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Runs in superpositions of renewal processes with applications to discrimination

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Abstract

Wald and Wolfowitz [Ann. Math. Statist. 11 (1940) 147–162] introduced the run test for testing whether two samples of i.i.d. random variables follow the same distribution. Here a run means a consecutive subsequence of maximal length from only one of the two samples. In this paper we contribute to the problem of runs and resulting test procedures for the superposition of independent renewal processes which may be interpreted as arrival processes of customers from two different input channels at the same service station. To be more precise, let $(S_n)_{n\geqslant 1}$ and $(T_n)_{n\geqslant 1}$ be the arrival processes for channel 1 and channel 2, respectively, and $(W_n)_{n\geqslant 1}$ their be superposition with counting process $N(t) \stackrel{\text{def}}{=} \sup\{n\geqslant 1 : W_n \leqslant t\}$. Let further R_n^* be the number of runs in $W_1, \ldots, W_n$ and $R_t = R^*_{N(t)}$ the number of runs observed up to time t. We study the asymptotic behavior of R_n^* and R_t, first for the case where $(S_n)_{n\geqslant 1}$ and $(T_n)_{n\geqslant 1}$ have exponentially distributed increments with parameters λ_1 and λ_2, and then for the more difficult situation when these increments have an absolutely continuous distribution. These results are used to design asymptotic level α tests for testing $\lambda_1 = \lambda_2$ against $\lambda_1 \neq \lambda_2$ in the first case, and for testing for equal scale parameters in the second.

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1. Introduction

Wald and Wolfowitz [11] introduced the *run test* for testing whether two samples follow the same distribution: Let $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ be two independent samples of i.i.d. real-valued random variables having continuous distribution functions F and G, respectively. Let R_{n_1,n_2} denote the number of runs in the pooled sample $X_1, \ldots, X_{n_1}, Y_1, \ldots, Y_{n_2}$ arranged in ascending order of magnitude, where a run is a subsequence of maximal length taken only from the X or the Y sample.

The run test rejects the hypothesis $F = G$ if R_{n_1,n_2} is less than some critical value. The distribution of R_{n_1,n_2} under the hypothesis is of course independent of the particular continuous distribution, and Wald and Wolfowitz [11] compute this distribution, derive asymptotic normality and show consistency of the test, as $n_1, n_2 \to \infty$ such that $n_1/(n_1+n_2) \to \beta \in (0,1)$. The distribution theory of runs is also treated in [12,9].

Let us now consider the case that only the joint sample size $n = n_1 + n_2$ is fixed and n_1 is a random variable having a binomial distribution with parameters n, p. Denoting by R'_n the resulting number of runs, we obtain from the explicit results for R_{n_1,n_2} that

$$\begin{aligned}\mathbb{P}(R'_n = k) &= \sum_{m=0}^{n} \mathbb{P}(R_{m,n-m} = k)\binom{n}{m} p^m (1-p)^{n-m}\\ &= \sum_{m=0}^{n} 2\binom{m-1}{l-1}\binom{n-m-1}{l-1} p^m (1-p)^{n-m}\end{aligned}$$

if $k = 2l$, and

$$\mathbb{P}(R'_n = k) = \sum_{m=0}^{n}\left[\binom{m-1}{l-1}\binom{n-m-1}{l} + \binom{n-m-1}{l-1}\binom{m-1}{l}\right] p^m (1-p)^{n-m}$$

if $k = 2l+1$. Under the hypothesis $F = G$, we have

$$\mathbb{E}R'_n \simeq 2np(1-p) \quad \text{and} \quad \operatorname{Var} R'_n \simeq 4np(1-p)(1-3p(1-p))$$

and

$$\hat{R}'_n \stackrel{\text{def}}{=} \frac{R'_n - 2np(1-p)}{2\sqrt{np(1-p)(1-3p(1-p))}} \stackrel{\text{d}}{\to} N(0,1), \tag{1.1}$$

as $n \to \infty$, see [9].

If $F \neq G$ the distributions of R_{n_1,n_2} and R'_n both depend of course on F and G. Given that F and G have continuous Lebesgue densities f and g, respectively, Henze and Voigt [5] showed that

$$\lim_{n\to\infty} \frac{R_{n_1,n_2}}{n_1+n_2} = 1 - \int \frac{\beta^2 f^2(x) + (1-\beta)^2 g^2(x)}{\beta f(x) + (1-\beta) g(x)}\,\mathrm{d}x \quad \text{a.s.}$$

as $n_1, n_2 \to \infty$ such that $n_1/(n_1+n_2) \to \beta \in (0,1)$. This easily implies

$$\frac{R'_n}{n} \stackrel{\text{P}}{\to} 1 - \int \frac{p^2 f^2(x) + (1-p)^2 g^2(x)}{p f(x) + (1-p) g(x)}\,\mathrm{d}x,$$

where $\stackrel{\text{P}}{\to}$ means convergence in probability.

In this paper we want to contribute to the problem of runs and resulting test procedures for superpositions of renewal processes which may be interpreted as arrival processes of customers from two different input channels at the same service station. So let us assume that $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ are independent samples of i.i.d. positive interarrival times, again with continuous distributions F and G, respectively. Denote the corresponding renewal processes by $(S_n)_{n \geqslant 1}$ (channel 1) and $(T_n)_{n \geqslant 1}$ (channel 2), i.e.

$$S_n = X_1 + \cdots + X_n \quad \text{and} \quad T_n = Y_1 + \cdots + Y_n \quad \text{for } n = 1, 2, \ldots .$$

We now consider runs in the superposition, $(W_n)_{n \geqslant 1}$ say, of $(S_n)_{n \geqslant 1}$ and $(T_n)_{n \geqslant 1}$, defined as subsequences of maximal length from the first or second of the these processes. We put

$$R_n^* \stackrel{\text{def}}{=} \text{number of runs in } W_1, \ldots, W_n$$

for $n = 1, 2, \ldots$ and

$$R_t \stackrel{\text{def}}{=} \text{number of runs in } W_1, \ldots, W_{N(t)} = R^*_{N(t)}$$

for $t > 0$, where $N(t) = \sum_{n \geqslant 1} \mathbf{1}_{(0,t]}(W_n)$.

If the interarrival times in both channels are exponentially distributed, the same holds true for the superposed arrival process. We will discuss this case in Section 2 and show that we may use the methods from the i.i.d. situation.

Section 3 deals with the more complicated situation of general interarrival distributions. We will derive the limiting behavior of R_n^* and R_t by drawing on the fact shown in [1] (see also [7]) that the superposition of absolutely continuous renewal processes constitutes a Markov renewal process.

2. Exponential interarrival times

We consider exponentially distributed interarrival times $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ with means $1/\lambda_1$ and $1/\lambda_2$, respectively. Then the following holds:

Theorem 1. *Put* $\lambda \stackrel{\text{def}}{=} \lambda_1 + \lambda_2$, $p \stackrel{\text{def}}{=} \lambda_1/\lambda$, *and let* R_n' *be as in Section* 1. *Then*

(i) R_n^* *has the same distribution as* R_n' *for each* $n = 1, 2, \ldots$.
(ii) $\mathbb{P}(R_t = k) = \sum_{m \geqslant 1} \mathbb{P}(R_m^* = k) \mathrm{e}^{\lambda t} \lambda^m t^m / m!$ *for each* $t > 0$.
(iii) $\hat{R}_n^* \stackrel{\text{def}}{=} \frac{R_n^* - 2p(1-p)n}{2\sqrt{p(1-p)(1-3p(1-p))n}} \stackrel{\text{d}}{\to} N(0,1)$, *as* $n \to \infty$.
(iv) $\hat{R}_t \stackrel{\text{def}}{=} \frac{R_t - 2p(1-p)N(t)}{2\sqrt{p(1-p)(1-3p(1-p))N(t)}} \stackrel{\text{d}}{\to} N(0,1)$, *as* $t \to \infty$.

