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Pseudo-Abelian integrals along Darboux cycles: A codimension one case ☆

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ABSTRACT

We investigate a polynomial perturbation of an integrable, non-Hamiltonian system with first integral of Darboux type. In the paper [M. Bobieński, P. Mardešić, Pseudo-Abelian integrals along Darboux cycles, Proc. Lond. Math. Soc., in press] the generic case was studied. In the present paper we study a degenerate, codimension one case. We consider 1-parameter unfolding of a non-generic case. The main result of the paper is an analog of Varchenko–Kchovanskii theorem for pseudo-Abelian integrals.

1. Introduction

We consider a polynomial planar vector field X having a Darboux type first integral:

$$H = H_\epsilon = (x-\epsilon)^a P \prod_{j=1}^{k} P_j^{a_j}, \quad P, P_j \in \mathbb{R}[x,y], \; a, a_j \in \mathbb{R}_+. \tag{1.1}$$

The integrable polynomial vector field X has the form

$$X = \frac{1}{M_\epsilon}(H_y \partial_x - H_x \partial_y), \tag{1.2}$$

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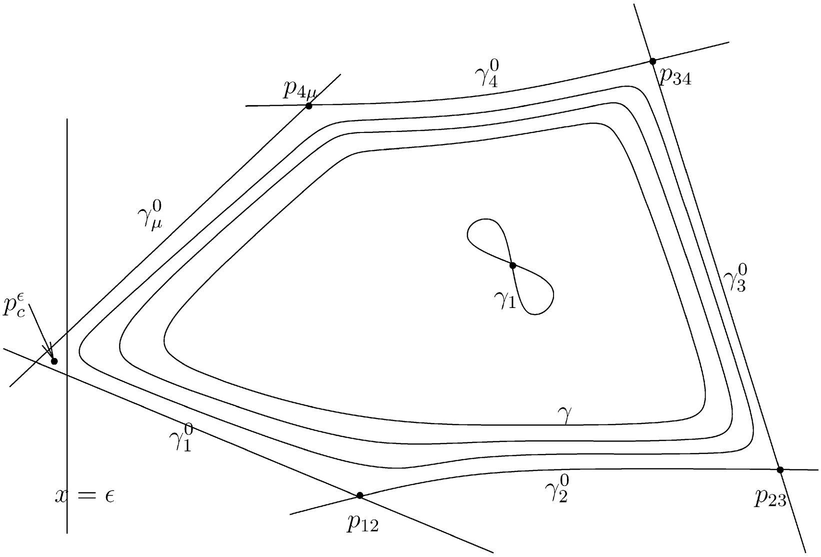

Fig. 1.1. The phase portrait of (1.2). The period annulus D bounded by polycycles γ_0 and γ_1.

where $M_\epsilon = (x-\epsilon)^{a-1}\prod_{j=1}^k P_j^{a_j-1}$ is the integrating factor. The phase portrait of the vector field (1.2) in the case we study in this paper is shown in Fig. 1.1. Let D be the open period annulus whose closure intersects the zero level curve $H^{-1}(0)$. Let the polycycle $\gamma_0 \subset H^{-1}(0)$ be a corresponding part of boundary of D. This polycycle γ_0 consists of edges γ_μ^0 which meet at vertices $p_{\mu\nu}$—see Fig. 1.1.

Now we consider a small polynomial deformation of X

$$X + \varepsilon Y, \qquad Y = R\partial_x + S\partial_y, \quad R, S \in \mathbb{R}[x, y]. \tag{1.3}$$

In general, most of the periodic orbits are broken, so to investigate limit cycles bifurcating from the annulus D, we consider the respective displacement function $\Delta(h)$

$$\Delta(h) = -\varepsilon \int\limits_{\gamma_h} M_\epsilon (S\,\mathrm{d}x - R\,\mathrm{d}y) + o(\varepsilon)$$

and its linearization given by the pseudo-Abelian integral. By the implicit function theorem, the limit cycles bifurcating in a compact domain $K \subset D$ are given by zeroes of the pseudo-Abelian integral

$$I(\epsilon, h) = I_\epsilon(h) = \frac{1}{h}\int\limits_{\gamma_h} M_\epsilon (S\,\mathrm{d}x - R\,\mathrm{d}y) = \int\limits_{\gamma_h} \frac{S\,\mathrm{d}x - R\,\mathrm{d}y}{(x-\epsilon)P\prod_{j=1}^k P_j}. \tag{1.4}$$

The aim of this paper is to prove the existence of local upper bound for the number of zeroes of pseudo-Abelian integral $I(h)$. This is an analog of the Varchenko–Kchovanskii theorem for pseudo-Abelian integrals. In the previous papers [1,7] the generic case was investigated. In this paper we consider a 1-parameter unfolding of the singular (non-generic) codimension 1 case. Another non-generic Darboux case was studied in [2].

