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# A novel definition of generalized synchronization on networks and a numerical simulation example☆

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ABSTRACT

This paper develops a novel definition of generalized synchronization on complex networks consisting of systems evolving in a chaotic or regular fashion. With two usual methods for detecting generalized synchronization, two criteria for generalized synchronization on networks are advanced. Some complex dynamical behaviors are discussed briefly on the basis of numerical simulations of a real network example.



## 1. Introduction

Following chaos synchronization being observed by Pecora and Carroll in the 1990s [1] and complex networks being intensively studied in many fields recently [2–6], the synchronization of complex dynamical networks has become a focus of attention [6–13].

However, most of the works mentioned above synchronize identical nodes. In fact, in many real-world cases such as biological systems and social activities (especially collective behaviors), it is rarely the case that every component can be assumed to be identical. As a result, more and more applications of chaos synchronization in secure communications have made it much more important to synchronize two different chaotic systems in recent years. In this regard, some works on generalized synchronization (GS) [14] of nonidentical systems have been performed.

As a sort of the synchronous chaotic behavior, generalized synchronization of unidirectionally coupled oscillators has been attracting special attention. For two coupled systems named the drive (master) system and response (slave) system with state variables $\boldsymbol{x}_d(t)$ and $\boldsymbol{x}_r(t)$, GS is defined as the existence of some smooth vector functional between the states of the drive and response systems, i.e. $\boldsymbol{x}_d(t)$ and $\boldsymbol{x}_r(t)$ have the relationship $\boldsymbol{x}_d(t) = \boldsymbol{H}[\boldsymbol{x}_r(t)]$ after a transient time. The form of the functional $\boldsymbol{H}[\cdot]$ (smooth or fractal) may be rather complicated, and two different dynamic systems may serve as coupled systems.

This article independently develops a definition of generalized synchronization on complex networks for the general case, and herein more abundant synchronized behavior on complex networks is shown. The paper is organized as follows. In Section 2, the definition of GS on networks is introduced. Section 3 proposes two criteria for detecting GS on coupled networks. Numerical simulations of a network sample are provided for illustration and verification in Section 4. Finally, conclusions are given in Section 5.

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## 2. The definition of generalized synchronization on networks

We consider a network with $l$ different sorts of nodes and $n$ nodes:

$$\dot{\boldsymbol{x}}_i = F_k(\boldsymbol{x}_i) + \sum_{j=1}^{n} \varepsilon_{ij} \boldsymbol{P} \boldsymbol{x}_j, \quad i = 1, \dots, n \text{ and } k = 1, \dots, l \tag{1}$$

where $\boldsymbol{x}_i = (x_i^1, \dots, x_i^d) \in \mathbb{R}^d$ is the $d$-vector containing the coordinates of the $i$th oscillator, and $F_k(\cdot) : D \subseteq \mathbb{R}^d \to \mathbb{R}^d$ is the (nonlinear) vector field controlling the dynamics of the $k$th sort of oscillators. The nonzero elements of the $d \times d$ matrix $\boldsymbol{P}$ determine which variables couple the oscillators. For clarity, we shall consider a vector version of coupling with the diagonal matrix $\boldsymbol{P} = \text{diag}(p_1, p_2, \dots, p_d)$, where $p_h = 1, h = 1, 2, \dots, s$ and $p_h = 0, h = s + 1, \dots, d$. Note that all the results that are obtained in this paper are also valid for other possible cases of scalar and vector couplings between the oscillators.

Let $\boldsymbol{G} = (\varepsilon_{ij}(t)) \in \mathbb{R}^{n \times n}$ be an $n \times n$ symmetric matrix with vanishing row-sums and nonnegative off-diagonal elements, i.e. $\varepsilon_{ij} = \varepsilon_{ji}, \varepsilon_{ij} \geq 0$, for $i \neq j$, and $\varepsilon_{ii} = -\sum_{j=1}^{n} \varepsilon_{ij}, j \neq i, i = 1, \dots, n$. The matrix $\boldsymbol{G}$ defines a graph with $n$ vertices and $m$ edges, and represents both the structure of the coupling and the coupling strengths. The vertices of the graph correspond to the individual oscillators and the edges to the off-diagonal elements of $\boldsymbol{G}$.

Therefore, we consider an arbitrary network of systems that are mutually coupled, due to the symmetry of the coupling matrix. The condition for vanishing row-sums is necessary for the existence of a generalized synchronous state.