Proof. We will pass from renewal processes to their corresponding renewal counting processes and use some well-known facts for Poisson processes; see e.g. [10].

(i) Let $(N_1(t))_{t \geqslant 0}$ and $(N_2(t))_{t \geqslant 0}$ denote the resulting counting processes for $S_1, S_2, \ldots$ and $T_1, T_2, \ldots$, which are Poisson processes with intensities λ_1 and λ_2, respectively. Then the counting process $(N(t))_{t \geqslant 0}$ of the superposition $W_1, W_2, \ldots$ satisfies

$$N(t) = N_1(t) + N_2(t), \quad t \geqslant 0$$

and is also Poisson with intensity λ. Put the mark $V_n = 0$ or 1 to each W_n according to whether W_n is an arrival epoch from channels 1 or 2. Hence

$$N_1(t) = \sum_{n \geqslant 1} \mathbf{1}_{\{W_n \leqslant t, V_n = 0\}} \quad \text{and} \quad N_2(t) = \sum_{n \geqslant 1} \mathbf{1}_{\{W_n \leqslant t, V_n = 1\}}.$$

It is well known that $V_1, V_2, \ldots$ are i.i.d. Bernoulli variables with parameter p, i.e. $\mathbb{P}(V_n = 1) = p = 1 - \mathbb{P}(V_n = 0)$, and they are independent of $(W_n)_{n \geqslant 1}$. With this notation

$$R_n^* = \text{number of runs in } V_1, \ldots, V_n = 1 + \sum_{i=2}^{n} \mathbf{1}_{\{V_{i-1} \neq V_i\}}.$$

We may thus resort to the combinatorial arguments of Wald and Wolfowitz [11] for the i.i.d. situation and obtain, with $U_n \overset{\text{def}}{=} V_1 + \cdots + V_n$ for $n \geqslant 1$,

$$\mathbb{P}(R_n^* = k) = \sum_{m=0}^{n} \mathbb{P}(R_{m,n-m} = k)\mathbb{P}(U_n = m) = \mathbb{P}(R_n' = k)$$

for each $n \geqslant 1$ and $k \geqslant 0$.

(ii) It suffices to note that

$$\mathbb{P}(R_t = k) = \sum_{m \geqslant 0} \mathbb{P}(R_m^* = k)\mathbb{P}(N(t) = m)$$

for all $t > 0$ and $m \geqslant 0$.

(iii) This follows immediately from the asymptotic normality result (1.1) of Mood together with (i).

(iv) Put $\mu(p) \overset{\text{def}}{=} 2p(1-p)$, $\sigma^2(p) \overset{\text{def}}{=} 4p(1-p)(1-3p(1-p))$, and let $m(t)$ be the largest integer less than or equal to λt. Note that $m(t)^{-1}N(t) \to 1$ a.s. and write

$$\hat{R}_t = \sqrt{\frac{m(t)}{N(t)}} \left(\frac{R_{m(t)}^* - m(t)\mu(p)}{\sigma(p)\sqrt{m(t)}} + \frac{R_{N(t)}^* - R_{m(t)}^* - (N(t) - m(t))\mu(p)}{\sigma(p)\sqrt{m(t)}} \right).$$

By (i), the first term in parentheses is equally distributed as $\hat{R}'_{m(t)}$ and hence, by (1.1), asymptotically standard normal as $t \to \infty$. Therefore it suffices to show that the second one converges to 0 in probability. To that end pick arbitrary $\varepsilon, \eta > 0$. Then

$$\begin{aligned} &\mathbb{P}\left(\left| \frac{R_{N(t)}^* - R_{m(t)}^* - (N(t) - m(t))\mu(p)}{\sigma(p)\sqrt{m(t)}} \right| > \varepsilon \right) \\ &\quad \leqslant \mathbb{P}(|N(t) - m(t)| > \eta t) \\ &\qquad + \mathbb{P}(|R_{N(t)}^* - R_{m(t)}^* - (N(t) - m(t))\mu(p)| > \varepsilon\sigma(p)\sqrt{m(t)}, |N(t) - m(t)| \leqslant \eta t). \end{aligned}$$

The first probability on the right-hand side of this inequality converges to 0 because $t^{-1}(N(t) - m(t)) \to 0$ a.s. as $t \to \infty$. The second one is bounded by

$$\mathbb{P}\left(\max_{k: |k - m(t)| \leqslant \eta t} |R_k^* - R_{m(t)}^* - (k - m(t))\mu(p)| > \varepsilon\sigma(p)\sqrt{m(t)} \right).$$

For $m(t) < k \leqslant m(t) + \eta t$, we have

$$R_k^* - R_{m(t)} - (k - m(t))\mu(p) = \sum_{i=m(t)+1}^{k} (\mathbf{1}_{\{V_{i-1} \neq V_i\}} - \mu(p)),$$

which is a sum of 1-dependent stationary random variables with mean zero. By summing over odd and even i separately, it can be decomposed into two sums of i.i.d. zero mean random variables. With an obvious modification the same can be said about $R_k^* - R_{m(t)} - (k - m(t))\mu(p)$ for $m(t) - \eta t \leqslant k < m(t)$. By combining these observations with an application of Kolmogorov's inequality (applied to the resulting i.i.d. sums) the conclusion

$$\mathbb{P}\left(\max_{k:|k-m(t)| \leqslant \eta t} |R_k^* - R_{m(t)} - (k - m(t))\mu(p)| > \varepsilon\sigma(p)\sqrt{m(t)} \right) < \varepsilon$$

for $\eta = \eta(\varepsilon)$ sufficiently small and t sufficiently large yields as in the proof of Theorem I.3.1 in [4]. Further details are omitted. □

A testing procedure: Using Mood's result (1.1) and the previous theorem we obtain tests for the hypothesis of equal intensities $\lambda_1 = \lambda_2$ against the alternative $\lambda_1 \neq \lambda_2$. For $\alpha \in (0, 1)$ define the critical value as

$$c(n, \alpha) \;=\; n/2 - v_\alpha \sqrt{n}/2,$$

where v_α is the α-fractile of the standard normal distribution. Let φ_n be the test based upon a sample of n observed arrivals which rejects the hypothesis for $R_n^* < c(n, \alpha)$, i.e.

$$\varphi_n \stackrel{\text{def}}{=} \mathbf{1}_{\{R_n^* < c(n,\alpha)\}}.$$

Let ϕ_t be the corresponding test when sampling from the fixed time interval $(0, t]$, defined as

$$\phi_t \stackrel{\text{def}}{=} \mathbf{1}_{\{R_t < c(N(t),\alpha)\}}.$$

Then the following corollary shows that φ_n and ϕ_t are asymptotically consistent level α tests.

Corollary 1. *In the situation of exponential interarrival times we have as $n \to \infty$, respectively $t \to \infty$:*

(i) $\mathbb{P}_{(v,v)}(R_n^* < c(n, \alpha)) \to \alpha$ *and* $\mathbb{P}_{(v,v)}(R_t < c(N(t), \alpha)) \to \alpha$ *for any* $v > 0$,
(ii) $\mathbb{P}_{(\lambda_1,\lambda_2)}(R_n^* < c(n, \alpha)) \to 1$ *and* $\mathbb{P}_{(\lambda_1,\lambda_2)}(R_t < c(N(t), \alpha)) \to 1$ *for any* $\lambda_1 \neq \lambda_2$.

Proof. From Theorem 1 we have for any λ_1, λ_2 with $p = \lambda_1/(\lambda_1 + \lambda_2)$

$$\mathbb{P}_{(\lambda_1,\lambda_2)}(R_n^* = \cdot) = \mathbb{P}_p(R_n' = \cdot)$$

now explicitly showing the parameters, in particular

$$\mathbb{P}_{(v,v)}(R_n^* = \cdot) = \mathbb{P}_{1/2}(R_n' = \cdot)$$

for each $v > 0$. Hence (i) follows from Theorem 1(iii) and (iv).

