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Orthogonal Newton Polynomials

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The problem is to determine all nonnegative measures on the Borel subsets of the complex plane with respect to which all polynomials are square integrable and with respect to which the Newton polynomials form an orthogonal set. The Newton polynomials do not belong to any classical scheme of orthogonal polynomials. The discovery that a plane measure exists with respect to which they form an orthogonal set was only recently made by T. L. Kriete and D. Trutt [*Amer. J. Math.* **93** (1971), 215–225]. A general structure theory for such measures is now obtained under hypotheses suggested by the expansion theory of Cesàro operators.

It has recently been discovered [1] that nonnegative plane measures exist with respect to which the Newton polynomials form an orthogonal set. Examples of such measures which appear in the expansion theory for Cesàro operators [2] are supported in the half-plane to the right of some vertical line and satisfy a restrictive condition on the growth of the norms of the Newton polynomials. A determination is now made of all nonnegative measures, satisfying such conditions, with respect to which the Newton polynomials form an orthogonal set. The known examples of such measures [3] turn out to be extreme points of the convex set of such measures, normalized by the condition of unit total mass. Some interesting Hilbert space of analytic functions [4] appear in the problem of mean square polynomial approximation with respect to these extremal measures.

If h is a given positive number, a unique Hilbert space $\mathcal{N}(h)$ exists, whose elements are functions analytic in the half-plane $z + \bar{z} > -h$, such that the Newton polynomials

$$(-1)^n \frac{z(z-1)\cdots(z-n+1)}{1\cdot 2\cdots n}$$

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form an orthogonal set in the space and such that the square of the norm of the nth polynomial is

$$\frac{h(h+1)\cdots(h+n-1)}{1\cdot 2\cdots n}.$$

The identity

$$2\pi\Gamma(h)\,\|F\|^2_{\mathcal{N}(h)} = \sum_{n=0}^{\infty}\frac{1}{n!}\int_{-\infty}^{+\infty}|\Gamma(n/2+h/2+it)\,F(n/2-h/2+it)|^2\,dt$$

holds for all polynomials $F(z)$. The norm-determining measure for $\mathcal{N}(h)$ is a fundamental example of a plane measure with respect to which the Newton polynomials form an orthogonal set.

The set of all nonnegative measures on the Borel subsets of the complex plane, of total mass one, with respect to which the Newton polynomials form an orthogonal set, is a compact convex set in the weak topology induced by the continuous functions of polynomial growth. By the Krein–Milman theorem [5], the set is the closed convex span of its extreme points. Interest in the structural properties of extremal measures is generated by their appearance in the Stone–Weierstrass theorem [6] and in problems of bounded polynomial approximation [7]. A related extreme point problem is now studied.

Motivation for the work is provided by an analogy with the theory of Pollaczek polynomials [8], which turns out to be essential. Let L_+, L_-, and D be the operators on polynomials defined by D takes $F(z)$ into $G(z)$ where

$$G(z) = z[F(z) - F(z-1)],$$

by L_- takes $F(z)$ into $G(z)$ where

$$G(z) = -[F(z+1) - F(z)],$$

and by L_+ takes $F(z)$ into $G(z)$ where

$$G(z) = -zF(z-1).$$

Then the identities

$$DL_- - L_-D = -L_-,$$

$$DL_+ - L_+D = L_+,$$

$$L_-L_+ - L_+L_- = 1,$$

and

$$L_+L_- + L_-L_+ - 2D = 1,$$

are satisfied. The Newton polynomial of degree n is an eigenfunction of D for the eigenvalue n. The operator L_- takes the Newton polynomial of degree $n+1$ into the Newton polynomial of degree n. And the operator L_+ takes the Newton polynomial of degree $n-1$ into n times the Newton polynomial of degree n.

If μ is a plane measure with respect to which all polynomials are square integrable, define an inner product on polynomials $F(z)$ and $G(z)$ by

$$\langle F(z), G(z)\rangle = \int F(z)\, \bar{G}(z)\, d\mu(z).$$

The orthogonality of the Newton polynomials in $L^2(\mu)$ is equivalent to the formal self-adjointness of D with respect to the inner product.

If the inner product is extended to an inner product on polynomials $F(z, \bar{z})$ and $G(z, \bar{z})$ in z and $\bar{z}$ by

$$\langle F(z, \bar{z}), G(z, \bar{z})\rangle = \int F(z, \bar{z})\, G(z, \bar{z})\, d\mu(z),$$

then the identity

$$\langle zF(z, \bar{z}), G(z, \bar{z})\rangle = \langle F(z, \bar{z}), \bar{z}G(z, \bar{z})\rangle$$

holds for all polynomials $F(z, \bar{z})$ and $G(z, \bar{z})$ in z and $\bar{z}$. The self-adjointness of D is equivalent to the identity

$$\begin{aligned} &\langle zF(z-1, \bar{z}), G(z, \bar{z}-1)\rangle - \langle F(z, \bar{z}-1), zG(z-1, \bar{z})\rangle \\ &\quad = \langle (z-\bar{z})\, F(z, \bar{z}), G(z, \bar{z})\rangle \end{aligned}$$

for all polynomials $F(z, \bar{z})$ and $G(z, \bar{z})$ in z and $\bar{z}$.