Let us recall some definitions, notation and general results from [1]. They will be useful later.

Definition 1.1. The Darboux function H given by (1.1) is *regular at infinity* if any (complex) level curve of the polynomial $(x-\epsilon)PP_1\cdots P_k$ is regular at infinity, i.e. crosses the line at infinity transversally in $1 + \deg P + \deg P_1 + \cdots + \deg P_k$ distinct points.

The condition for function to be regular at infinity is stable under a small perturbation of H (inside the Darboux type with fixed number of factors). This condition is satisfied by generic k-tuples of polynomials. It was proved in [1] that under this hypothesis the pseudo-Abelian integrals I are endlessly continuable [3–5]. That is, each path of length $\ell \leqslant \infty$ can meet only a finite set of singular points. This means that on the Riemann surface of the integral I the ramification points are discrete, even though their projection to the complex plane can have accumulation points. Moreover, all possible ramification points are located on finite number of circles centered at 0. The last statement is a consequence of fact that all ramification points, except 0, correspond to the critical points of the polynomial vector field X. Due to different determinations of the Darboux function H any critical point can generate infinite sequence of ramification points. Since $|H|$ is a uni-valued function, all of these determinations are located on the circle.

We prove the existence of local bound for the number of zeroes of pseudo-Abelian integral. Below we define an open subset in which this property holds. Let $X_k(\epsilon; n_0, n_1, \ldots, n_k; n)$ be the following, finite-dimensional space of polynomial system with Darboux type first integral:

$$X_k(\epsilon; n_0, n_1, \ldots, n_k; n) := \Bigg\{ \frac{1}{M_\epsilon}(H_y \partial_x - H_x \partial_y) + \varepsilon(R\partial_x + S\partial_y): \\ H_\epsilon = (x-\epsilon)^a P \prod_{j=1}^{k} P_j^{a_j},\ \deg P \leqslant n_0,\ \deg P_j \leqslant n_j,\ \deg(R, S) \leqslant n \Bigg\},$$

where $M_\epsilon = (x-\epsilon)^{a-1} \prod_{j=1}^{k} P_j^{a_j - 1}$. The parameters of the space $X_k(\epsilon; n_0, \ldots, n_k; n)$ are positive exponents $(a, a_1, \ldots, a_k)$, coefficients of polynomials P, P_j and coefficients of the polynomial perturbation (R, S). The equation $P(0,0) = 0$ distinguishes a *codimension one* subset $Y_1 \subset X_k(\epsilon; n_0, \ldots, n_k; n)$. We define an open subset $X_k^0(\epsilon; n_0, \ldots, n_k; n) \subset Y_1$ by the condition that the unperturbed system admits a period annulus D with boundary component $\gamma_\epsilon^0 \subset H^{-1}(0)$ and the following genericity assumptions (in Y_1) are satisfied:

(1) The Darboux function H_ϵ is regular at infinity.
(2) For $\epsilon = 0$, the polycycle γ_0^0 consists of edges γ_μ^0 contained in a smooth part of the level curve $P_{j_\mu}^{-1}(0)$ for some j_μ. Any vertex $p_{\mu\nu}$, except $p = (0,0)$, corresponds to the transversal intersection of level curves $P_{j_\mu}^{-1}(0)$ and $P_{j_\nu}^{-1}(0)$.
(3) The polynomial P has a critical point of Morse type $p = (0,0)$, i.e. $P(x,y) = y^2 - x^2 + h.o.t.$ Other polynomials P_j satisfy $P_j(0,0) \neq 0$, $j = 1, \ldots, k$.

Remark 1.2. It is worth to notice that the above genericity assumptions guarantee that for sufficiently small, non-zero ϵ the genericity condition in the sense of [1] is satisfied. Indeed the intersection in all vertices $p_{\mu\nu}$ except $p = (0,0)$ remain transversal for sufficiently small ϵ. The zero vertex $(0,0)$ either becomes transversal or bifurcates into a pair of transversal vertices, depending on the sign of ϵ.

Theorem 1.3. *Let the system $X + \varepsilon Y \in X_k^0(\epsilon; n_0, \ldots, n_k; n)$. Assume that the pseudo-Abelian integral $I(\epsilon, h)$ given by* (1.4) *is not identically zero. There exists an upper bound $Z(X_0; n_0, \ldots, n_k, n)$ for the number of isolated zeros of pseudo-Abelian integrals generated by vector fields in $X_k^0(\epsilon; n_0, \ldots, n_k; n)$ sufficiently close to X.*

2. Proof of Theorem 1.3

The proof is based on Gabrielov's theorem [6] and its continuous generalization recalled in the following lemma.