For (ii) it is enough to note that $2p(1-p)<\frac{1}{2}$ for all $p\neq\frac{1}{2}$, again using the asymptotic normality results. □

Corollary 1 shows that the run statistic provides asymptotically consistent level α tests for the problem of testing equal intensities, i.e. equal scale parameters of the interarrival times.

For homogeneous Poisson processes (exponential interarrival times) as treated in this section one may want to use the uniformly most powerful unbiased level α test for the i.i.d. Bernoulli sample $V_1,\dots,V_n$. It is given by

$$\varphi_n^* = \mathbf{1}_{\{|U_n-n/2|>v_{\alpha/2}\sqrt{n}/2\}}$$

when using that $(U_n-n/2)/\sqrt{n}/2$ is asymptotically standard normal under each $\mathbb{P}_{(\lambda,\lambda)}$ and therefore normal approximation for the critical value.

On the other hand, leaving homogeneous Poisson processes simple tests as φ_n^* are no longer available. But the run statistic still makes sense in more general situations, and it is our opinion that useful discrimination tests can be built upon this statistic. The results in the following section will demonstrate this in the problem of testing for equal scale parameters for general renewal processes.

3. Superpositions of renewal processes

In this section we will consider the number of runs R_n^*, resp. R_t for the superposition of two absolutely continuous renewal processes which no longer forms a renewal process unless the interarrival times in both channels are exponentially distributed. However, it is shown in [1] and briefly summarized below that it forms a Markov renewal process and can thus be analyzed within the framework of Markov renewal theory.

Given the interarrival times $X_1, X_2,\dots$ and $Y_1, Y_2,\dots$ for the two channels with generic copies X, Y, finite means $\xi \stackrel{\text{def}}{=} \mathbb{E}X$, $\zeta \stackrel{\text{def}}{=} \mathbb{E}Y$ and associated renewal processes $(S_n)_{n\geqslant 0}$ and $(T_n)_{n\geqslant 0}$, respectively, let X^*, Y^* denote two generic random variables having the stationary renewal distributions for the respective channels, defined as

$$\mathbb{P}(X^*\in \mathrm{d}x)=\xi^{-1}\mathbb{P}(X>x)\,\mathrm{d}x \quad\text{and}\quad \mathbb{P}(Y^*\in \mathrm{d}y)=\zeta^{-1}\mathbb{P}(Y>y)\,\mathrm{d}y.$$

Let further $\mathbb{B}=(\mathbb{B}_n)_{n\geqslant 0}$ denote the sequence of backward recurrence times associated with the superposition $(W_n)_{n\geqslant 0}$. This means that $\mathbb{B}_n=(\mathbb{B}_n^X,\mathbb{B}_n^Y)$ gives the elapsed times since the last renewal from channel 1, respectively channel 2 at W_n, in particular $\mathbb{B}_0=(0,0)$. It is well known that $\mathbb{B}$ forms a Markov chain with state space $\mathscr{S}\stackrel{\text{def}}{=}\{0\}\times[0,\infty)\cup[0,\infty)\times\{0\}$. The absolute continuity of X and Y implies that $\mathbb{B}$ is further positive Harris recurrent with unique stationary distribution

$$\pi \stackrel{\text{def}}{=} \frac{\xi}{\xi+\zeta}\mathbb{P}(X^*\in\cdot)\otimes\delta_0+\frac{\zeta}{\xi+\zeta}\delta_0\otimes\mathbb{P}(Y^*\in\cdot), \tag{3.1}$$

where δ_0 is Dirac measure at 0 and $\otimes$ denotes product measure, see [1]. As one can readily verify, the increments of $(W_n)_{n\geqslant 0}$ are conditionally independent given $\mathbb{B}$ and

$$\mathbb{P}(W_n-W_{n-1}\in\cdot|\mathbb{B})=Q(\mathbb{B}_{n-1},\mathbb{B}_n,\cdot)$$

for all $n \geqslant 1$ and a suitable kernel Q, see [1]. Therefore, $(\mathbb{B}_n, W_n)_{n \geqslant 0}$ constitutes a Markov renewal process with Harris recurrent driving chain $\mathbb{B}$.

We next observe that for $n \geqslant 2$

$$\{V_{n-1} \neq V_n\} = \{\mathbb{B}^X_{n-1} = \mathbb{B}^Y_n = 0\} \cup \{\mathbb{B}^Y_{n-1} = \mathbb{B}^X_n = 0\} \quad \text{a.s.} \tag{3.2}$$

The two sets on the right-hand side are a.s. disjoint because the absolute continuity of X and Y in combination with the independence of $(S_n)_{n \geqslant 0}$ and $(T_n)_{n \geqslant 0}$ guarantees that the event $\{W_k = W_{k+1}$ for some $k \geqslant 1\}$ of multiple renewals in the superposition has probability 0. Eq. (3.2) shows that, conditioned upon $\mathbb{B}$, the indicators $\mathbf{1}_{\{V_{n-1} \neq V_n\}}$ are deterministic and thus independent and that

$$\mathbb{P}(V_{n-1} \neq V_n | \mathbb{B}) = \mathbf{1}_{\{(0,0)\}}(\mathbb{B}^X_{n-1}, \mathbb{B}^Y_n) + \mathbf{1}_{\{(0,0)\}}(\mathbb{B}^Y_{n-1}, \mathbb{B}^X_n) \quad \text{a.s.} \tag{3.3}$$

Since these indicators are the increments of $(R^*_n)_{n \geqslant 1}$ we have proved

Lemma 1. *Under the given assumptions* $(\mathbb{B}_n, R^*_n)_{n \geqslant 1}$ *forms a Markov renewal process.*

Theorem 2. *Under the given assumptions*,

$$\lim_{n \to \infty} \frac{R^*_n}{n} = \frac{2}{\xi + \zeta} \int_0^\infty \mathbb{P}(X > t)\mathbb{P}(Y > t)\, \mathrm{d}t \quad a.s. \tag{3.4}$$

and

$$\lim_{t \to \infty} \frac{R_t}{t} = \frac{2}{\xi\zeta} \int_0^\infty \mathbb{P}(X > t)\mathbb{P}(Y > t)\, \mathrm{d}t \quad a.s. \tag{3.5}$$

Proof. By the strong law of large numbers for Markov renewal processes, R^*_n/n converges a.s. to $E_\pi(R^*_2 - R^*_1) = \mathbb{P}_\pi(V_1 \neq V_2)$, where $\mathbb{P}_\pi$ denotes the probability measure under which $\mathbb{B}$ has initial distribution π and is hence stationary. Now use (3.1) and (3.3) to infer

$$\begin{aligned}\mathbb{P}_\pi(V_1 \neq V_2) &= \mathbb{P}_\pi(\mathbb{B}^X_1 = 0, \mathbb{B}^Y_2 = 0) + \mathbb{P}_\pi(\mathbb{B}^Y_1 = 0, \mathbb{B}^X_2 = 0)\\ &= \frac{\zeta}{\xi + \zeta}\mathbb{P}(X > Y^*) + \frac{\xi}{\xi + \zeta}\mathbb{P}(Y > X^*)\\ &= \frac{2}{\xi + \zeta} \int_0^\infty \mathbb{P}(X > t)\mathbb{P}(Y > t)\, \mathrm{d}t.\end{aligned}$$

Of course, the occurring generic variables X, X^*, Y and Y^* are here assumed to be mutually independent. We have thus proved (3.4). Next, by the elementary renewal theorem (see e.g. [10])

$$\lim_{t \to \infty} \frac{N(t)}{t} = \lim_{t \to \infty} \frac{N_1(t)}{t} + \lim_{t \to \infty} \frac{N_2(t)}{t} = \frac{1}{\xi} + \frac{1}{\zeta} = \frac{\xi + \zeta}{\xi\zeta} \quad \text{a.s.,}$$

which combined with (3.4) yields

$$\frac{R_t}{t} = \frac{R^*_{N(t)}}{t} = \frac{N(t)}{t}\frac{R_{N(t)}}{N(t)} \to \frac{2}{\xi\zeta} \int_0^\infty \mathbb{P}(X > t)\mathbb{P}(Y > t)\, \mathrm{d}t,$$

i.e. (3.5). □

Our next result shows that suitable normalizations of R_n^* and R_t converge to a standard normal distribution. This will subsequently be used to derive asymptotically consistent level α tests for the problem of testing for equal scale parameters.