It is now shown that an equivalent identity holds for a larger class of functions if the measure satisfies an additional condition called regularity.

The moments λ_n of a nonnegative measure μ on the Borel subsets of the complex plane with respect to which the Newton polynomials form an orthogonal set are defined by

$$\lambda_n = \int \left| \frac{z(z-1)\cdots(z-n+1)}{1\cdot 2\cdots n} \right|^2 d\mu(z).$$

The moment-generating function for μ is the formal power series $\sum \lambda_n z^n$. The measure μ is said to be regular if it is supported in a half-plane $z+\bar{z} \geqslant -h$ and if its moment-generating function converges in the unit disk.

THEOREM 1. *Let μ be a nonnegative measure on the Borel subsets of the* *complex plane, with respect to which the Newton polynomials form an orthogonal set, which is supported in the half-plane $z + \bar{z} \geqslant -k$. Then the moment-generating function for μ converges in the unit disk if, and only if, the identity*

$$\int (x+iy)\, F(x-\tfrac{1}{2}, y+\tfrac{1}{2}i)\, \bar{G}(x-\tfrac{1}{2}, y-\tfrac{1}{2}i)\, d\mu(x+iy)$$

$$-\int (x-iy)\, F(x-\tfrac{1}{2}, y-\tfrac{1}{2}i)\, \bar{G}(x-\tfrac{1}{2}, y+\tfrac{1}{2}i)\, d\mu(x+iy)$$

$$= 2i \int yF(x,y)\, \bar{G}(x,y)\, d\mu(x+iy)$$

holds for all functions $F(x,y)$ and $G(x,y)$ of x and y which are finite linear combinations of the functions $\exp(iay)$, *a real, with bounded continuous functions of x as coefficients.*

The norm-determining measure for $\mathcal{N}(h)$ is an example of a regular measure. The regular measures of total mass one which are supported in the closed half-plane to the right of a given vertical line form a compact convex set in the weak topology induced by the continuous functions which are of polynomial growth in the half-plane. By the Krein–Milman theorem, the set is the closed convex span of its extreme points. A structural property of extremal measures is obtained by methods taken from the proof of the Stone–Weierstrass theorem.

THEOREM 2. *If μ is an extreme point of the set of regular nonnegative measures of total mass one on the Borel subsets of the complex plane with respect to which the Newton polynomials form an orthogonal set, then the support of μ is contained in the union of a sequence of vertical lines whose intersections with the real axis are congruent modulo one-half.*

The analysis of such measures requires some information from the theory of Pollaczek polynomials. If h is a given positive number and if an inner product on polynomials is defined by

$$2^{1-2h}\Gamma(2h)\langle F(t), G(t)\rangle = \int_{-\infty}^{+\infty} F(t)\, \bar{G}(t)\, \Gamma(h-it)\, \Gamma(h+it)\, dt,$$

then the identity

$$\langle (h-it)\, F(t+i), G(t)\rangle = \langle F(t), (h-it)\, G(t+i)\rangle$$

holds for all polynomials $F(z)$ and $G(z)$. These properties essentially characterize the inner product.

Assume that μ is a nonnegative measure on the Borel subsets of the real line with respect to which all polynomials are square integrable. Define an inner product on polynomials by

$$\langle F(t), G(t) \rangle = \int F(t)\, \bar{G}(t)\, d\mu(t).$$

If the above identity holds for all polynomials $F(z)$ and $G(z)$ and if the constant function one has norm one, then the identity

$$2^{1-2h}\Gamma(2h)\,\mu(E) = \int_E \Gamma(h - it)\,\Gamma(h + it)\, dt$$

holds for every Borel set E.

These results have an immediate application.

THEOREM 3. *Let μ be a regular nonnegative measure on the Borel subsets of the complex plane with respect to which the Newton polynomials form an orthogonal set. Assume that a real number h exists such that the support of μ is contained in the union of the sequence of vertical lines which intersect the real axis at points $n/2 - h/2$ for nonnegative integers n, and that the line which intersects at $-h/2$ does contain a point of the support of μ. Then h is nonnegative. If h is zero, then the mass of μ is concentrated at the origin. If h is positive, then a positive number p exists such that for every real Borel set E the μ-measure of the set of points of the form $-h/2 + it$ with t in E is*

$$p \int_E \Gamma(h/2 - it)\,\Gamma(h/2 + it)\, dt.$$

A generalization of the theory of Pollaczek polynomials is needed for a further analysis of the measures.

