various occasion in his later philosophical works. One important manifestation was the direction of time, itself treated as a serial relation.

(5) The duality used in projective geometry had led to the conceptions of lines as the intersection of collinear planes and of points as the intersection of concurrent lines. Giving it prominence in the 1906 paper, Whitehead generalized the latter procedures in his later philosophy into the method of extensive abstraction and the construal of events as the abstracted nonempty *minimum sensibile* of "spatio-temporal happenings." The theory of limits, long the standard means of handling mathematical analysis and the calculus and a major heritage for set theory and thus for logicism, must also have played a role in this insight, although the limiting case was an "ideal" (Sect. 5.4).

(6) Throughout these themes a common mathematical factor exists, namely algebras: their laws, their symbolism, their manifestations, their properties. Whitehead passed from detailed technical experience in the varieties of algebras through a grandiose symbolic logical program to an ambitious metaphysics concerning both human knowledge and understanding of nature.

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[37] This book has a peculiar history. A collection of previous writings, it appeared during Whitehead's last year. Lowe told me that Whitehead knew little of it and that it was edited anonymously by the American philosopher Dagobert D. Runes (born 1902), maybe with help from Whitehead's wife. Separate editions were published by the American and British publishers; but then the American house also republished the British edition in 1948!!