Theorem 3. *Given the previous assumptions,* $\mathbb{E}X^2 < \infty$ *and* $\mathbb{E}Y^2 < \infty$,

$$\hat{R}_n^* \stackrel{\text{def}}{=} \frac{R_n^* - \mu n}{\sigma\sqrt{n}} \stackrel{\text{d}}{\to} N(0,1), \quad \text{as } n \to \infty \tag{3.6}$$

and

$$\hat{R}_t \stackrel{\text{def}}{=} \frac{R_t - \mu N(t)}{\sigma\sqrt{N(t)}} \stackrel{\text{d}}{\to} N(0,1), \quad \text{as } t \to \infty, \tag{3.7}$$

where $\mu \stackrel{\text{def}}{=} \frac{2}{\xi+\zeta}\int_0^\infty \mathbb{P}(X > t)\mathbb{P}(Y > t)\,\mathrm{d}t$ *and*

$$\sigma^2 \stackrel{\text{def}}{=} \mu(1-\mu) + 2\sum_{n\geqslant 2}(\mathbb{P}_\pi(V_1 \neq V_2, V_n \neq V_{n+1}) - \mu^2) > 0.$$

The proof of this result is not only more difficult than the one of its counterpart Theorem 1 in the Poisson case but also rather long. It is therefore provided in the next section.

Testing for equal scale parameters: Consider an absolutely continuous positive random variable Z with mean $EZ = 1$. Let us assume that, for $\xi, \zeta > 0$, the X_i are distributed as ξZ and the Y_i are distributed as ζZ. We want to consider the problem of testing for equal scale parameters $\xi = \zeta$ based on the run statistic R_n^*. Clearly, R_n^* does not change if we multiply the X_i's and Y_i's by the same positive constant. Hence the limiting constant of (3.4)

$$\kappa(\xi,\zeta) \stackrel{\text{def}}{=} \frac{2}{\xi+\zeta}\int_0^\infty \mathbb{P}(\xi Z > t)\mathbb{P}(\zeta Z > t)\,\mathrm{d}t \tag{3.8}$$

depends on ξ and ζ only through their ratio $\rho \stackrel{\text{def}}{=} \zeta/\xi$ or, equivalently, $p \stackrel{\text{def}}{=} \xi/(\xi+\zeta) = 1/(\rho+1)$ and may be written as

$$\mu(p) \stackrel{\text{def}}{=} \kappa\left(\frac{1-p}{p}, 1\right) = 2p\int_0^\infty \mathbb{P}(Z > t)\mathbb{P}\left(\frac{1-p}{p}Z > t\right)\mathrm{d}t, \tag{3.9}$$

which is also immediate by a change of variables in the integral in (3.8). The obvious inequality

$$\mu(p) \leqslant 2p \min\left\{1, \frac{1-p}{p}\right\}$$

shows that $\mu(p)$ becomes small whenever p is close to its boundary values 0 or 1. In fact, $\mu(p)$ attains its absolute maximum at $p = \frac{1}{2}$ which provides the basis for using the run statistic for discrimination purposes.

Lemma 2. *Let Z be a positive random variable with finite mean and* $\mu(p)$ *be as defined in* (3.9) *for* $p \in (0, 1)$. *Then*

$$\mu(p) < \mu(1/2) = \int_0^\infty \mathbb{P}(Z > t)^2 \, \mathrm{d}t$$

for all $p \neq 1/2$.

Proof. Using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
\mu(p) &\leqslant 2p\left(\int_0^\infty \mathbb{P}(Z > t)^2 \, \mathrm{d}t\right)^{1/2}\left(\int_0^\infty \mathbb{P}\left(\frac{1-p}{p} Z > t\right)^2 \mathrm{d}t\right)^{1/2} \\
&= 2p\left(\int_0^\infty \mathbb{P}(Z > t)^2 \, \mathrm{d}t\right)^{1/2}\left(\frac{1-p}{p}\int_0^\infty \frac{p}{1-p}\mathbb{P}\left(Z > \frac{pt}{1-p}\right)^2 \mathrm{d}t\right)^{1/2} \\
&= 2p^{1/2}(1-p)^{1/2}\int_0^\infty \mathbb{P}(Z > t)^2 \, \mathrm{d}t \\
&= 2p^{1/2}(1-p)^{1/2}\mu(1/2).
\end{aligned}$$

Since $2p^{1/2}(1-p)^{1/2}$ has its unique maximum 1 at $p = \frac{1}{2}$ the lemma is proved. □

Level α run tests for $p = \xi/(\xi + \zeta) = \frac{1}{2}$ against $p \neq \frac{1}{2}$, either based upon a sample of n arrivals or upon the $N(t)$ arrivals within a time interval $(0, t]$, can now be defined along the same lines as in Section 2 for the Poisson case. We write $\mathbb{P}_p$ for the situation that p is the underlying parameter. For $\alpha \in (0, 1)$ the critical value here takes the form

$$c(n, \alpha) = \mu(1/2)n - \upsilon_\alpha \sigma(1/2)\sqrt{n},$$

where as before υ_α is the α-fractile of the standard normal distribution. The following corollary shows that $\varphi_n \stackrel{\text{def}}{=} \mathbf{1}_{\{R_n^* < c(n,\alpha)\}}$ and $\phi_t \stackrel{\text{def}}{=} \mathbf{1}_{\{R_t < c(N(t),\alpha)\}}$ are again asymptotically consistent level α tests.

Corollary 2. *In the described situation of testing for equal scale parameters we have as* $n \to \infty$, *respectively* $t \to \infty$:

(i) $\mathbb{P}_{1/2}(R_n^* < c(n, \alpha)) \to \alpha$ *and* $\mathbb{P}_{1/2}(R_t < c(N(t), \alpha)) \to \alpha$,
(ii) $\mathbb{P}_p(R_n^* < c(n, \alpha)) \to 1$ *and* $\mathbb{P}_p(R_t < c(N(t), \alpha)) \to 1$ *for any* $p \neq 1/2$.

The proof is essentially a copy of the proof of Corollary 1 when substituting Theorem 1 with Theorem 3 there. It is therefore omitted.

4. Proof of Theorem 3

We begin with some further notation and put $\mathbb{P}_{x,y} \stackrel{\text{def}}{=} \mathbb{P}(\cdot | \mathbb{B}_0^X = x, \mathbb{B}_0^Y = y)$ (so $\mathbb{P} = \mathbb{P}_{0,0}$) with expectation operator $\mathbb{E}_{x,y}$. The transition kernel of $(\mathbb{B}_n)_{n \geqslant 0}$ is denoted by P and we write $Pg(x, y)$ for

$\int g(u, v)P((x, y), \mathrm{d}(u, v))$. For a measurable $A \subset \mathscr{S}$, the first hitting and first return time of $(\mathbb{B}_n)_{n \geqslant 0}$ to A are denoted as $\tau_0(A)$ and $\tau(A)$, respectively, i.e.

$$\tau_0(A) \stackrel{\text{def}}{=} \inf\{n \geqslant 0 : \mathbb{B}_n \in A\} \quad \text{and} \quad \tau(A) \stackrel{\text{def}}{=} \inf\{n \geqslant 1 : \mathbb{B}_n \in A\}.$$

Proof of Theorem 3. Note that

$$\hat{R}_n^* = \frac{\Sigma_n(h)}{\sigma\sqrt{n}} \stackrel{\text{def}}{=} \frac{\sum_{k=1}^n h(\mathbb{B}_k, R_k^* - R_{k-1}^*)}{\sigma\sqrt{n}}$$

with $h(x, y, e) \stackrel{\text{def}}{=} e - \mu$, whence (3.6) and (3.7) are central limit theorems for an additive functional of the temporally homogeneous Markov chain $(\mathbb{B}_n, R_n^* - R_{n-1}^*)_{n \geqslant 0}$ with state space $\mathscr{S} \times \{0, 1\}$. Its transition kernel $\tilde{P}((x, y, e), \cdot)$, say, is independent of $e \in \{0, 1\}$ because, by Lemma 1, $(\mathbb{B}_n, R_n^*)_{n \geqslant 0}$ is a Markov renewal process. This property implies that $(\mathbb{B}_n, R_n^* - R_{n-1}^*)_{n \geqslant 0}$ inherits the Harris ergodicity from $(\mathbb{B}_n)_{n \geqslant 0}$ and that $A \times \{0, 1\}$ is a small set (see [8, p. 106]) whenever A is a small set for $(\mathbb{B}_n)_{n \geqslant 0}$. Let $\tilde{\pi}$ be the stationary distribution of $(\mathbb{B}_n, R_n^* - R_{n-1}^*)_{n \geqslant 0}$.