THEOREM 4. *Let h be a given positive number. Let σ be a nonnegative measure on the Borel subsets of the real line with respect to which all polynomials are square integrable. Then the function*

$$\varphi(z) = \int_{-\infty}^{+\infty} \frac{\Gamma(1 - it)\,\Gamma(1 + it)}{\Gamma(h + \frac{1}{2} - it)\,\Gamma(h + \frac{1}{2} + it)} \operatorname{sech}(\pi z - \pi t)\, d\sigma(t)$$

is analytic and has a nonnegative real part in the strip $-\frac{1}{2} < y < \frac{1}{2}$. The integral

$$\int_{-\infty}^{+\infty} |F(t)|^2 \frac{\Gamma(h - it)\,\Gamma(h + it)}{\Gamma(\frac{1}{2} - it)\,\Gamma(\frac{1}{2} + it)}\, \varphi(t)\, dt$$

converges for every polynomial $F(z)$. The identity

$$\int_{-\infty}^{+\infty} F(t+i)\,\bar{G}(t)\,\frac{\Gamma(h+1-it)\,\Gamma(h+it)}{\Gamma(\frac{1}{2}-it)\,\Gamma(\frac{1}{2}+it)}\,\varphi(t)\,dt$$

$$-\int_{-\infty}^{+\infty} F(t)\,\bar{G}(t+i)\,\frac{\Gamma(h-it)\,\Gamma(h+1+it)}{\Gamma(\frac{1}{2}-it)\,\Gamma(\frac{1}{2}+it)}\,\varphi(t)\,dt$$

$$=-2i\int_{-\infty}^{+\infty} tF(t+\tfrac{1}{2}i)\,\bar{G}(t+\tfrac{1}{2}i)\,d\sigma(t)$$

holds for all entire functions $F(z)$ and $G(z)$ of exponential type which are of polynomial growth on the real axis.

These properties essentially characterize such pairs of measures when h is positive.

THEOREM 5. *Let h be a given positive number. If μ and ν are nonnegative measures on the Borel subsets of the real line, with respect to which all polynomials are square integrable, such that the identity*

$$\int_{-\infty}^{+\infty} (h-it)\,F(t+i)\,\bar{G}(t)\,d\nu(t)$$

$$-\int_{-\infty}^{+\infty} (h+it)\,F(t)\,\bar{G}(t+i)\,d\nu(t)$$

$$=-2i\int_{-\infty}^{+\infty} tF(t+\tfrac{1}{2}i)\,\bar{G}(t+\tfrac{1}{2}i)\,d\mu(t)$$

holds for all entire functions $F(z)$ and $G(z)$ of exponential type which are square integrable on the real axis, then a nonnegative measure σ exists on the Borel subsets of the real line, which agrees with μ on every Borel set which does not contain the origin, such that the identity

$$\nu(E)+p\int_E \Gamma(h-it)\,\Gamma(h+it)\,dt=\int_E \frac{\Gamma(h-it)\,\Gamma(h+it)}{\Gamma(\frac{1}{2}-it)\,\Gamma(\frac{1}{2}+it)}\,\varphi(t)\,dt$$

holds for every Borel set E, where p is a nonnegative number and $\varphi(z)$ is defined for σ as in Theorem 4.

Contiguous relations are used to extend these results to nonpositive values of h. Let h be a given real number and let μ_+ and μ_- be nonnegative measures on the Borel subsets of the real line, with respect to which all polynomials are square integrable, such that the identity

$$\int_{-\infty}^{+\infty} (h-it)\,F(t+i)\,\bar{G}(t)\,d\nu_-(t)$$
$$-\int_{-\infty}^{+\infty} (h+it)\,F(t)\,\bar{G}(t+i)\,d\nu_-(t)$$
$$=-2i\int_{-\infty}^{+\infty} tF(t+\tfrac{1}{2}i)\,\bar{G}(t+\tfrac{1}{2}i)\,d\mu_-(t)$$

holds for all entire functions $F(z)$ and $G(z)$ of exponential type which are of polynomial growth on the real axis. If μ_+ and ν_+ are nonnegative measures on the Borel subsets of the real line defined by

$$\mu_+(E)=\int_E (t^2+h^2+h+1/4)\,d\mu_-(t)$$

and

$$\nu_+(E)=\int_E (t^2+h^2)\,d\nu_-(t)$$

for every Borel set E, then all polynomials are square integrable with respect to μ_+ and ν_+. The identity

$$\int_{-\infty}^{+\infty} (h+1-it)\,F(t+i)\,\bar{G}(t)\,d\nu_+(t)$$
$$-\int_{-\infty}^{+\infty} (h+1+it)\,F(t)\,\bar{G}(t+i)\,d\nu_+(t)$$
$$=-2i\int_{-\infty}^{+\infty} tF(t+\tfrac{1}{2}i)\,\bar{G}(t+\tfrac{1}{2}i)\,d\mu_+(t)$$

holds for all entire functions $F(z)$ and $G(z)$ of exponential type which are of polynomial growth on the real axis.

Conversely if the identity for μ_+ and ν_+ is assumed to hold for all entire functions $F(z)$ and $G(z)$ of exponential type which are of polynomial growth on the real axis, then the identity for μ_- and ν_- holds for all entire functions $F(z)$ and $G(z)$ of exponential type which are of polynomial growth on the real axis and which vanish at $-ih$. A uniqueness theorem follows.