In the following, we first present a rigorous definition of generalized synchronization of a network.

**Definition 1.** Suppose $\boldsymbol{x}_k(t, \boldsymbol{X}_0)$ (where $\boldsymbol{X}_0 = ((\boldsymbol{x}_1^0)^{\mathrm{T}}, \dots, (\boldsymbol{x}_n^0)^{\mathrm{T}})^{\mathrm{T}}$ and $k = 1, 2, \dots, l$) is a solution for the network (1). If there exists a nonempty open subset $\boldsymbol{E} \subseteq \boldsymbol{D}$, with $\boldsymbol{x}_k^0 \in E$ $(k = 1, 2, \dots, l)$, such that $\boldsymbol{x}_k(t, \boldsymbol{X}_0) \in \boldsymbol{D}$, for all $t \geq 0$, $k = 1, 2, \dots, l$, and

$$\lim_{t \to \infty} \|\boldsymbol{x}_g(t, \boldsymbol{X}_0) - \boldsymbol{H}_h[\boldsymbol{x}_f(t, \boldsymbol{X}_0)]\| = 0, \quad \text{for } 1 \leq g, f \leq l, 1 \leq h \leq n \tag{2}$$

where the functional $\boldsymbol{H}_h[\cdot]$ denotes the relation of two nodes, and $\|\cdot\|$ means any norm on the space of real matrices (or vectors), then the network (1) is said to realize generalized synchronization and $\boldsymbol{E} \times \cdots \times \boldsymbol{E}$ is called the region of synchrony for the network (1).

**Remark.**
- The definition above can be used to analyze behaviors of GS among all sorts of nodes of networks, not just describing the state of GS between two systems.
- According to the definition, we can comprehend the implication of GS for complex networks. One network consists of diverse nodes; moreover all nodes can be classified into some distinct communities in terms of their natures and relations with other nodes around themselves. It is intuitively obvious that the nodes in the same community can realize complete synchronization (which implies strong synchronization) under strong enough coupling, while the nodes between different communities may attain GS (which suggests weak synchronization) under stricter conditions than those for complete synchronization.
- Considering the case of two coupled nodes, the functional $\boldsymbol{H}_h[\cdot]$ corresponds to the existence of an attracting *synchronization manifold* $\boldsymbol{M}$ (given by $\boldsymbol{H}_h[\cdot]$) in the full state space of the coupled systems. In view of the complexity and smoothness of the function $\boldsymbol{H}_h[\cdot]$ and the corresponding synchronization manifold $\boldsymbol{M}$, we find it difficult to resolve the concrete functional $\boldsymbol{H}_h[\cdot]$ directly. Accordingly, we develop some indirect methods as in the following section. ◇

## 3. Two criteria for detecting GS on networks

As outlined in the preceding section, the existence of a functional relation depends on stability features of the response system. This aspect can be used to formulate another definition of generalized synchronization in terms of asymptotic stability (entrainment) of the response system. There are basically two approaches for testing for synchronization in this sense: the auxiliary systems approach and that of conditional Lyapunov exponents. Therefore, two criteria for detecting GS on networks will be stated analogously.

### 3.1. The auxiliary systems approach

One direction for studying asymptotic stability is by the auxiliary system method introduced by Abarbanel et al. [15], using an identical duplication of the response system that is driven by the same driving signal. We consider thus the system

$$\begin{cases} \dot{\boldsymbol{x}} = F(\boldsymbol{x}) \\ \dot{\boldsymbol{y}} = G(\boldsymbol{x}, \boldsymbol{y}) \\ \dot{\boldsymbol{y}}' = G(\boldsymbol{x}, \boldsymbol{y}'). \end{cases} \tag{3}$$

Generalized synchronization occurs if $\lim_{t \to \infty} \|\boldsymbol{y}(t) - \boldsymbol{y}'(t)\| = 0$, i.e. if the response system and the auxiliary system show complete synchronization.

In the mutually coupled case, every individual node of the network is both the drive system and the response system. Therefore we have this criterion for detecting GS on networks:

**Criterion 2.** Suppose $\boldsymbol{x}_i^j(t)$ is the $j$th auxiliary system (there exist $m$ auxiliary systems in total, namely $1 \le j \le m$) of $\boldsymbol{x}_i(t)$ in network (1). If $\lim_{t\to\infty} \|\boldsymbol{x}_i(t) - \boldsymbol{x}_i^j(t)\| = 0$ for all $1 \le i \le n$ and $1 \le j \le m$, then the network (1) achieves the GS state after a transient time.