Lemma 2.1. *Let $U \subset \mathbb{C}$ be an open subset and $I_\epsilon : U \to \mathbb{C}$ be a continuous family of holomorphic functions (in uniform convergence on compacts). Let $l \subset U$ be a path in U. Assume that I_ϵ does not vanish on l. Then the increment of argument along $\Delta \operatorname{Arg}_l I_\epsilon$ is uniformly bounded for sufficiently small ϵ.*

The polycycle D is mapped by H to a segment $(0, h_1)$. We split this segment into three pieces: $(0, r)$, $[r, h_1 - r]$ and $(h_1 - r, h_1)$ for sufficiently small, ϵ*-independent* r. The number of zeroes of I_ϵ on $[r, h_1 - r]$ is locally bounded by Gabrielov's theorem.

To prove the existence of estimate on $(h_1 - r, h_1)$, we observe that the first integral function H (each branch) is holomorphic function in a neighborhood of any point except the zero level curve $H^{-1}(0)$. Thus, the investigation of monodromy and singularities of I_ϵ is analogous to the Hamiltonian case. So, the upper bound for the number of zeroes of pseudo-Abelian integral I_ϵ is a consequence of classical Varchenko–Kchovanskii theorem.

Below we prove the existence of ϵ*-independent* bound for the number of zeroes close to $h = 0$, i.e. on $(0, r)$.

First we investigate the case $\epsilon = 0$. We make the blow-up transformation $y = x\eta$. Due to the assumptions on H made in the definition of the space X_k^0, the first integral H has the following local form

$$H_0(x, \eta) = x^{a+2} \widehat{H}(x, \eta) = x^{a+2} \big(\eta^2 - 1\big) \hat{g}^*(x, \eta),$$

where $\hat{g}^*$ is a germ of holomorphic function such that $\hat{g}^*(0, 0) \neq 0$. Thus, the intersections of respective complex curves in vertices $(0, \pm 1)$ are transversal. The intersections in all other vertices are also transversal by the assumption on H. So, the system satisfies the genericity condition in the sense of [1]. Thus, the number of isolated zeroes of pseudo-Abelian integrals is locally bounded on the hyperplane $\{\epsilon = 0\} \cap X_k^0(\ldots)$. The main idea of the proof of this fact was the following (see [1] for details). In a disk $D(0, r)$ of positive, uniform with respect to parameters radius r the origin is the only singularity (ramification point) of the pseudo-Abelian integral I. Moreover, certain iterated variation of I around $h = 0$ vanishes identically. Using these two properties we are able to control the singularity of I at $h = 0$ uniformly with respect to parameters.

Now, for non-zero ϵ, the integrable Darboux system X_ϵ has an additional singular point p_c^ϵ which tends to $(0, 0)$ as $\epsilon \to 0$. This singular point p_c^ϵ corresponds to a small center bounded by curves $P = 0$ and $x = \epsilon$—see Fig. 1.1. It generates a possible ramification points of I_ϵ located on a circle whose radius is of order $|\epsilon|^{a+2}$—see Lemma 2.2 below. Thus, we cannot directly repeat the argument from [1] to get an ϵ-independent estimation. To overcome this problem we investigate the asymptotic behavior of integral I_ϵ at $h = 0$, using coordinate $u = h/\epsilon^{a+2}$—see Proposition 2.3.

We fix some useful notation for the rescaled variation, iterated of variations and variation with bounded radius.

$$\begin{aligned}
\mathcal{V}ar_\alpha I(h) &= I\big(e^{2\pi i/\alpha} h\big) - I(h), \\
\mathcal{V}ar_\alpha^{r_0} I(h) &= I\big(e^{2\pi i/\alpha} h\big) - I(h), \quad |h| > r_0, \\
\mathcal{V}ar_{\alpha_1, \ldots, \alpha_k} &:= \mathcal{V}ar_{\alpha_1} \circ \mathcal{V}ar_{\alpha_2} \circ \cdots \circ \mathcal{V}ar_{\alpha_k}.
\end{aligned} \tag{2.1}$$

The following lemma describes the set of ramification points of the pseudo-Abelian integral I_ϵ.

Lemma 2.2. *The pseudo-Abelian integral $I_\epsilon(h)$ is an endlessly continuable, multi-valued function on $\mathbb{C}^*$. All ramification points are localized on finite number of circles centered at 0 of radius $r_0(\epsilon), r_1(\epsilon), \ldots, r_N(\epsilon)$. The radius functions $r_j(\epsilon)$ for $j = 1, \ldots, N$ are germs of analytic functions of ϵ and $r_j(0) \neq 0$. Ramification points located on circle of radius r_0 (bifurcating from 0) has the form*

$$h_\mu(\epsilon)(\epsilon) = \epsilon^{a+2} u_\mu(\epsilon), \quad u_\mu(0) \neq 0,$$

where $u_\mu(\epsilon)$ are analytic functions of ϵ.