THEOREM 6. *Let h be a nonpositive real number. Let μ be a nonnegative measure on the Borel subsets of the real line with respect to which all polynomials are square integrable. Then at most one nonnegative measure ν*

exists on the Borel subsets of the real line, with respect to which all polynomials are square integrable, such that the identity

$$\int_{-\infty}^{+\infty} (h - it)\, F(t+i)\, \bar{G}(t)\, d\nu(t)$$

$$- \int_{-\infty}^{+\infty} (h + it)\, F(t)\, \bar{G}(t+i)\, d\nu(t)$$

$$= -2i \int_{-\infty}^{+\infty} tF(t + \tfrac{1}{2}i)\, \bar{G}(t + \tfrac{1}{2}i)\, d\mu(t)$$

holds for all entire functions $F(z)$ and $G(z)$ of exponential type which are of polynomial growth on the real axis.

Examples of such measures appear in the orthogonality problem for Newton polynomials.

THEOREM 7. *If h is a positive number and if n is a nonnegative integer, then the identity*

$$\int_{-\infty}^{+\infty} \Gamma(n/2 + 1/2 + h/2 - it)\, \Gamma(n/2 + 1/2 + h/2 + it)$$

$$\times (h/2 - 1/2 - n/2 - it)\, F(t+i)\, \bar{G}(t)\, dt$$

$$- \int_{-\infty}^{+\infty} \Gamma(n/2 + 1/2 + h/2 - it)\, \Gamma(n/2 + 1/2 + h/2 + it)$$

$$\times (h/2 - 1/2 - n/2 + it)\, F(t)\, \bar{G}(t+i)\, dt$$

$$= -2i(n+1) \int_{-\infty}^{+\infty} \Gamma(n/2 + h/2 - it)\, \Gamma(n/2 + h/2 + it)$$

$$\times tF(t + \tfrac{1}{2}i)\, \bar{G}(t + \tfrac{1}{2}i)\, dt$$

holds for all entire functions $F(z)$ and $G(z)$ of exponential type which are of polynomial growth on the real axis.

An extreme point computation follows.

THEOREM 8. *The extreme points of the set of regular nonnegative measures on the Borel subsets of the complex plane, of total mass one and supported in the half-plane $z + \bar{z} \geqslant -1$, with respect to which the Newton polynomials form an orthogonal set, are the norm-determing measures for the space $\mathcal{N}(h)$, $0 < h \leqslant 1$, and the measure with mass one concentrated at the origin.*

The problem is more complicated for a half-plane $z + \bar{z} \geqslant -k$ where $k > 1$ because a finite combination of the norm-determing measures for the spaces $\mathcal{N}(h)$ can be nonnegative without having all the coefficients be nonnegative.

A determination of moment-generating functions follows from the extreme point computation.

THEOREM 9. *A formal power series $\sum \lambda_n z^n$ is the moment-generating function of a regular nonnegative measure μ on the Borel subsets of the complex plane, which is supported in the half-plane $z + \bar{z} \geqslant -1$ and with respect to which the Newton polynomials form an orthogonal set if, and only if,*

$$\sum \lambda_n z^n = \int (1 - z)^{-t} \, d\sigma(t)$$

for a nonnegative measure σ on the Borel subsets of the real line which is supported in the interval $[0, 1]$.

A convolution structure holds in the space of nonnegative plane measures with respect to which the Newton polynomials form an orthogonal set.

THEOREM 10. *Let μ_1 and μ_2 be nonnegative measures on the Borel subsets of the complex plane with respect to which the Newton polynomials form an orthogonal set. Define the convolution $\mu = \mu_1 * \mu_2$ to be the nonnegative measure on the Borel subsets of the complex plane whose value for a Borel set E is the $\mu_1 \times \mu_2$-measure of the set of pairs (z_1, z_2) of complex numbers such that $z_1 + z_2$ belongs to E. Then the Newton polynomials form an orthogonal set with respect to μ and the moment-generating function for μ is the product of the moment-generating functions for μ_1 and μ_2.*

The moment-generating function for the norm-determining measure of $\mathcal{N}(h)$ is $(1 - z)^{-h}$. The norm-determining measures for the spaces $\mathcal{N}(h)$ are well behaved under convolution.

THEOREM 11. *The convolution of the norm-determining measure for $\mathcal{N}(h_1)$ and the norm-determining measure for $\mathcal{N}(h_2)$ is the norm-determining measure for $\mathcal{N}(h_1 + h_2)$.*

The nonnegative measures with respect to which the Newton polynomials form an orthogonal set are those with respect to which the operator D is self-adjoint. An interesting problem is to determine whether this situation is always derived from one in which the operators L_+ and L_- are adjoints. This is the case for the measures in the closed convex span of the norm-determining measures for the spaces $\mathcal{N}(h)$.