**Remark.**
- By virtue of the alternative method for testing the predictability of the response system from knowledge of $\boldsymbol{x}_d(t)$ and, therefore, for detecting synchronous behavior in nontrivial cases, we can indirectly judge the functional relation between the driving system and the driven system.
- The auxiliary system need not be implemented in an actual closed-loop system, if circumstances do not allow.
- The bottom-up approach analyzes whether every auxiliary system of all nodes can completely synchronize itself with its response system before drawing a conclusion on whether the network realizes the GS state. ◇

### *3.2. The largest Lyapunov exponents approach*

On the other hand, a sufficient and necessary condition for GS (or CS) is the largest conditional Lyapunov exponent (LCLE) being negative [1,16–18].

Conditional Lyapunov exponents (also called *transversal Lyapunov exponents*) characterize the dynamics of small perturbations $\boldsymbol{y}' = \boldsymbol{y} + \boldsymbol{e}$ of response state $\boldsymbol{y}$, resulting in an error dynamics

$$\begin{aligned} \boldsymbol{e} &= G(\boldsymbol{x}, \boldsymbol{y}) - G(\boldsymbol{x}, \boldsymbol{y}') \\ &= G(\boldsymbol{x}, \boldsymbol{y}) - G(\boldsymbol{x}, \boldsymbol{y} - \boldsymbol{e}) \end{aligned}$$

where linearization results in

$$\dot{\boldsymbol{e}} = DG_r(\boldsymbol{x}, \boldsymbol{y})\dot{\boldsymbol{e}}. \tag{4}$$

The conditional Lyapunov exponents (CLEs) are computed from the Jacobian $DG_r(\boldsymbol{x}, \boldsymbol{y})$ and characterize the asymptotic local stability of the response system. If all CLEs are negative, the response system is stable and shows GS. With the extended approach, another criterion for detecting GS on networks is developed:

**Criterion 3.** The network (1) is said to achieve a GS state when all LCLEs of the system state of network (1) become negative.

**Remark.**
- According to the theorem in [19], namely that GS occurs if for all initial values $(\boldsymbol{x}_0, \boldsymbol{y}_0^1)$ and $(\boldsymbol{x}_0, \boldsymbol{y}_0^2)$ one attains $\lim_{t\to\infty} \|\boldsymbol{y}(t, \boldsymbol{x}_0, \boldsymbol{y}_0^1) - \boldsymbol{y}(t, \boldsymbol{x}_0, \boldsymbol{y}_0^2)\| = 0$, asymptotical stability can be proved analytically using Lyapunov functions. Numerically, this condition can be checked by computing the (largest) Lyapunov exponent of the response system.
- Thinking extensively, GS on the networks with many coupled nodes can be judged by the largest Lyapunov exponents approach, using the same method as for validating GS on a system with two coupled subsystems. Therefore, one can draw the conclusion that the network will realize a GS state if all LCLEs of the network are below zero asymptotically.
- The Criterion 3 is used more widely in practice than Criterion 2 technically, because of using many more replica systems in the auxiliary systems approach, which can increase the calculation cost. ◇

## 4. Numerical simulations of a network example

In this section, Definition 1 and two criteria for detecting GS are illustrated by using Lorenz systems with different system parameters. The equation of motion for a single Lorenz oscillator is written as

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = rx - y - xz \\ \dot{z} = -bz + xy. \end{cases} \tag{5}$$

Here $\sigma = 10$, $b = 8/3$, and $r$ differs for different oscillators. For this case $l = 2$, which means that there are two sorts of node in the network; we use $r_1 = 35$ and $r_2 = 28$, which are displayed in Fig. 1.

The system Jacobian is

$$DG_r(\boldsymbol{x}_i) = \begin{pmatrix} -\sigma & \sigma & 0 \\ r_i - z & -1 & -x \\ y & x & -b \end{pmatrix}. \tag{6}$$

For simplicity, we consider a small-world network (1) with ten nodes and two sorts of chaotic nodes, as shown in Fig. 2.

