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ON APPROXIMATION METHODS OF LEONARDO FIBONACCI

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SUMMARIES

As is well known, Leonardo da Pisa gave a very precise approximation for the only irrational root of the equation $x^3 + 2x^2 + 10x = 20$. *Two hypotheses concerning his method were put forward in the XIX century. With good reason they were criticized by M. Cantor in his* Vorlesungen. *In the present paper it is argued that Leonardo calculated his approximate value using step-by-step the rule of two false positions. Our argument is based on an analysis of all approximation methods used by him.*

Хорошо известно, что Леонардо Пизанский дал чрезвычайно точное приближение для иррационального корня уравнения $x^3 + 2x^2 + 10x = 20$. В прошлом веке были выдвинуты две реконструкции его метода. М. Кантор в своих "Лекциях" вполне обоснованно их критиковал. В настоящей статье показано, что Леонардо вычислял это приближенное значение, используя шаг за шагом метод двух ложных положений. Наш вывод основывается на анализе всех методов приближения, которые он использовал.

Approximate values were given for some magnitudes in works of classical Greek mathematics, but there were no indications as to how they had been found. For example, in the beginning of his "Measurement of the Circle," Archimedes drew an isosceles triangle with base touching a given circle and with vertex coinciding with the circle's center. Then, without revealing his reasons, he said that the ratio of the height of the triangle to half its base is greater than 265/153 but smaller than 1351/780. This is a very good approximation to $\sqrt{3}$. However, we now can only conjecture his method. There exist at least nine different reconstructions. [E. J. Hofmann 1930]

Approximation methods as well as calculation methods in general were supposed to play an auxiliary role and therefore were not considered proper for inclusion in purely mathematical works. [1] But surely it is not false to maintain that the needs of pure mathematics itself (to say nothing of practical measuring) for reliable and accurate approximations was great enough to insure that approximation methods were undoubtedly studied and developed in addition to theoretical science. They appeared in some mathematical works beginning with Ptolemy's time. Still, the tradition survived of not explaining methods which gave rise to this or that approximation.

I think that this tradition is the reason Leonardo da Pisa, otherwise called Leonardo Fibonacci (1180-1250?), passed over in silence the method that allowed him to calculate with extraordinary accuracy the root of a certain cubic equation. The present paper gives a possible reconstruction of his method.

I now examine all the seven cases in which Leonardo made use of approximations.

A. *Liber abaci*, ch. 12 [Leonardo 1857, vol. 1, 173-175; 178-212]

B. *Liber abaci*, ch. 13, [*ibid.*, 318-350]

C. *Liber abaci*, ch. 14, [*ibid.*, 352-356]

D. *Liber abaci*, ch. 14, [*ibid.*, 378-384]

E. *Practica geometriae* [Leonardo 1857, vol. 2, 86-91]

F. *Epistola Leonardi ad magistrum Theodorum* [Leonardo 1857, vol. 2, 249-250] and in *Practica geometriae*, [*ibid.*, 218-224]

G. *Flos Leonardi Bigolli*, [*ibid.*, 234-240]

Unlike the other sources, the *Liber abaci* was a textbook containing many examples in which the most important rules of arithmetic were explained and their use demonstrated. The passages A, B, C and D are located near the end of this book, meaning that the subjects involved were considered to be difficult. They are followed by a single chapter about irrationalities in which the tenth book of Euclid's "Elements" is examined (which in the Middle Ages was supposed to be of extreme complexity).

Case A

The so called *Regula falsi* is expounded here. One of Leonardo's examples is a problem that is reduced to the equation $x/3 + x/4 = 21$. Take $y = 12$ and get $y/3 + y/4 = 7$. Then the solution 36 may be obtained from the proportion $7/21 = 12/x$. In the general case, if rendered in symbols, this rule says that the exact solution x of an equation $ax = b$ may be found from the proportion $ay/b = y/x$ with an arbitrary y.

Case B

The voluminous 13th chapter of *Liber abaci* is dedicated to the Arab rule *elchatayn* [2]. The heading of the chapter shows that Leonardo highly appreciated this rule. It reads "On the rule *elchatayn* and how it solves almost all questions of abacus." The Arab word *alkhat'ayni* means "two mistakes." Accordingly the Latin name of this rule, which was used in the Middle Ages and remained in literature of the history of mathematics, was *regula* *duorum falsorum positionum*. It makes use of two approximations ("false positions") in order to get the third and more accurate one. We see that it is the modern linear interpolation $y = (a(f(b)-c)+b(c-f(a)))/(f(b)-f(a))$ and yields an approximate solution of the equation $f(x) = c$. The "false positions" are a and b. The proportion $(y-a)/(b-a) = (c-f(a))/(f(b)-f(a))$ gives the value of y. Since Fibonacci did not know negative numbers he had to formulate the rule in different ways depending on the signs of $f(a)$ and $f(b)$.