THEOREM 12. *Let μ_- be a nonnegative measure on the Borel subsets of the complex plane with respect to which the Newton polynomials form an orthogonal set. Assume that the moment-generating function for μ_- is of the form $\int (1-z)^{-t}\, d\sigma_-(t)$ for a nonnegative measure σ_- on the Borel subsets of the nonnegative half-line with respect to which all polynomials are square integrable. Then a nonnegative measure μ_+ on the Borel subsets of the complex plane exists, with respect to which the Newton polynomials form an orthogonal set, such that the identity*

$$\int zF(z-1)\,\bar{G}(z)\,d\mu_-(z) = \int F(z)[\bar{G}(z+1) - \bar{G}(z)]\,d\mu_+(z)$$

holds for all polynomials $F(z)$ and $G(z)$. The moment-generating function for μ_+ is

$$\int (1-z)^{-t}\, d\sigma_+(t),$$

where σ_+ is the nonnegative measure on the Borel subsets of the nonnegative half-line such that the identity

$$\int t\chi(t+1)\, d\sigma_-(t) = \int \chi(t)\, d\sigma_+(t)$$

holds for the characteristic function χ of every real Borel set.

Such a measure μ_+ is conjectured to exist without any hypothesis on the moment-generating function of μ_-. The integral representations of the moment-generating functions are taken in the formal power series sense.

Proof of Theorem 1. Since the support of μ is contained in the half-plane $z + \bar{z} \geqslant k$, the function c^z is square integrable with respect to μ when $0 < c < 1$. If the moment-generating function of μ is assumed to converge in the unit disk, the expansion

$$c^z = \sum (1-c)^n (-1)^n \frac{z(z-1)\cdots(z-n+1)}{1 \cdot 2 \cdots n}$$

is valid in the mean square sense. The self-adjointness of D implies the identity

$$\int z a^{z-1} b^{\bar{z}}\, d\mu(z) - \int \bar{z} a^z b^{\bar{z}-1}\, d\mu$$

$$= \int (z - \bar{z})\, a^z b^{\bar{z}}\, d\mu(z)$$

when $0 < a < 1$ and $0 < b < 1$. The desired identity follows for functions $F(x, y)$ and $G(x, y)$ of the form $c^x \exp(iay)$ where a is real and $0 < c < 1$. By the Stone Weierstrass theorem, every continuous function of real x in the half-line $[-k/2, \infty)$ which has limit zero at infinity is a uniform limit of finite linear combinations of the functions c^x with $0 < c < 1$. By the Lebesgue dominated convergence theorem, the identity follows when $F(x, y)$ and $G(x, y)$ are of the form $f(x) \exp(iay)$, where a is real and $f(x)$ is a bounded continuous function of real x. By linearity, the identity holds whenever $F(x, y)$ and $G(x, y)$ are finite linear combinations of the functions $\exp(iay)$ with bounded continuous functions of x as coefficients.

Conversely, it will be shown that the moment-generating function for μ converges in the unit disk if the identity is satisfied for all such functions. The identity

$$\int za^{z-1}b^{\bar{z}}\bar{z}^n \, d\mu(z) - \int \bar{z}a^z b^{\bar{z}-1}(\bar{z}-1)^n \, d\mu(z)$$

$$= \int (z - \bar{z}) \, a^z b^{\bar{z}} \bar{z}^n \, d\mu(z)$$

is obtained inductively for positive integers n by differentiation with respect to b and using the Lebesgue dominated convergence theorem. By the same theorem, the identity holds in the limit $b = 1$. It follows that the identity

$$\int zc^{z-1}\bar{f}(z) \, d\mu(z) - \int \bar{z}c^z \bar{f}(z-1) \, d\mu(z)$$

$$= \int (z - \bar{z}) \, c^z \bar{f}(z) \, d\mu(z)$$

holds for every polynomial $f(z)$ when $0 < c < 1$. When $f(z)$ is the nth Newton polynomial, the identity reads

$$\int z(c^{z-1} - c^z) \, \bar{f}(z) \, d\mu(z) = -n \int c^z \bar{f}(z) \, d\mu(z).$$

By the Lebesgue dominated convergence theorem, the identity can be written

$$(1 - c) \, d/dc \int c^z \bar{f}(z) \, d\mu(z) = -n \int c^z \bar{f}(z) \, d\mu(z).$$

It follows that

$$\int c^z \bar{f}(z) \, d\mu(z) = k(1 - c)^n,$$

where k is independent of c. The identity

$$\int c^z f(z) \bar{f}(z)\, d\mu(z) = kc^n$$

is obtained on differentiating n times with respect to c using the Lebesgue dominated convergence theorem. By the same theorem, the identity

$$\int f(z) \bar{f}(z)\, d\mu(z) = k$$

holds in the limit $c = 1$. It follows that

$$\int c^z \bar{f}(z)\, d\mu(z) = (1-c)^n \int f(z) \bar{f}(z)\, d\mu(z).$$

Since

$$\sum (1-c)^{2n} \int \left| \frac{z(z-1)\cdots(z-n+1)}{1 \cdot 2 \cdots n} \right|^2 d\mu(z)$$

$$\leqslant \int c^{z+\bar{z}}\, d\mu(z) < \infty$$

by Bessel's inequality, the moment-generating function for μ converges in the unit disk.