As illustrated in Fig. 2, all black nodes represent Lorenz systems with parameters $\sigma = 10$, $b = 8/3$, $r_1 = 35$, while all white nodes represent Lorenz systems with parameters $\sigma = 10$, $b = 8/3$, $r_2 = 25$. The network's *diameter*

$$D = \max_{i,j} d_{ij} = 2,$$

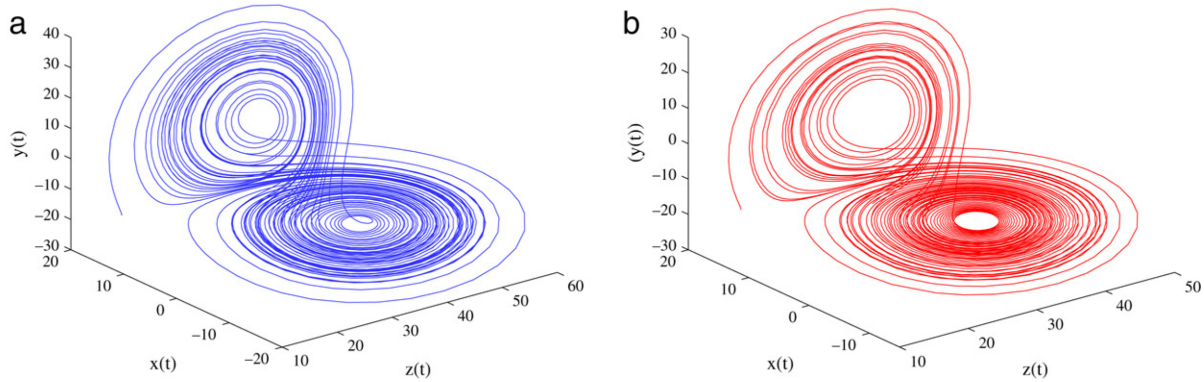



**Fig. 1.** Lorenz chaotic attractor. (a) $\sigma = 10, b = 8/3, r_1 = 35$. (b) $\sigma = 10, b = 8/3, r_2 = 25$.

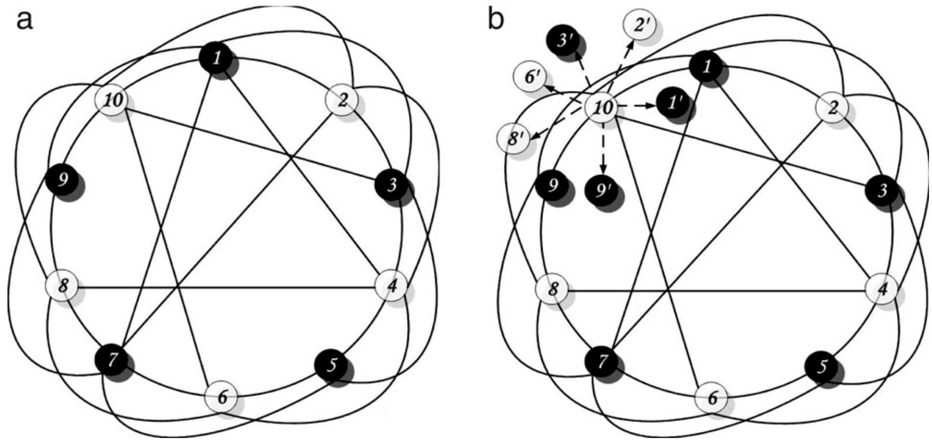



**Fig. 2.** Network (1) with ten nodes (two sorts of nodes). (a) Small-world network. (b) Small-world case with auxiliary systems on the tenth node.

where $d_{ij}$ is the shortest length between node $i$ and node $j$. The *average path lengths* of network (1)

$$L = \frac{1}{\frac{1}{2}n(n+1)} \sum_{i \geq j} d_{ij} = \frac{64}{45} = 1.42,$$

and the network *clustering coefficients* are such that

$$L = \frac{\sum_i C_i}{n} = \frac{26}{45} = 0.58,$$

where $C_i$ is the clustering coefficient of node $i$. In this case, the network (1) is studied, as shown in Fig. 2, with inner-coupled matrix $\boldsymbol{P} = \mathrm{diag}(p_1, p_2, \ldots, p_d) = \mathrm{diag}(1, 0, 0)$. From Eq. (1), the associated feedback system is

$$\begin{pmatrix} \dot{x}_i \\ \dot{y}_i \\ \dot{z}_i \end{pmatrix} = \begin{pmatrix} \sigma(y_i - x_i) \\ r_k x_i - y_i - x_i z_i \\ -bz_i + x_i y_i \end{pmatrix} + \varepsilon \boldsymbol{P} \sum_{j \neq i} \begin{pmatrix} x_j - x_i \\ y_j - y_i \\ z_j - z_i \end{pmatrix}, \tag{7}$$

where $\varepsilon$ is the coupling strength, for $k = 1, 2$ and $1 \leq i, j \leq 10$.