If f is a linear function, then y is the exact solution. All Leonardo's examples are of that kind, e.g. $f(x) = 2\ (2(2x-12)-12,\ c=12$ (problem of a merchant) or $f(x) = (50-x)^2-x^2,\ c = 30^2-40^2$ (problem of two birds).

Case C

A rule for calculation of square roots is given here. I mention only the algorithm. The greatest natural number whose square is not greater than c is taken as the first approximation of $\sqrt{c}$, i.e. $u = [\sqrt{c}]$. The rule says further

$v = u + (c-u^2)/2u,\quad x = v-(c-u^2)^2/(2u)^2 2v$. The reader can check that $v = (u+c/u)/2,\ x = (v+c/v)/2$.

This means that the first coincides with the well-known Babylonian method of extracting square roots and the second step repeats the first. This fact is itself interesting and will help us in the following to reconstruct another Leonardo method that used iteration.

Case D

This passage gives a method of solving the simplest cubic $x^3 = c$. Leonardo seldom ascribes anything to himself in *Liber abaci* (his authorship is implied in other works), but here there is such an indication. Let us consider the first and the most important of his examples (Leonardo 1857, vol. 1, 380). Here, for the verbal explanations of Leonardo, we have substituted arithmetic symbols (even though alternating equality and inequality signs is unconventional). $3^3 = 27<27+20 = 47<64 = 4^3$. $\sqrt[3]{47}$ is 3 with remainder 20. Now $20>(64-27)/2$. Trying $3 + 1/2$, $(3+1/2)^3 = 27+1/8+3\cdot 3^2/2+3\cdot 3/4 = 42+7/8$, and $47-(42-7/8) = 4+1/8$. So $\sqrt[3]{47}$ is $3+1/2$ with remainder $4+1/8$. But $3(3+1/2)4 = 42$ and $4+1/8$ divided by 42 is approximately $1/10$. So $\sqrt[3]{47}$ is $3+1/2+1/10 = 3+3/5$ with remainder about $1/3$.

After that Leonardo writes: "...*quam terciam si proportionaveris ad numerum, qui provenit ex triplo a.d. in 4, propius nimirum ad radicem de 47 devenis,*" i.e. if you divide this one third by the tripled second approximation times four, you will of course come nearer to the root of 47. These words show how to take the next step.

In all the other examples Leonardo is content with the second approximation which is apparently obtained by linear interpolation. Indeed, the rule *elchatayn* if applied to $f(x) = x^3$ would give the values $a=x$, $b=a+1$, $(y-a)/1 = (c^3-a^3)/(3a(a+1)^1+1)$. (See Case B.) In the example cited above the result of each step is rounded off. Put $c=47$. $a=[\sqrt[3]{47}]=3$. $y=3+1/2$ instead of $3+20/37$. $z=y+(c-y^3)/3y(a+1)\approx 3+1/2+1/10$. $x=z+(c-z^2)/3z(a+1)$ (this step is outlined in the quotation). Note that the *elchatayn* rule is used here repeatedly [3].

Case E

After having calculated the side of the regular polygon with 96 angles, Leonardo gives for π the approximate value 4320/1375 (four significant digits). He adds that the ancient scholars gave for π the less accurate value of 22/7.

Case F

In this passage from a letter to Master Theodor, the problem is to inscribe an equilateral pentagon in an isosceles triangle of base 12 and equal sides 10, so that one vertex of the pentagon is at the apex of the triangle and the other four lie one on each equal side and two on the base. Leonardo takes one side as the unknown and after some simple geometric considerations (which involve applying similarity and the Pythagorean theorem to right triangles) he obtains the equation $(7/20)x^2+(64/5)x=64$ [p. 250, line 14]. Herewith, he says, the problem is reduced to the rule of algebra [4].