Proof of Theorem 2. The identity

$$1 = \cos^2(2\pi x - 2\pi h) + \sin^2(2\pi x - 2\pi h)$$

is applied for real numbers h. Choose h so that the support of μ is not contained in the union of the vertical lines which intersect the real axis in points which are congruent to h module one-half and so that the support of μ is not contained in the union of the vertical lines which intersect the real axis in points which are congruent to $h - \frac{1}{4}$ modulo one-half. Then a real number t exists, $0 < t < 1$, such that

$$1 - t = \int \cos^2(2\pi x - 2\pi h)\, d\mu(x+iy)$$

and

$$t = \int \sin^2(2\pi x - 2\pi h)\, d\mu(x+iy).$$

Let μ_+ and μ_- be the nonnegative measures on the Borel subsets of the complex plane such that

$$(1-t)\,\mu_+(E) = \int_E \cos^2(2\pi x - 2\pi h)\, d\mu(x+iy)$$

and

$$t\mu_-(E) = \int_E \sin^2(2\pi x - 2\pi h)\, d\mu(x+iy)$$

for every Borel set E. By Theorem 1, μ_+ and μ_- are regular nonnegative measures with respect to which the Newton polynomials form an orthogonal set. By construction, μ_+ and μ_- have total mass one and

$$\mu = (1-t)\mu_+ + t\mu_-.$$

Since μ is assumed to be an extreme point, μ_+ and μ_- are equal to μ. It follows that the support of μ is contained in the union of the vertical lines which intersect the real axis in points x such that

$$1-t = \cos^2(2\pi x - 2\pi h) \qquad \text{and} \qquad t = \sin^2(2\pi x - 2\pi h).$$

These points are congruent modulo one-half.

Proof of Theorem 3. Let σ be the nonnegative measure on the Borel subsets of the real line such that the σ-measure of every real Borel set E is the μ-measure of the set of points $h/2 + it$ with t in E. By Theorem 1, the regularity of μ implies the identity

$$\int (h/2 - it)\, F(t+i)\, \bar{G}(t)\, d\sigma(t)$$

$$= \int (h/2 + it)\, F(t)\, \bar{G}(t+i)\, d\sigma(t)$$

for all polynomials $F(z)$ and $G(z)$. If h is positive, the desired form of σ is given by the theory of Pollaczek polynomials.

When h is not positive, the case $F(z)$ and $G(z)$ equal to one in the identity gives

$$\int t\, d\sigma(t) = 0$$

and the case $F(z) = z$ and $G(z) = 1$ gives

$$\int (t^2 - h/4)\, d\sigma(t) = 0.$$

It follows that h is zero and that the mass of the measure σ is concentrated at the origin. The mass of the measure μ must also be concentrated at the origin, for otherwise μ is the sum of such a measure and a measure which satisfies the hypotheses of the theorem with h negative. It has been shown that no such measure exists.

Proof of Theorem 4. The function $\varphi(z)$ is clearly positive on the real axis. Since every polynomial is assumed to be square integrable with respect to σ the finiteness of the integral

$$\int |F(t)|^2\, \varphi(t)\, dt$$

for polynomials $F(z)$ is obtained by substituting the definition of $\varphi(z)$ and making the obvious rearrangement of integrals. It follows that the integrals appearing in the desired identity are absolutely convergent.

The desired identity is verified by substituting the definition of $\varphi(z)$ and making changes in the order of integration, which are permissible by absolute convergence. The problem is thereby reduced to the case in which μ is the measure with mass one concentrated at some real point λ. Then

$$\varphi(z) = \frac{\Gamma(1-i\lambda)\,\Gamma(1+i\lambda)}{\Gamma(h+\frac{1}{2}-i\lambda)\,\Gamma(h+\frac{1}{2}+i\lambda)}\operatorname{sech}(\pi z - \pi\lambda).$$

Let r be a given number, $0 < r < \frac{1}{2}$. By Cauchy's formula, the identity

$$\begin{aligned}
&\int_{-\infty}^{+\infty} F(t+i)\,\bar{G}(t)\,\frac{\Gamma(h+1-it)\,\Gamma(h+it)}{\Gamma(\frac{1}{2}-it)\,\Gamma(\frac{1}{2}+it)}\,\varphi(t)\,dt \\
&\quad = \int_{-\infty}^{\lambda-r} F(t+\tfrac{1}{2}i)\,\bar{G}(t+\tfrac{1}{2}i)\,\frac{\Gamma(h+\frac{1}{2}-it)\,\Gamma(h+\frac{1}{2}+it)}{\Gamma(-it)\,\Gamma(1+it)} \\
&\qquad \times \varphi(t-\tfrac{1}{2}i)\,dt - ir\int_0^{\pi} F(\lambda+\tfrac{1}{2}i+re^{it})\,\bar{G}(\lambda+\tfrac{1}{2}i+re^{-it}) \\
&\qquad \times \frac{\Gamma(h+\frac{1}{2}-i\lambda-ire^{it})\,\Gamma(h+\frac{1}{2}+i\lambda+ire^{it})}{\Gamma(-ire^{it}-i\lambda)\,\Gamma(1+ire^{it}+i\lambda)} \\
&\qquad \times \varphi(\lambda-\tfrac{1}{2}i+re^{it})e^{it}\,dt + \int_{\lambda+r}^{\infty} F(t+\tfrac{1}{2}i)\,\bar{G}(t+\tfrac{1}{2}i) \\
&\qquad \times \frac{\Gamma(h+\frac{1}{2}-it)\,\Gamma(h+\frac{1}{2}+it)}{\Gamma(-it)\,\Gamma(1+it)}\,\varphi(t-\tfrac{1}{2}i)\,dt
\end{aligned}$$