We will conclude (3.6) and the asserted form of σ^2 from Theorem 17.5.3 in [8] after the verification of the drift condition

$$P\tilde{\mathbb{G}}(x, y, e) - \tilde{\mathbb{G}}(x, y, e) \leqslant -1 + b\mathbf{1}_{\tilde{C}}(x, y, e), \quad (x, y, e) \in \mathscr{S} \times \{0, 1\} \tag{4.1}$$

for $(\mathbb{B}_n, R_n^* - R_{n-1}^*)_{n \geqslant 0}$, where the function $\tilde{\mathbb{G}} \geqslant 1$ satisfies $\int \tilde{\mathbb{G}}^2 \,\mathrm{d}\tilde{\pi} < \infty$, $\tilde{C}$ is a small set and $b \in (0, \infty)$ a constant. For the proof of (3.7) we will first show that $\Sigma_{N(t)}(h)$ has the same limiting behavior as another additive functional possessing stationary, 1-dependent increments. Asymptotic normality of this second functional is then rather easily obtained by an application of Anscombe's theorem. The positivity of σ^2 will be proved in Lemma 6 in Section 4.

Proof of (3.6): Lemmas 3 and 4 below show that the set $C_a \stackrel{\text{def}}{=} \{0\} \times (0, a]$ is small for $(\mathbb{B}_n)_{n \geqslant 0}$ and satisfies $\sup_{(x,y) \in \mathscr{S}} \mathbb{E}_{x,y}\tau(C_a) < \infty$ for each $a > 0$ with $\mathbb{P}(Y > a) > 0$. These two facts imply that C_a is (1-)regular (see [8, p. 333 and Theorem 14.2.4 on p. 339]). Consequently, by Theorem 14.2.3 in [8], the drift condition

$$P\mathbb{G}_a(x, y) - \mathbb{G}_a(x, y) \leqslant -1 + b\mathbf{1}_{C_a}(x, y), \quad (x, y) \in \mathscr{S} \tag{4.2}$$

holds true for some $b \in (0, \infty)$, where $\mathbb{G}_a(x, y) \stackrel{\text{def}}{=} \mathbb{E}_{x,y}\tau_0(C_a)$ for $(x, y) \in \mathscr{S}$. Note that $\tau_0(C_a) \leqslant \tau(C_a)$ and Lemma 4 ensure that $\mathbb{G}_a$ is a bounded function with supremum $\|\mathbb{G}_a\|_\infty$. Putting $\mathbb{V}_a \stackrel{\text{def}}{=} \mathbf{1} + \mathbb{G}_a$, (4.2) even implies the stronger geometric drift condition

$$P\mathbb{V}_a - \mathbb{V}_a \leqslant -1 + b\mathbf{1}_{C_a} \leqslant -\lambda\mathbb{V}_a + b\mathbf{1}_{C_a} \tag{4.3}$$

with $\lambda \stackrel{\text{def}}{=} (1 + \|\mathbb{G}_a\|_\infty)^{-1} \in (0, 1)$ and therefore the geometric ergodicity of $(\mathbb{B}_n)_{n \geqslant 0}$, see [8, Theorem 15.0.1].

Next put $\tilde{\mathbb{G}}_a(x, y, e) \stackrel{\text{def}}{=} \mathbb{G}_a(x, y)$ for $(x, y, e) \in \mathscr{S} \times \{0, 1\}$ and observe that $\tilde{P}\tilde{\mathbb{G}}_a = P\mathbb{G}_a$. Combining this fact with (4.2) we infer validity of (4.1) with $\tilde{\mathbb{G}} = \tilde{\mathbb{G}}_a$ and $\tilde{C} = C_a \times \{0, 1\}$ for any a with $\mathbb{P}(Y > a) > 0$. Furthermore $\int \tilde{\mathbb{G}}_a^2 \,\mathrm{d}\tilde{\pi} = \int \mathbb{G}_a^2 \,\mathrm{d}\pi < \infty$ trivially holds by the boundedness of $\mathbb{G}_a$. Since $(R_n^*)_{n \geqslant 0}$ has increments bounded by 1, we conclude (3.6) and the asserted form of σ^2 from Theorem 17.5.3 in [8].

Proof of (3.7): Using Nummelin's split chain (see [8, p. 101f]), each small set induces a renewal process $(v_n)_{n\geqslant 1}$ of regeneration epochs for $(\mathbb{B}_n, R_n^* - R_{n-1}^*)_{n\geqslant 0}$ such that, under every initial distribution, the cycles $(\mathbb{B}_j, R_j^* - R_{j-1}^*)_{v_{n-1}\leqslant j<v_n}$, $n\geqslant 1$ are 1-dependent and for $n\geqslant 2$ also stationary ($v_0 \stackrel{\text{def}}{=} 0$) with the same distribution as the first cycle under $\mathbb{P}_\phi$, $\phi \stackrel{\text{def}}{=} \mathbb{P}(\mathbb{B}_{v_1} \in \cdot)$. Consequently, $\Sigma_n^*(h) \stackrel{\text{def}}{=} \Sigma_{v_n}(h)$, $n\geqslant 0$, forms a random walk with 1-dependent increments which are further stationary for $n\geqslant 2$. Its stationary drift $\mathbb{E}(\Sigma_2^*(h) - \Sigma_1^*(h))$ equals $\mathbb{E}_\phi v_1 \mathbb{E}_\pi h(\mathbb{B}_1, R_1^*) = 0$. The geometric ergodicity of $(\mathbb{B}_n)_{n\geqslant 0}$ ensures that $\mathbb{E}_\phi v_1^2 < \infty$ (see [8, Theorem 15.0.1]) and thus $\mathbb{E}(\Sigma_2^*(h) - \Sigma_1^*(h))^2 < \infty$because $|h|\leqslant 1$. Put $\tau^*(n) \stackrel{\text{def}}{=} \inf\{k : v_k \geqslant n\}$. Then

$$\frac{|\Sigma_{\tau^*(n)}^*(h) - \Sigma_n(h)|}{\sqrt{n}} \leqslant \frac{v_{\tau^*(n)} - n}{\sqrt{n}} \stackrel{\text{P}}{\to} 0, \quad \text{as } n \to \infty, \tag{4.4}$$

where $\stackrel{\text{P}}{\to}$ means convergence in probability (under every initial distribution). Combining this result with (3.6) we infer

$$\frac{\Sigma_{\tau^*(n)}^*(h)}{\sigma\sqrt{n}} \stackrel{\text{d}}{\to} N(0,1), \quad \text{as } n \to \infty.$$

With $(\Sigma_n^*(h))_{n\geqslant 1}$ having stationary, 1-dependent increments it is not difficult to verify that Anscombe's theorem applies to $\Sigma_{\tau(N(t))}^*(h)$ and gives

$$\frac{\Sigma_{\tau^*(N(t))}^*(h)}{\sigma\sqrt{N(t)}} \stackrel{\text{d}}{\to} N(0,1), \quad \text{as } t \to \infty.$$

The details are omitted. Since (4.4) remains true when n is replaced with $N(t)$ giving

$$\left| \frac{\Sigma_{\tau^*(N(t))}^*(h)}{\sigma\sqrt{N(t)}} - \hat{R}_t \right| \stackrel{\text{P}}{\to} 0 \quad \text{as } t \to \infty,$$

we finally infer (3.7). □

Lemma 3. *The set* $C_a = \{0\} \times (0, a]$ *is small for each* $a > 0$ *with* $\mathbb{P}(Y > a) > 0$.