Since the (positive) solution is irrational, Leonardo gives an approximation of it: $x=4;\ 27,\ 24,\ 40,\ 50$ in the sexagesimal system used by Leonardo. All the given sexagesimal significant digits are correct. Leonardo does not describe the method that has given such accuracy, but his words "reduced to the rule of algebra" mean that after a standard procedure the root is expressed in form $x=(16/7)(3\sqrt{11}-8)$. Now to calculate $\sqrt{11}$ Leonardo had at his disposal the method described in Case C. This gives excellent accuracy (in modern terms, the process converges rapidly). This explains the accuracy of the solution.

Case G

It is well known that at a mathematical tournament in the presence of the emperor Frederick II, Fibonacci was challenged to solve the cubic equation $x^3+2x^2+10x = 20$. He found the (positive) root with a marvelous accuracy as 1; 22, 7, 42, 33, 4, 40, but it has remained unknown how he managed to reach his result.

I suggest that Leonardo made use of iterated linear interpolation by the *elchatayn* rule. The single argument against this view is that all the arguments for it are indirect. They are enumerated below.

1. Two other problems were proposed to Leonardo together with the mentioned cubic: the problem of three merchants exchanging money and that of a square (of a rational number) which, after adding or subtracting five, produced a square again. Both of these are very similar to some problems solved by Leonardo earlier [Vygodskii 1960, 238-245]. It is natural to suggest that the method used to solve the cubic should be also looked for among the methods found in other places in his work. It is unbelievable that Leonardo could invent a new method and accomplish difficult calculations "in the presence of the emperor," i.e. in less that a day, and then never use it again.

2. Iterated linear interpolation can be employed with success to solve the cubic in question unlike the two other approximation methods we have seen Leonardo use--the Babylonian method for extraction of square roots and the geometrical one for π.

3. Leonardo made use of linear interpolation previously as in Cases A and B.

4. He used iteration as in C, D and F.

5. He repeatedly used linear interpolation by the *elchatayn* rule as in D.

6. In D, Leonardo used iterated linear interpolation to extract cube roots. This circumstance is important because it suggests a comparison with cases C (square roots) and F (quadratics) in both of which the approximation method was the same.

I have carried out the calcuations in sexagesimal fractions which, provided that my hypothesis is correct, must have been accomplished by Leonardo. Step 1. $a=1$, $b=2$; $x_1=1;\ 18$. Step 2. $a=x_1$, $b=2$; $x_2=1;\ 20$. Step 3. $a=x_1$, $b=x_2$; $x_3=1;\ 22,\ 58$. Step 4. $a=x_2$, $b=x_3$; $x_4=1;\ 21,\ 13$. Step 5. $a=x_4$, $b=x_3$; $x_5=1;\ 22,\ 06,\ 32$, *and so on.*

The 18th step (which took about three quarters of an hour) gave precisely Leonardo's result. Of course, it is impossible to determine the values of a and b he preferred at each step. He could possibly somewhat diminish the total number of steps by enlarging at each of them the number of significant digits, but this would have complicated the calculations and increased the time needed.

In any case, the result can be obtained in less than a day with a certain skill in handling of sexagesimal fractions. It is not to be excluded that Leonardo improved the answer when preparing the manuscript. In case he knew a better method (which is difficult to imagine) his answer would be more accurate

NOTES

1. A referee suggests rather that they may not have been explicitly discussed because they were too well known.

2. M. Ya. Vygodskii [1960] demonstrated that this rule is of Chinese origin.

3. This fact, as well as others that I do not specify was not noticed earlier. Incidentally, fragments from M. Cantor [1892] remain the most detailed contribution to the investigation of Leonardo's work. However, no special attention is paid there to approximation methods.

4. This problem may be historically interesting from other points of view in relation to the "golden ratio."

CITATIONS

Cantor, M 1892 *Vorlesungen über Geschichte der Mathematik* Leipzig vol. 2

Hofmann, J E 1930 Erklärungsversuche für Archimedes Berechnung von $\sqrt{3}$ *Archiv für Gesch. der Math.* 12(4), 387-408

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Vygodskii, M Ya 1960 On the origin of the rule of double false position *Ist.-Mat. Issl.* 13, 231-252

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IMPOSITION OF DECIMAL NUMERATION ON PRIMITIVE TRIBES

Barbabus Hughes, O.F.M. (School of Education, California State University, Northridge, California 91324, USA) writes that he has found another instance in Hawaii of the phenomenon reported in *HM* 1, 82 of missionaries imposing decimal numeration on natives He would appreciate hearing of other examples of an invading culture imposing a new number system.