holds for all entire functions $F(z)$ and $G(z)$ of exponential type which are of polynomial growth on the real axis. A similar identity is obtained by interchanging $F(z)$ and $G(z)$ and conjugating each side of the equation. Since

$\varphi(z)$ is real for real z and periodic of period i, the desired identity can be written

$$r\int_0^{2\pi}(-ire^{it}-i\lambda)\,F(\lambda+\tfrac{1}{2}i+re^{it})\,\bar{G}(\lambda+\tfrac{1}{2}i+re^{-it})$$
$$\times\frac{\Gamma(h+\frac{1}{2}-i\lambda-ire^{it})\,\Gamma(h+\frac{1}{2}+i\lambda+ire^{it})}{\Gamma(1-ire^{it}-i\lambda)\,\Gamma(1+ire^{it}+i\lambda)}$$
$$\times\,\varphi(\lambda-\tfrac{1}{2}i+re^{it})\,e^{it}\,dt$$
$$=2\lambda F(\lambda+\tfrac{1}{2}i)\,\bar{G}(\lambda+\tfrac{1}{2}i).$$

By the definition of $\varphi(z)$, the desired identity can be written

$$\int_0^{2\pi}(\lambda+re^{it})\,F(\lambda+\tfrac{1}{2}i+re^{it})\,\bar{G}(\lambda+\tfrac{1}{2}i+re^{-it})$$
$$\times\frac{\Gamma(h+\frac{1}{2}-i\lambda-ire^{it})\,\Gamma(h+\frac{1}{2}+i\lambda+ire^{it})}{\Gamma(1-i\lambda-ire^{it})\,\Gamma(1+i\lambda+ire^{it})}\,\frac{ire^{it}\,dt}{\sin(\pi ire^{it})}$$
$$=2\lambda F(\lambda+\tfrac{1}{2}i)\,\bar{G}(\lambda+\tfrac{1}{2}i)\,\frac{\Gamma(h+\frac{1}{2}-i\lambda)\,\Gamma(h+\frac{1}{2}+i\lambda)}{\Gamma(1-i\lambda)\,\Gamma(1+i\lambda)}.$$

It is sufficient to verify the identity in the limiting case $r=0$, in which case it holds by uniform convergence of the integrand.

Proof of Theorem 5. Let σ be a nonnegative measure on the Borel subsets of the real line which agrees with μ on any real Borel set which does not contain the origin. The mass of σ at the origin is for the moment undetermined. If $\varphi(z)$ is defined for σ as in Theorem 4 and if p is a nonnegative number, the expression

$$\int_{-\infty}^{+\infty}(h-it)\,F(t+i)\,\bar{G}(t)\,d\nu(t)$$
$$-\int_{-\infty}^{+\infty}(h+it)\,F(t)\,\bar{G}(t+i)\,d\nu(t)$$
$$+\,p\int_{-\infty}^{+\infty}(h-it)\,F(t+i)\,\bar{G}(t)\,\Gamma(h-it)\,\Gamma(h+it)\,dt$$
$$-\,p\int_{-\infty}^{+\infty}(h+it)\,F(t)\,\bar{G}(t+i)\,\Gamma(h-it)\,\Gamma(h+it)\,dt$$
$$-\int_{-\infty}^{+\infty}(h-it)\,F(t+i)\,\bar{G}(t)\,\frac{\Gamma(h-it)\,\Gamma(h+it)}{\Gamma(\frac{1}{2}-it)\,\Gamma(\frac{1}{2}+it)}\,\varphi(t)\,dt$$
$$+\int_{-\infty}^{+\infty}(h+it)\,F(t)\,\bar{G}(t+i)\,\frac{\Gamma(h-it)\,\Gamma(h+it)}{\Gamma(\frac{1}{2}-it)\,\Gamma(\frac{1}{2}+it)}\,\varphi(t)\,dt$$

vanishes for all entire functions $F(z)$ and $G(z)$ of exponential type which are of polynomial growth on the real axis.