Proof. If $\mathbb{P}(Y > a) > 0$ then $\mathbb{P}(Y^* \leqslant a) > 0$ and thus $\pi(C_a) = (\zeta/(\xi + \zeta))\mathbb{P}(Y^* \leqslant a) > 0$. For C_a to be small it hence remains to verify that

$$\inf_{u\in(0,a]} \mathbb{P}_{0,u}(\mathbb{B}_k \in \cdot) \geqslant \beta\Gamma$$

for some $k\geqslant 1$, $\beta \in (0, 1]$ and a probability measure Γ concentrated on C_a.

Since, for all $m, n\geqslant 1$ and $u \in (0, \infty)$, S_m and T_n are independent and absolutely continuous with respect to Lebesgue measure $\lambda\!\!\!\lambda$ under $\mathbb{P}_{0,u}$, a technical but straightforward argument shows that for some $a > 0$ there exist $m, n \geqslant 1$ and $\alpha \in (0, 1)$ such that

$$\inf_{u\in(0,a]} \mathbb{P}_{0,u}(S_m - T_n \in \mathrm{d}v \cap (0, a)) \geqslant \alpha \mathbf{1}_{(0,a)}(v)\lambda\!\!\!\lambda(\mathrm{d}v).$$

Consequently,

$$\begin{aligned}\mathbb{P}_{0,u}(\mathbb{B}^X_{m+n}=0,\mathbb{B}^Y_{m+n}\in B) &\geqslant \mathbb{P}_{0,u}(S_m-T_n\in B\cap(0,\infty), Y_{n+1}>S_m-T_n)\\ &=\int_0^\infty \mathbb{P}_{0,u}(S_m-T_n\in B\cap(0,y))\mathbb{P}(Y\in \mathrm{d}y)\\ &\geqslant\int_a^\infty \mathbb{P}_{0,u}(S_m-T_n\in B\cap(0,a))\mathbb{P}(Y\in \mathrm{d}y)\\ &\geqslant \alpha\mathbb{P}(Y>a)\lambda\!\!\lambda(B\cap(0,a))\end{aligned}$$

for all $u\in(0,a]$, i.e.

$$\inf_{u\in(0,a]}\mathbb{P}_{0,u}(\mathbb{B}_{m+n}\in\cdot)\geqslant\alpha\mathbb{P}(Y>a)\delta_0\otimes\lambda\!\!\lambda(\cdot\cap(0,a)).$$

This shows that C_a is a small set whenever a satisfies $\mathbb{P}(Y>a)>0$. □

Lemma 4. *The set* $C_a=\{0\}\times(0,a]$ *satisfies* $\sup_{(x,y)\in\mathscr{S}}\mathbb{E}_{x,y}\tau(C_a)<\infty$ *for each* $a>0$ *with* $\mathbb{P}(Y>a)>0$.

Proof. We first note that, having proved $\sup_{(x,y)\in\mathscr{S}}\mathbb{E}_{x,y}\tau(C_a)<\infty$, regularity is a direct consequence of Theorem 14.2.4 in [8] because C_a is also small.

Let $\|\cdot\|$ denote total variation distance. Fix any a with $\mathbb{P}(Y>a)>0$. By absolute continuity of X and Y,

$$\lim_{t\to\infty}\|\mathbb{P}(S_{N_1(t)+1}-t\in\cdot)-\mathbb{P}(X^*\in\cdot)\|=0,$$

$$\lim_{t\to\infty}\|\mathbb{P}(T_{N_2(t)+1}-t\in\cdot)-\mathbb{P}(Y^*\in\cdot)\|=0,$$

which implies that

$$\inf_{t\geqslant t_0}\mathbb{P}(S_{N_1(t)+1}-t\leqslant a)\geqslant\mathbb{P}(X^*\leqslant a)/2\geqslant 2\gamma>0,$$

$$\inf_{t\geqslant t_0}\mathbb{P}(T_{N_2(t)+1}-t\leqslant a)\geqslant\mathbb{P}(Y^*\leqslant a)/2\geqslant 2\gamma>0$$

for some $t_0>0$, where $\gamma\stackrel{\text{def}}{=}\min\{\mathbb{P}(X^*\leqslant a),\mathbb{P}(Y^*\leqslant a)\}/4$. Choose m large enough so that $\min\{\mathbb{P}(S_m\geqslant t_0),\mathbb{P}(T_m\geqslant t_0)\}\geqslant 1/2$. Define $W_{-1}\stackrel{\text{def}}{=}T_0$, $\hat{W}_0\stackrel{\text{def}}{=}S_{N_1(\hat{W}_{-1})+1}$ and

$$\begin{aligned}&\hat{W}_1\stackrel{\text{def}}{=}S_{N_1(\hat{W}_{-1})+m+1},\quad \hat{W}_2\stackrel{\text{def}}{=}T_{N_2(\hat{W}_1)+1},\quad \hat{W}_3\stackrel{\text{def}}{=}T_{N_2(\hat{W}_1)+m+1},\\ &\hat{W}_4\stackrel{\text{def}}{=}S_{N_1(\hat{W}_3)+1},\quad \hat{W}_5\stackrel{\text{def}}{=}S_{N_1(\hat{W}_3)+m+1},\quad \hat{W}_6\stackrel{\text{def}}{=}T_{N_2(\hat{W}_5)+1},\ldots,\\ &\hat{W}_{0:n}\stackrel{\text{def}}{=}(\hat{W}_0,\ldots,\hat{W}_n)\end{aligned}$$

and

$$D_n\stackrel{\text{def}}{=}\hat{W}_n-\hat{W}_{n-1}$$

for $n \geqslant 1$. Then the conditional distribution under $\mathbb{P}_{x,y}$ of D_{2n} given $\hat{W}_{0:2n-1}$ depends only on (D_{2n-2}, D_{2n-1}) and $\mathbb{P}_{x,y}(D_{2n} \in \cdot | D_{2n-2}=u, D_{2n-1}=v)$ either equals $\mathbb{P}(S_{N_1(u+v)+1}-u-v \in \cdot)$ or $\mathbb{P}(T_{N_2(u+v)+1}-u-v \in \cdot)$ for $u, v > 0$, $(x, y) \in \mathscr{S}$ and $n \geqslant 1$. Furthermore, D_{2n-1} is independent of $\hat{W}_{0:2n-2}$ for $n \geqslant 1$, and its distribution (under each $\mathbb{P}_{x,y}$) equals either that of S_m or of T_m (under $\mathbb{P} = \mathbb{P}_{0,0}$). Noting that $\mathbb{E}X^2 < \infty$ and $\mathbb{E}Y^2 < \infty$ ensures

$$\int_0^\infty \sup_{t \geqslant 0} \mathbb{P}(S_{N_1(t)+1} - t > s)\,\mathrm{d}s < \infty$$

and a similar result for $\sup_{t \geqslant 0} \mathbb{P}(T_{N_2(t)+1} - t > s)$, see e.g. [13, Theorem 2.4], we hence infer the existence of an integrable distribution G on $[0, \infty)$ such that

$$\mathbb{P}_{x,y}(D_n > t | \hat{W}_{0:n-1}) \leqslant 1 - G(t) \quad \text{a.s.} \tag{4.5}$$

for all $t > 0$, $(x, y) \in \mathscr{S}$ and $n \geqslant 1$. By choice of t_0 and m, we further obtain