Numerical results illustrated by Fig. 3 show that the system (7) attaches the GS state for the negative of the largest conditional Lyapunov exponents of all nodes of the network according to Criterion 3, when the coupling strength falls in the exponentially stable region, namely $\varepsilon \in [22, 25]$.

In Fig. 3, it is found easily that 10 nodes cannot attach a common GS state at the same time, and abundant dynamic behaviors, which include partial generalized synchronization and dynamic bursting, are emerging with coupling strength enhancing. When we select $\varepsilon = 0$, which means that all oscillators are not coupled, all Lyapunov exponents of nodes are positive, which means that all oscillators are not in generalized synchronization at all. As we increase the coupling strength to $\varepsilon < 21$, some nodes attach GS; however others are not realizing GS. We call this behavior a *partial GS* [20]. This result is very

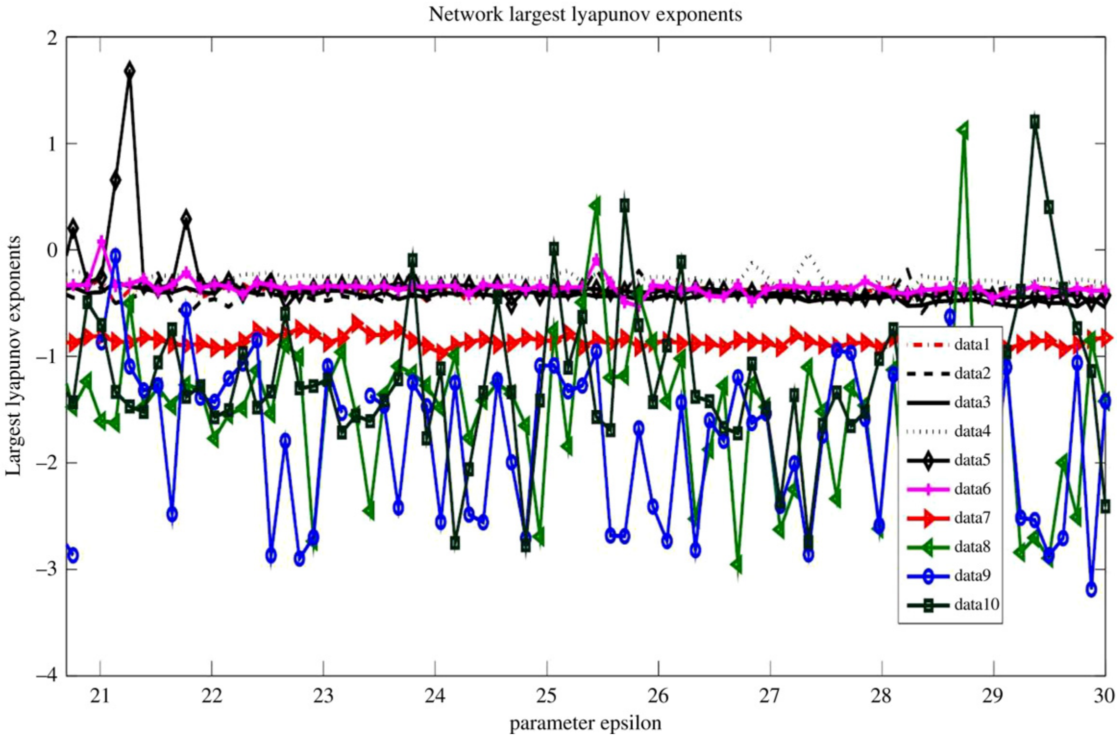



**Fig. 3.** All largest Lyapunov exponents of 10 nodes in networks against coupling strength $\varepsilon$, where different curves correspond to different nodes.