Moreover, the first branch of integral I_ϵ is holomorphic in a neighborhood of the real segment $(0, r)$.

The following proposition describes the asymptotic analytic behavior of pseudo-Abelian integral I_ϵ in a sufficiently small disc centered at 0 (the radius of disc is of order $R\epsilon^{a+2}$).

Proposition 2.3. *Let $u = h/\epsilon^{a+2}$. Then the variation admits the following decomposition $\mathcal{V}ar_1 I_\epsilon = I_1(\epsilon, h) + I_1(\epsilon, u)$, where the functions f_0 and f_1 depend in an analytic way on ϵ and they have the following analytic properties.*

(1) *The function $I_1(\epsilon, h)$ is holomorphic multi-valued in a disc $D(0, R_1)$; the origin $h = 0$ is the only ramification point in the disc. The growth of I_1 at $h = 0$ is at most polynomial (i.e. $|I_1| \leqslant C|h|^{-N}$ for some constants C, N). Moreover, the integral I_1 satisfies the following variation equation around zero*

$$\mathcal{V}ar_{1/a_1,\dots,1/a_k} I_1 \equiv 0. \tag{2.2}$$

(2) *The function $I_0(\epsilon, u)$ is holomorphic, multi-valued in $u \in D(0, R_0)$. The function I_0 is ramified at the origin and in singularities located close to the circle of radius $\rho_0 < R_0$. The function I_0 in a punctured disc $D(0, \frac{\rho_0}{2})$ is given by a convergent power series*

$$\sum_{n \geqslant n_0} a_n(\epsilon) u^{n/a}.$$

Outside the ramification circle, e.g. in a ring $R(2\rho_0, R_0)$, it is given by

$$\sum_{n \in \mathbb{Z}} b_n(\epsilon) u^{n/(a+2)} = \sum b_n(\epsilon) \epsilon^{-n} h^{n/(a+2)}.$$

Moreover, if the integral $I(\epsilon, h)$ is not identically zero, there exists a pair (A, α), $A \in \mathbb{R}$, $\alpha = 0, 1, \dots, k$, such that $\epsilon^{-A}(\log \epsilon)^{-\alpha} I(\epsilon, u) \xrightarrow{\epsilon \to 0} F(u) \not\equiv 0$. The limit function $F(u)$ is a multi-valued, holomorphic function satisfying the following variation relation

$$\mathcal{V}ar_{1,1/a,1/a_1,\dots,1/a_k} F(u) \equiv 0. \tag{2.3}$$

Proof of Theorem 1.3. The main idea of proof is to reduce problem to the one which was previously considered in [1]. In that paper we have shown that the iterated variation equation of type (2.2) implies local bound for the number of zeroes. In our case, by two additional steps we reduce the problem of estimating the number of zeroes of I_ϵ to the analogous problem for the variation of I_1. The latter function satisfies the iterated variation equation so one can use previously proved results.

In the first step we use the argument principle for the function I_ϵ and the contour Γ_1 shown in Fig. 2.1.

(1) The increment of argument of I_ϵ along big circle C_r is bounded by Gabrielov's theorem.
(2) By Proposition 2.3 the function $\epsilon^{-A} \log^{-l} \epsilon I_\epsilon(\epsilon, \epsilon^{a+2} u)$ (for certain real A and integer l) is non-zero, holomorphic in a neighborhood of $C_{\epsilon^{a+2}}$ and depends in continuous way on ϵ. Thus, the increment of argument along the circle $C_{\epsilon^{a+2}}$ is bounded by Lemma 2.1. The same argument works for these parts of segments $s_\pm$ which are "far" from $u = 0$ (i.e. for $|u| > \rho_0$).
(3) The limit function $F(u) = \lim_{\epsilon \to 0} \epsilon^{-A} \log^{-l} \epsilon I_\epsilon(\epsilon, \epsilon^{a+2} u)$ has a non-zero principal part at $u = 0$ of the form $u^{\alpha_0} \log^j u$, where $\alpha_0 \in \mathbb{R}$, $j \in \mathbb{Z}$, $0 \leqslant j \leqslant k+1$ (see [1]). Thus, for sufficiently small ϵ, the leading term of $I_\epsilon(\epsilon, \epsilon^{a+2} u)$ at $u = 0$ has the form $u^\alpha \log^j u$, where $\alpha \leqslant \alpha_0$, $0 \leqslant j \leqslant k+1$. Since there is only finite number of such terms, there exists an upper bound for the increment of argument along small circle around zero.
(4) To estimate the increment of argument along these pieces of $s_\pm$ which are close to $u = 0$ we use Schwartz's principle $\operatorname{Im}(I_\epsilon)|_{s_\pm} = C_\pm \mathcal{V}ar_{u^\beta} I_\epsilon$, for some constants $C_\pm, \beta$. Thus, the increment of argument is bounded by the number of zeroes of variation. By commutativity of different variations around zero and by Proposition 2.3, the iterated variation $\mathcal{V}ar_{1,1/a,1/a_1,\dots,1/a_k}$ of the