$$\begin{aligned}
\mathbb{P}_{x,y}\left(\min_{1 \leqslant k \leqslant n} D_{4k} > a\right) &= \int_{(a,\infty)} \int_{(0,\infty)} \mathbb{P}_{x,y}(D_{4n} > a | D_{4n-2} = u, D_{4n-1} = v) \\
&\quad \times \mathbb{P}_{x,y}(D_{4n-1} \in \mathrm{d}v) \mathbb{P}_{x,y}\left(D_{4n-2} \in \mathrm{d}u, \min_{1 \leqslant k \leqslant n-1} D_{4k} > a\right) \\
&\leqslant ((1 - 2\gamma)\mathbb{P}_{x,y}(D_{4n-1} > t_0) + \mathbb{P}_{x,y}(D_{4n-1} \leqslant t_0)) \\
&\quad \times \mathbb{P}_{x,y}\left(\min_{1 \leqslant k \leqslant n-1} D_{4k} > a\right) \\
&= (1 - 2\gamma\mathbb{P}_{x,y}(D_{4n-1} > t_0))\mathbb{P}_{x,y}\left(\min_{1 \leqslant k \leqslant n-1} D_{4k} > a\right) \\
&\leqslant (1 - \gamma)\mathbb{P}_{x,y}\left(\min_{1 \leqslant k \leqslant n-1} D_{4k} > a\right) \\
\cdots &\leqslant (1 - \gamma)^n
\end{aligned}$$

for all $n \geqslant 1$ and $(x, y) \in \mathscr{S}$. We thus see that $\hat{\tau} \stackrel{\text{def}}{=} \inf\{n : D_{4n} \leqslant a\}$ has geometrically decreasing tails of order less than $1 - \gamma$ under each $\mathbb{P}_{x,y}$, $(x, y) \in \mathscr{S}$. In particular,

$$\sup_{(x,y) \in \mathscr{S}} \mathbb{E}_{x,y} \hat{\tau} < \infty. \tag{4.6}$$

Now $\tau(C_a) \stackrel{\text{def}}{=} \inf\{n \geqslant 1 : \mathbb{B}_n \in C_a\}$ is clearly bounded by $N_1(\hat{W}_{4\hat{\tau}})$ which may be rewritten as

$$\tau(C_a) \leqslant \sum_{k=1}^{4\hat{\tau}} N_1(\hat{W}_{k-1}, \hat{W}_k], \tag{4.7}$$

where $N_1(s, t] \stackrel{\text{def}}{=} N_1(t) - N_1(s)$ for $s \leqslant t$. We claim that there exists an integrable distribution H such that

$$\mathbb{P}_{x,y}(N_1(\hat{W}_{k-1}, \hat{W}_k] > n | \hat{W}_{0:k-1}) \leqslant 1 - H(n) \quad \text{a.s.} \tag{4.8}$$

for all $k, n \geqslant 1$. For the proof we will use (4.5) and the inequality

$$\mathbb{P}_{x,y}(N_1(s, s+t] > n) \leqslant \mathbb{P}(N_1(t) > n-1) \tag{4.9}$$

for all $(x, y) \in \mathscr{S}$, $s, t \geqslant 0$ and $n \geqslant 1$, see e.g. [3, p. 810]. Now it is readily seen that $N_1(\hat{W}_{k-1}, \hat{W}_k]$ either equals *m* or 1, or satisfies

$$\begin{aligned} &\mathbb{P}_{x,y}(N_1(\hat{W}_{k-1}, \hat{W}_k] > n | \hat{W}_{0:k-1}) \\ &\quad = \int_{[0,\infty)} \int_{[0,\infty)} \mathbb{P}_{x,y}(N_1(\hat{W}_{k-1}, \hat{W}_{k-1}+t] > n) \mathbb{P}_{x,y}(D_k \in \mathrm{d}t | \hat{W}_{0:k-1}) \\ &\quad \leqslant \int_{[0,\infty)} \mathbb{P}(N_1(t) > n-1) \mathbb{P}_{x,y}(D_k \in \mathrm{d}t | \hat{W}_{0:k-1}) \quad \text{a.s.,} \end{aligned}$$

where (4.9) was used for the last inequality. Now use (4.5) and the fact that $\mathbb{P}(N_1(t) > n-1)$ is increasing in *t* to conclude that either $N_1(\hat{W}_{k-1}, \hat{W}_k] \in \{1, m\}$, or

$$\mathbb{P}_{x,y}(N_1(\hat{W}_{k-1}, \hat{W}_k] > n) \leqslant \int_{[0,\infty)} \mathbb{P}(N_1(t) > n-1) G(\mathrm{d}t).$$

This proves (4.8) for some distribution *H*, and since

$$\sum_{n \geqslant 0} \mathbb{P}(N_1(t) > n) \leqslant \mathbb{E} N_1(t) \leqslant c(t+1)$$

for a suitable constant $c \in (0, \infty)$, we further see that *H* can be chosen as an integrable distribution with mean μ_H, say.

Finally, by combining (4.6), (4.8) and Lemma 5 below, we obtain

$$\sup_{(x,y) \in \mathscr{S}} \mathbb{E}_{x,y} \tau(C_a) \leqslant 2\mu_H \sup_{(x,y) \in \mathscr{S}} \mathbb{E}_{x,y} \hat{\tau} < \infty,$$

which is the asserted result. □

Lemma 5. *Let* $0 = Z_0 \leqslant Z_1 \leqslant Z_2 \leqslant \cdots$ *be an increasing sequence of nonnegative random variables whose increments* $Z_n - Z_{n-1}$ *are stochastically bounded by an integrable distribution, i.e., there exists a distribution function H such that* $H(0) = 0$, $\mu_H \stackrel{\text{def}}{=} \int_0^\infty (1 - H(t))\, \mathrm{d}t < \infty$ *and*

$$\mathbb{P}(Z_n - Z_{n-1} > t | Z_0, \ldots, Z_{n-1}) \leqslant 1 - H(t) \quad a.s. \tag{4.10}$$

for all $t \geqslant 0$ *and* $n \geqslant 1$. *Then*

$$\mathbb{E} Z_\tau \leqslant \mu_H \mathbb{E} \tau$$

for each stopping time τ *for* $(S_n)_{n \geqslant 0}$.

Proof. Integration of (4.10) with respect to t gives $\mathbb{E}(Z_n - Z_{n-1}|Z_1, \ldots, Z_{n-1}) \leqslant \mu_H$ a.s. for all $n \geqslant 1$. Hence the assertion follows from

$$\begin{aligned}\mathbb{E}Z_\tau &= E\left(\sum_{n\geqslant 1} \mathbb{E}(Z_n - Z_{n-1}|Z_1, \ldots, Z_{n-1})\mathbf{1}_{\{\tau \geqslant n\}}\right)\\ &\leqslant \mathbb{E}\left(\sum_{n\geqslant 1} \mu_H \mathbf{1}_{\{\tau\geqslant n\}}\right)\\ &= \mu_H \mathbb{E}\tau. \qquad \square\end{aligned}$$

Lemma 6. *The asymptotic variance* σ^2 *in Theorem* 3 *is positive*.