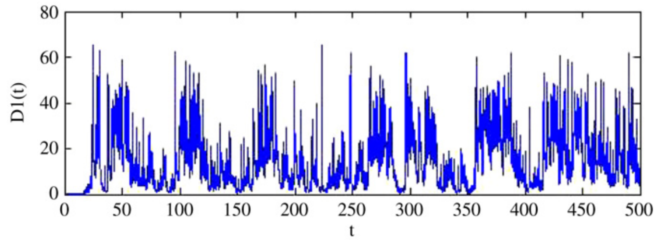
interesting because for mutually coupled nonidentical chaotic oscillators, every pair of coupling directions are inequivalent, that is, synchronizations in the two directions cannot occur at the same minimum threshold, because the two coupled nodes do not have the same degree of chaotic behavior, which is determined by the instinctive dynamics. That is to say, for systems with a larger maximum Lyapunov exponent, higher stochasticity prevents it from being the first to be entrained. When one further increases the coupling strength, one fairly confidently expects all nodes to achieve a GS state, for the negative of all largest conditional Lyapunov exponents of ten nodes. To further observe the effect of strengthening the value of $\varepsilon$ beyond the exponentially stable regions, we attain $\varepsilon = 25.5$. Meanwhile, the largest Lyapunov exponents of node 8 and node 10 exceed zero, which implies desynchronization and node dynamical behavior bursting out. This illustrates that the node dynamics behaviors are influenced by the network topological structure because of there being enough coupling strength. When one increases the coupling strength, the global generalized synchronizations are realized again.

To verify the correctness of Criterion 2, we let $\varepsilon = 23$, and introduce the instantaneous distance between the $i$th node and its auxiliary node as

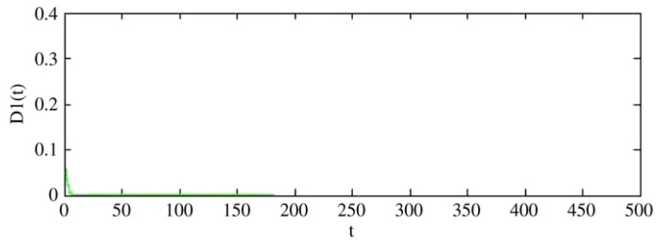
$$D_i(t) = \|X_i(t) - X_i(t)\| = \sqrt{(x_i' - x_i)^2 + (y_i' - y_i)^2 + (z_i' - z_i)^2}. \tag{8}$$

Considering all auxiliary nodes linked to the 10th node, we find that all the instantaneous distances of the auxiliary node attach zero. For simplicity, we merely take $D_1(t)$, the instantaneous distance between the $l$th node and its auxiliary system, into account, as illustrated by Fig. 4. In Fig. 4(a), for $\varepsilon = 10$, it can be found that the differences oscillate irregularly around a nonzero value. This strong fluctuation indicates that node 1 and its auxiliary node do not attain the GS state. On increasing $\varepsilon$ to $\varepsilon = 23$, one may find that the GS state is achieved, as exhibited by Fig. 4(b). On the basis of the above studies, one can carry out the simulation for all other distances between other nodes and their auxiliary systems. For simplicity, we omit these simulations for other nodes.

**Remark.** The collective behavior is an excellent example of such realistic complex systems. Individual connectivity is organized into some distinct communities, from the microscopic individual level, via the mesoscopic level of local cliques or communities, to the macroscopic level of collective behaviors within a social field. Different communities means different sorts of nodes, while all nodes in the same community display the same behavioral properties. The experimentally observed social activity is characterized by GS phenomena over a wide range of spatial and temporal scales, just because of the relations between some distinct communities. Namely the correlative behaviors embody the 'functional' relation between all groups, and thus the GS pattern arises. The rich set of interactions between individuals in society results in complex community structure, and, conversely, complex structures affect the GS phenomena, i.e. as in collective behavior in society.




(a) Epsilon $= 10$.




(b) Epsilon $= 23$.

**Fig. 4.** The evolutions of the difference function $D_i(t)$ of node 1 and its auxiliary node with respect to the 10th node; (a) weak coupling where $\varepsilon = 10$, (b) strong coupling where $\varepsilon = 23$.

## 5. Conclusion

In this paper, we have presented an innovative definition of GS on complex networks. Furthermore, we analyze complex dynamical behavior of nodes in networks attributable to the dynamics of nodes and the topological structure of networks by analyzing a simple real network example. Accordingly, we draw the conclusion that the GS behavior on networks may be realized and have more abundant dynamics and real physical sense, including partial GS, global GS, dynamical behavior bursts so on.

The proposed definition and two criteria provide theoretical ideas and a framework for the further investigation of the dynamical behaviors and topological structures of many real-world cases, such as the simulation research into the collective behaviors within virtual societies. We foresee the development of this definition to yield open problems for general complex dynamical networks.

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