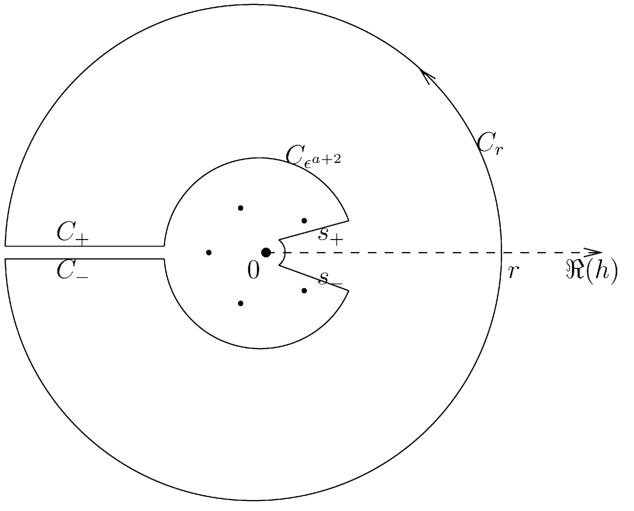

Fig. 2.1. The contour Γ_1.

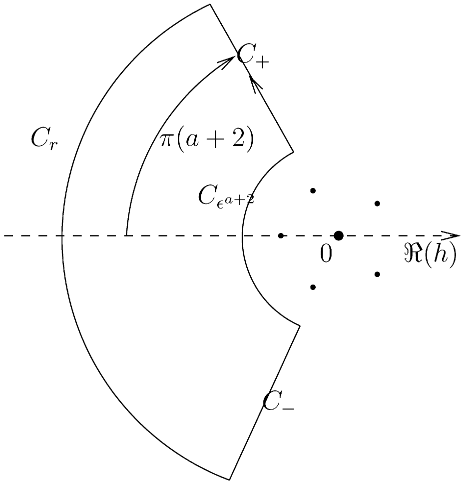

Fig. 2.2. The contour Γ_2.

function $\mathcal{V}ar_{u^\beta} I_\epsilon$ vanishes identically. Thus, using argument principle $k+2$ number of times along series of sector shape contours we prove the existence of the upper bound for the increment of argument. For more details see [1, Proposition 4.2].

(5) Finally, by Schwartz's principle

$$\operatorname{Im}(I_\epsilon)|_{C_\pm} = \mp\frac{1}{2i}\mathcal{V}ar_1^{2\rho_0\epsilon^{a+2}} I_\epsilon,$$

the increments of argument along segments $C_\pm$ are bounded by the zeroes of the variation $\mathcal{V}ar_1^{2\rho_0\epsilon^{a+2}} I_\epsilon$ on segment $(-r, -2\rho_0)$.

In the second step we use the contour Γ_2 presented in Fig. 2.2. By the same arguments as in step one, the increment of argument along arcs C_r and $C_{\epsilon^{a+2}}$ are locally bounded and so the problem reduces to bound the number of zeroes of the iterated variation $\mathcal{V}ar^{2\rho_0\epsilon^{a+2}}_{1,1/(a+2)} I_\epsilon$ on segments $C_\pm$. This variation of the function I_0 vanishes identically by Proposition 2.3 and so

$$\mathcal{V}ar^{2\rho_0\epsilon^{a+2}}_{1,1/(a+2)} I_\epsilon = \mathcal{V}ar_{1,1/(a+2)} I_1.$$

The function $I_1(h)$ is a multi-valued holomorphic function in a disc $D(0,r)$ with 0 as the only ramification point. Moreover, it satisfies the variation relation (2.2). Thus, by Proposition 4.2 from [1], the number of zeroes is locally bounded on the parameter space. □

3. Analytical properties of integrals near $h = 0$

In this section we prove Lemma 2.2 and Proposition 2.3.

Proof of Lemma 2.2. The fact that the pseudo-Abelian integral is an endlessly continuable multi-valued holomorphic function was proved in [1]. Let us shortly recall the main idea. We define a continuation of the function I_ϵ as the integral along transport of the real oval γ along a path l in the h-plane. We define the transport of γ by mean of the gradient flow of H, modified in a neighborhood of the line at infinity. The only obstructions in this continuation is $H^{-1}(0)$ and singular points (zeroes) $p_1, \ldots, p_N$ of the polynomial vector field X_ϵ. Since the first integral H is a multi-valued function, any of these critical points can generate infinite sequence of critical values (potential ramification points). Only finite number of these ramification are realized as we restrict to paths of bounded length.