Proof. For each a with $\mathbb{P}(Y > a) > 0$, the set $C_a = \{0\} \times 0, a]$ is small for $(\mathbb{B}_n)_{n\geqslant 0}$ (Lemma 3). It is also small for the chain $(\mathbb{B}_n, R_n^* - R_{n-1}^*)_{n\geqslant 1}$ because $(\mathbb{B}_n, R_n^* - R_{n-1}^*)$ depends on $(\mathbb{B}_j, R_j^* - R_{j-1}^*)_{1\leqslant j\leqslant n-1}$ only through $\mathbb{B}_{n-1}$. For a fixed and $k \geqslant 1$, let $c_k \in [0, 1]$ be the maximal value so that

$$\inf_{u\in(0,a]} \mathbb{P}_{(0,u)}(\mathbb{B}_k \in \cdot) \geqslant c_k \delta_0 \otimes \lambda\!\!\!\lambda(\cdot \cap (0, a]).$$

Choose $k \geqslant 1$ and $e \in \{0, 1\}$ such that $c_k > 0$ and

$$D_a \stackrel{\text{def}}{=} \left\{ y \in (0, a] : \int_{(0,a]} \mathbb{P}_{0,x}(R_k^* - R_{k-1}^* = e|\mathbb{B}_k = (0, y))\lambda\!\!\!\lambda(\mathrm{d}x) \geqslant a/2 \right\}$$

is $\lambda\!\!\!\lambda$-positive. It follows

$$\begin{aligned}&\mathbb{P}_{0,u}(\mathbb{B}_{2k}^X = 0, \mathbb{B}_{2k}^Y \in A, R_{2k}^* - R_{2k-1}^* = e)\\ &\quad\geqslant c_k^2 \int_{A\cap(0,a]} \int_{(0,a]} \mathbb{P}_{0,u}(R_{2k}^* - R_{2k-1}^* = e|\mathbb{B}_k = (0, x), \mathbb{B}_{2k} = (0, y))\lambda\!\!\!\lambda(\mathrm{d}x)\lambda\!\!\!\lambda(\mathrm{d}y)\\ &\quad= c_k^2 \int_{A\cap(0,a]} \int_{(0,a]} \mathbb{P}_{0,x}(R_k^* - R_{k-1}^* = e|\mathbb{B}_k = (0, y))\lambda\!\!\!\lambda(\mathrm{d}x)\lambda\!\!\!\lambda(\mathrm{d}y)\\ &\quad\geqslant \frac{ac_k^2}{2}\lambda\!\!\!\lambda(A \cap D_a)\end{aligned}$$

for all $u \in (0, a]$ and all measurable $A \subset \mathbb{R}$ and thus with $\Psi \stackrel{\text{def}}{=} \mathbf{1}_{D_a}(y)\lambda\!\!\!\lambda(\mathrm{d}y)/\lambda\!\!\!\lambda(D_a)$.

$$\inf_{u\in(0,a]} \mathbb{P}_{0,u}(\mathbb{B}_{2k}^X = 0, \mathbb{B}_{2k}^Y \in \mathrm{d}y, R_{2k}^* - R_{2k-1}^* = e) \geqslant c_{2k}^* \Psi(\mathrm{d}y) \tag{4.11}$$

for some $c_{2k}^* > 0$.

Now let $\mathcal{N}$ be the set of all $l \in \mathbb{N}$ for which (4.11) holds true if $2k$ is replaced with l and c_{2k}^* with some $c_l^* > 0$ (keeping e and D_a fixed). We claim that $\mathcal{N}$ contains $l_0 + \mathbb{N}$ for some $l_0 \geqslant 1$. In fact, since $(\mathbb{B}_n, R_n^* - R_{n-1}^*)_{n\geqslant 1}$ is aperiodic, we have that $\{l \geqslant 1 : \mathbb{P}_\Psi(\mathbb{B}_l \in C_a) > 0\}$ contains $l_1 + \mathbb{N}$ for some $l_1 \geqslant 1$. Consequently, for all $l \geqslant 2k + l_1$ and all $u \in (0, a]$

$$\mathbb{P}_{0,u}(\mathbb{B}_l \in C_a) \geqslant \mathbb{P}_{0,u}(\mathbb{B}_{2k} \in C_a, \mathbb{B}_l \in C_a) \geqslant c_{2k}^* \mathbb{P}_\Psi(\mathbb{B}_{l-2k} \in C_a) > 0$$

and then, for all $l \geqslant l_0 \overset{\text{def}}{=} 4k + l_1$, $u \in (0, a]$ and all measurable $A \subset \mathbb{R}$,

$$\begin{aligned}
&\mathbb{P}_{0,u}(\mathbb{B}_l^X = 0, \mathbb{B}_l^Y \in A, R_l^* - R_{l-1}^* = e) \\
&\quad \geqslant \int_{(0,u]} \mathbb{P}_{0,x}(\mathbb{B}_{2k}^X = 0, \mathbb{B}_{2k}^Y \in A, R_{2k}^* - R_{2k-1}^* = e)\mathbb{P}_{0,u}(\mathbb{B}_{l-2k} \in \{0\} \times \mathrm{d}y) \\
&\quad \geqslant (c_{2k}^*)^2 \mathbb{P}_{\Psi}(\mathbb{B}_{l-4k} \in C_a)\Psi(A)
\end{aligned}$$

which proves our claim.

Let $\tilde{\pi}$ be the stationary distribution of $(\mathbb{B}_n, R_n^* - R_{n-1}^*)_{n \geqslant 1}$. Since C_a is a small set for $(\mathbb{B}_n)_{n \geqslant 0}$ with minorizing measure $\delta_0 \otimes \Psi$, it is well known that

$$\tilde{\pi} = \frac{1}{\mathbb{E}_{\delta_0 \otimes \Psi} \tau} \mathbb{E}_{\delta_0 \otimes \Psi} \left(\sum_{k=0}^{\tau-1} \mathbf{1}_{\{(\mathbb{B}_k, R_k^* - R_{k-1}^*) \in \cdot\}} \right)$$

for some regeneration epoch τ (see e.g. [2] for the construction) and thus $\tilde{\pi} \geqslant c(\delta_0 \otimes \Psi \otimes \delta_e)$ for $c \overset{\text{def}}{=} (\mathbb{E}_{\delta_0 \otimes \Psi} \tau)^{-1} > 0$. A combination with (4.11), with $l \in \mathcal{N}$ instead of $2k$, then implies

$$\mathbb{P}_{\tilde{\pi}}((\mathbb{B}_1, R_1^*, \mathbb{B}_{l+1}, R_{l+1}^* - R_l^*) \in \cdot) \geqslant \tilde{c}_l (\delta_0 \otimes \Psi \otimes \delta_e)^2 \tag{4.12}$$

for all $l \in \mathcal{N}$ and suitable $\tilde{c}_l > 0$.

Now assume $\sigma^2 = 0$ and observe that $\mu \in (0, 1)$. Since $(\mathbb{B}_n, R_n^* - R_{n-1}^*)_{n \geqslant 1}$ is Harris ergodic and satisfying the drift condition (4.1) with bounded $\tilde{\mathbb{G}}$, Proposition 2.4 in [6] implies the existence of a measurable function $\Lambda : \mathcal{S} \times \mathbb{N}_0 \to \mathbb{R}$ such that

$$\Sigma_n(h) = \Lambda(\mathbb{B}_n, R_n^* - R_{n-1}^*) - \Lambda(\mathbb{B}_1, R_1^*) \quad \mathbb{P}_{\tilde{\pi}}\text{-a.s.} \tag{4.13}$$

for all $n \geqslant 1$. Note that $\Sigma_n(h)$ takes values in $\{j - n\mu; 0 \leqslant j \leqslant n\}$. Choose $m \geqslant 1$ such that $m - 1$ and m are both elements of $\mathcal{N}$. Combining (4.12) and (4.13), we see that there exists a value s such that

$$\mathbb{P}_{\tilde{\pi}}(\Sigma_i(h) = s) \geqslant \tilde{c}_i \int \mathbf{1}_{\{s\}}(\Lambda(b_2, r_2) - \Lambda(b_1, r_1))(\delta_0 \otimes \Psi \otimes \delta_e)^2(\mathrm{d}b_1, \mathrm{d}r_1, \mathrm{d}b_2, \mathrm{d}r_2) > 0$$

for both $i = m$ and $i = m + 1$. Hence there must be $m_1, m_2 \in \mathbb{N}_0$ such that $s = m_1 - m\mu = m_2 - (m + 1)\mu$, i.e. $\mu = m_2 - m_1 \in \mathbb{Z}$. Since $\mu \in (0, 1)$, we have produced a contradiction. □

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