The only thing which remain to prove concerns localization of these singularities $h_j(\epsilon)$ tending to zero as $\epsilon \to 0$. By assumption on the first integral H, provided in the definition of the X^0_k space, the only singular point of X_ϵ located in a neighborhood of γ^0_ϵ is the center p^ϵ_c—see Fig. 1.1. Its coordinates have the form $p^\epsilon_c = (\frac{2\epsilon}{a+2}, 0) + O(\epsilon^2)$. Thus, the respective critical values has the form given in Lemma 2.2. □

Proof of Proposition 2.3. The idea of the proof is the following. We split the loop γ into four pieces: The part γ_1 which is in finite distance from the origin, γ_0 which is close to the origin and two segments $l_\pm$ which join these two components—see Fig. 3.1. We fix the cutting lines $x = \epsilon\xi_0$ and $x = x_0$ more precisely later. The segments $l_\pm$ passes close to the zero level curve of P.

We investigate the analytic properties of the respective integrals along these (relative) cycles.

We have $I_\epsilon = \tilde{I}_0 + \tilde{I}_1 + \tilde{I}_l$, where

$$\tilde{I}_0 = \int\limits_{\gamma_0} \omega, \qquad \tilde{I}_1 = \int\limits_{\gamma_1} \omega,$$

$$\tilde{I}_l = \int\limits_{l_+} \omega + \int\limits_{l_-} \omega.$$

We prove the following properties:

(a) The function $\tilde{I}_1(\epsilon, h)$ is analytic in ϵ. It is holomorphic, multi-valued in h and in a disk of sufficiently small (ϵ-independent) radius it does not have any other singularities except $h = 0$. It satisfies the variation relation $\mathcal{V}ar_{1,1/a_1,\ldots,1/a_k} \tilde{I}_1 \equiv 0$, i.e. variation $\mathcal{V}ar_1$ followed by variation as in (2.2).

(b) Let $u = h/\epsilon^{a+2}$. The function $\tilde{I}_0(\epsilon, u)$ is analytic in ϵ. It is holomorphic, multi-valued in u with ramification at $u = 0$ and a sequence of ramifications $u_\mu(\epsilon)$: $|u_\mu| = \rho_0 + O(\epsilon)$ (ramification circle). The integral $\tilde{I}_0$ satisfies the following variation relation around zero $\mathcal{V}ar_{1,1/a}\tilde{I}_0 \equiv 0$ and the relation $\mathcal{V}ar^{2\rho_0}_{1,1/(a+2)}\tilde{I}_0 \equiv 0$ outside the ramification circle.

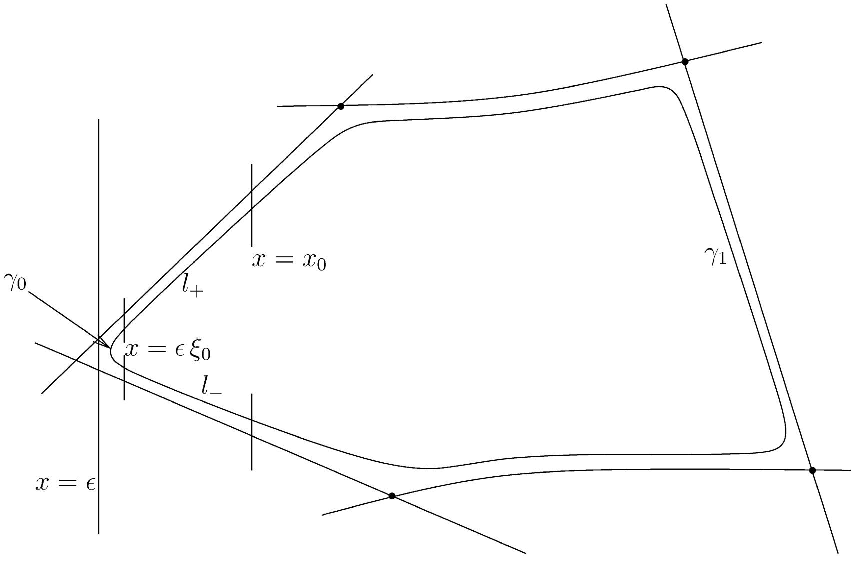

Fig. 3.1. The splitting of loop γ into pieces.

(c) The integral along the segment $l_\pm$ reads $\tilde{I}_l = f_1(\epsilon, h) + f_0(\epsilon, u) + F$, where f_0, f_1 are analytic in ϵ with at most polar singularity of order 3. The functions f_1 and f_0 are meromorphic with at most polar singularity in h and u respectively. The function F is given by the following convergent power series

$$F = \sum_{k \geqslant -1,\, l \geqslant -3} c_{jk} \left(\frac{h}{\epsilon^{a+2}} \right)^k \epsilon^l \begin{cases} \alpha_{kl}^{-1}(\epsilon^{\alpha_{kl}} - 1) & \text{for } \alpha_{kl} \neq 1, \\ \log \epsilon & \text{for } \alpha_{kl} = 0, \end{cases} \tag{3.1}$$

where the real exponents α_{kl} satisfy $|\alpha_{kl}| < 1/10$.

The above properties prove Proposition 2.3. Indeed, since $\mathcal{V}ar_1 \tilde{I}_l \equiv 0$, we have $\mathcal{V}ar_1 I = \mathcal{V}ar_1 \tilde{I}_1 + \mathcal{V}ar_1 \tilde{I}_0$. Denote $I_0 = \mathcal{V}ar_1 \tilde{I}_0$ and $I_1 = \mathcal{V}ar_1 \tilde{I}_1$. Properties listed in point (1) of Proposition 2.3 follows the point (a) listed above. On the other hand, variation relations satisfied by the function $\tilde{I}_0$ imply that I_0 is a meromorphic function of $u^{1/a}$ inside the ramification circle and a uni-valued holomorphic function of $u^{1/(a+2)}$ outside the ramification circle. Thus the power series expansions given in the point (2) of Proposition 2.3 follow.

To prove the existence of leading term of I as $\epsilon \to 0$, we observe the following identity $\mathcal{V}ar_{1,1/a} I = \mathcal{V}ar_{1/a} I_1(\epsilon, h)$; the latter function is analytic in ϵ. Consider the case $\mathcal{V}ar_{1,1/a} I \not\equiv 0$. Then, there exists a non-zero leading term (in ϵ) of the form $\epsilon^B (\log \epsilon)^l \tilde{G}(u)$. By analytic properties of functions $\tilde{I}_0$, $\tilde{I}_1$, $\tilde{I}_l$, there is only finitely many terms of order (in ϵ) $\leqslant B + 1$. We take the lowest order (leading term) in the decomposition $I = \tilde{I}_0 + \tilde{I}_1 + \tilde{I}_l$.

Consider now the case $\mathcal{V}ar_{1,1/a} I \equiv 0$ but $\mathcal{V}ar_1 I = I_1(\epsilon, h) + I_0(\epsilon, u) \not\equiv 0$. Due to the first identity, the function $\mathcal{V}ar_1 I$ is an holomorphic in $u^{1/a}$ and so $\mathcal{V}ar_1 I = \sum_{n \geqslant n_0, j \geqslant 0} (a_{nj} \epsilon^j \epsilon^{(a+2)n} + b_{nj} \epsilon^j) u^{n/a}$. We order these terms according to the power of ϵ and choose the lowest order (leading) term of order ϵ^B. Then we follow the previous idea and we consider all terms of order $\leqslant B + 1$ in the decomposition of I.

Finally if $\mathcal{V}ar_1 I \equiv 0$, then $I = \hat{I}_1(\epsilon, h) + \hat{I}_0(\epsilon, u) + F$, where $\hat{I}_0, \hat{I}_1$ are analytic functions and F is given by (3.1). We take the lowest order term in u variable: $u^k (g_0(\epsilon) + \sum_{n=0}^{N_0} a_n(\epsilon) \epsilon^{\alpha_n} + b_0 \epsilon^\beta \log \epsilon)$, where $g_0, a_0, \ldots, a_{N_0}$ are analytic functions of ϵ. We take the ϵ-leading term of the coefficient, say of order ϵ^{κ_0}, and then we consider all terms of the I expansion of order $\leqslant \kappa_0$ (in ϵ-variable). We choose a non-zero leading term.

Now we prove properties (a)–(c) listed above.

In point (a) we have the situation very similar to that considered in [1]. The only difference relies in the fact that now we deal with relative cycle instead of true cycle. The proof of variation relation (2.2) is exactly the same as previously. Actually, after the first variation $\mathcal{V}ar_h I_1$ with respect to h we deal with a real (closed) cycle. Indeed, the exponent of term P in the expression of H_ϵ is 1 and so the parts of path γ_1 passing close to the edge γ_1^0 (Fig. 1.1) have parametrization of the form $\varphi^-(t,h)$, for an analytic function φ^-. So, the respective terms in $\mathcal{V}ar_h I_1$ cancel out. The equivalent argument works for the second branch of $P^{-1}(0)$ passing through $(0,0)$ (i.e. edge γ_μ^0 in Fig. 1.1). Thus, the cycle $\mathcal{V}ar_h I_1$ is localized in the neighborhood of the polycycle γ^0, except edges γ_1^0 and γ_μ^0. This part of polycycle γ^0 satisfies the genericity assumptions from [1] and so one can apply results from that paper about vanishing of iterated variation of the pseudo-Abelian integral.

The analytic dependence of the integral I_1 on parameter ϵ is a consequence of the Implicit Function Theorem. The only critical points of the system appearing in a neighborhood of γ_1 are normal crossing vertices $p_{\mu\nu}$. This configuration is stable under small deformation.

To prove the point (b) we make a blow up transformation

$$x = \epsilon\xi, \qquad y = \epsilon\eta, \qquad h = \epsilon^{a+2}u.$$

The equation for γ_0 reads $(\xi-1)^a(\eta^2-\xi^2+\epsilon(\ldots))F(\epsilon\xi,\epsilon\eta) = u$, where $F(0,0)\neq 0$ and $(\ldots)$ denotes polynomial of higher order in ϵ,ξ,η. Thus, for sufficiently small ϵ we deal with the situation just considered in point (a). We have a relative cycle contained in the level curve of the Darboux function with exponents $(1,a)$. Thus, the function $\tilde{I}_0$ satisfies the following variation relation around zero: $\mathcal{V}ar_{1,1/a}\tilde{I}_0 \equiv 0$. The localization of singular (ramification) points $u_j(\epsilon)$ is a consequence of Lemma 2.2. Now it remains to prove the variation relation outside the ramification circle. We make another blow-up transformation $\eta = \xi\zeta$. The level curve equation reads $\xi^2(\xi-1)^a(\zeta^2-1+\epsilon\xi(\ldots))F(\epsilon\xi,\epsilon\xi\zeta) = u$. The relative cycle γ_0, as a part of the real oval γ, satisfies $|\xi| \leqslant |\xi_0|$ (by definition of γ_0) and $|\zeta| \leqslant 1+|\epsilon\xi_0 C_0| \leqslant \widetilde{C}_0$ (by the form of polynomial P). For sufficiently large $|u|$ (geometrically for u outside the ramification circle) one has for $(\xi,\zeta)\in\gamma_0$:

$$\left|\xi^2(\xi-1)^a\right| > 2^a = \sup_{\xi\in D(0,1)}\left|\xi^2(\xi-1)^a\right|,$$

so $|\xi|>1$ on γ_0. The formula $\xi(1-\xi^{-1})^{a/(a+2)}$ defines a uni-valued holomorphic function for $|\xi|>1$ and so in considered region we have the situation of Darboux function with exponents $(1,a+2)$. Thus the function $\tilde{I}_0$ satisfies the variation relation $\mathcal{V}ar^{2\rho_0}_{1,1/(a+2)}\tilde{I}_0 \equiv 0$ outside the ramification circle.

Finally, the proof of point (c) is a result of direct calculations. We parametrize $l_\pm$, take a power series expansion and integrate explicitly.

The equation for level curve reads $(x-\epsilon)^a(y^2-x^2)G^*(x,y) = h$, where $G(0,0)\neq 0$. In the region of $l_\pm$ i.e. for $x\in[\epsilon\xi_o, x_0]$, h and ϵ small, by the implicit function theorem, the solution has the form

$$y = xg_\pm\left(x, \frac{h}{x^2(x-\epsilon)^a}\right),$$

where $g_\pm$ are (germs of) holomorphic functions. We put this expression into the $I_{l_\pm}$ integral, make a substitution $v = \epsilon/x$ and we get

$$I_{l_\pm} = \int_{\epsilon/x_0}^{\xi_0^{-1}} \left(\frac{h}{\epsilon^{a+2}}\right)^{-1} \epsilon^{-3} v^2(1-v)^{-2} G\left(\epsilon/v, \frac{h}{\epsilon^{a+2}} v^{a+2}(1-v)^{-a}\right) \mathrm{d}v,$$

where G is a germ of holomorphic function. One can assume that the cutting points $x_0, \epsilon\xi_0$ (Fig. 3.1) were chosen in such a way that $|\epsilon/x_0| < |\xi_0^{-1}| < 1$. Thus, the integrated function forms a convergent power series of powers of v

$$I_{l_\pm} = \sum_{j\geqslant -1,\, m\geqslant -3,\, n\geqslant 0} b_{jmn}\left(\frac{h}{\epsilon^{a+2}}\right)^j \epsilon^m \int_{\epsilon/x_0}^{\xi_0^{-1}} v^{j(a+2)-m+n}\,\mathrm{d}v$$

$$= \sum_{|j(a+2)-m+n+1|\geqslant 1/10} \left(\tilde{b}_{jn}(u)^j\epsilon^n - \hat{b}_{jn}h^j\epsilon^n\right) + \sum_{|\alpha_{jn}|<1/10} c_{jn}\left(\frac{h}{\epsilon^{a+2}}\right)^j \epsilon^m \int v^{\alpha_{jn}-1}\,\mathrm{d}v.$$

We split the sum into terms which power of v is in finite distance (say $\geqslant 1/10$) from -1 and the others; first terms generate functions f_0, f_1 and the others form the function F. $\square$

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