But every function which is bounded and analytic in the upper half-plane is a bounded limit in the half-plane of entire functions of exponential type which are bounded in the half-plane. It follows that the expression vanishes when

$$F(z) = \frac{\Gamma(-\frac{1}{2} - iz)}{\Gamma(h - iz)} f(z) \qquad \text{and} \qquad G(z) = \frac{\Gamma(-\frac{1}{2} - iz)}{\Gamma(h - iz)} g(z)$$

for entire functions $f(z)$ and $g(z)$ of exponential type which are bounded in the upper half-plane and vanish at $\frac{1}{2}i$. So the expression

$$\begin{aligned}
&\int_{-\infty}^{+\infty} h(t+i) \frac{\Gamma(\frac{1}{2} - it)\,\Gamma(\frac{1}{2} + it)}{\Gamma(h - it)\,\Gamma(h + it)}\, d\nu(t) \\
&\quad + \int_{-\infty}^{+\infty} h(t) \frac{\Gamma(\frac{1}{2} - it)\,\Gamma(\frac{1}{2} + it)}{\Gamma(h - it)\,\Gamma(h + it)}\, d\nu(t) \\
&\quad + p \int_{-\infty}^{+\infty} h(t+i)\, \Gamma(\tfrac{1}{2} - it)\,\Gamma(\tfrac{1}{2} + it)\, dt \\
&\quad + p \int_{-\infty}^{+\infty} h(t)\, \Gamma(\tfrac{1}{2} - it)\,\Gamma(\tfrac{1}{2} + it)\, dt \\
&\quad - \int_{-\infty}^{+\infty} h(t+i)\, \varphi(t)\, dt - \int_{-\infty}^{+\infty} h(t)\, \varphi(t)\, dt
\end{aligned}$$

vanishes for every entire function $h(z)$ of exponential type which is bounded on the real axis and vanishes at $-\frac{1}{2}i$. The choice of p and of the mass of σ at the origin can be made so that the expression vanishes for all entire functions $h(z)$ of exponential type which are bounded on the real axis. The resulting identity is applied when $h(z)$ is an exponential. Since a bounded measure on the Borel subsets of the real line is uniquely determined by its Fourier transform, the identity

$$\int_E \frac{\Gamma(\frac{1}{2} - it)\,\Gamma(\frac{1}{2} + it)}{\Gamma(h - it)\,\Gamma(h + it)}\, d\nu(t) + p \int_E \Gamma(\tfrac{1}{2} - it)\,\Gamma(\tfrac{1}{2} + it)\, dt$$
$$= \int_E \varphi(t)\, dt$$

holds for every real Borel set E. The theorem follows.

Proof of Theorem 6. By the remarks which precede the statement of the theorem, it is sufficient to consider the case in which $h+1$ is positive. The hypotheses imply that the identity

$$\int_{-\infty}^{+\infty} (h+1-it)\, F(t+i)\, \bar{G}(t)(t^2+h^2)\, d\nu(t)$$

$$-\int_{-\infty}^{+\infty} (h+1+it)\, F(t)\, \bar{G}(t+i)(t^2+h^2)\, d\nu(t)$$

$$= -2i \int_{-\infty}^{+\infty} tF(t+\tfrac{1}{2}i)\, \bar{G}(t+\tfrac{1}{2}i)(t^2+h^2+h+1/4)\, d\mu(t)$$

holds for all entire functions $F(z)$ and $G(z)$ of exponential type which are of polynomial growth on the real axis. By Theorem 5, a nonnegative number p and a nonnegative measure σ on the Borel subsets of the real line exist such that σ agrees with μ on every Borel set which does not contain the origin and such that the identity

$$\int_E (t^2+h^2)\, d\nu(t) + p \int_E \Gamma(h+1-it)\, \Gamma(h+1+it)\, dt$$

$$= \int_E \frac{\Gamma(h+1-it)\, \Gamma(h+1+it)}{\Gamma(\frac{1}{2}-it)\, \Gamma(\frac{1}{2}+it)}\, \varphi(t)\, dt$$

holds for every Borel set E, where $\varphi(z)$ is defined for σ as in Theorem 4.

An application of Cauchy's formula as in the proof of Theorem 4 will now show that p and the mass of σ at the origin must be chosen so that

$$\frac{\Gamma(h-iz)\, \Gamma(h+1+iz)}{\Gamma(\frac{1}{2}-iz)\, \Gamma(\frac{1}{2}+iz)}\, \varphi(z) - p\Gamma(h-iz)\, \Gamma(h+1+iz)$$

is analytic in the strip $0 < y < \frac{1}{2}$. The only possible singularity is a simple pole at $-ih$ if $-\frac{1}{2} < h < 0$ and at $i(h+1)$ if $-1 < h < -\frac{1}{2}$. When p and the mass of σ at the origin are not chosen so as to remove this singularity, the desired identity does not hold for all polynomials $F(z)$ and $G(z)$, but instead a different identity holds in which there is an added real multiple of

$$F(i-ih)\, \bar{G}(-ih) - F(-ih)\, \bar{G}(i-ih).$$

A similar situation holds in the limiting cases $h=0$ and $h=-\frac{1}{2}$. The choice of p and of the mass of σ at the origin are to be made so as to remove this extra term. This completes the proof of uniqueness when $-1 < h \leqslant 0$, and hence in all cases in which h is not positive.

Proof of Theorem 7. By Cauchy's formula,

$$\int_{-\infty}^{+\infty} \Gamma(n/2+1/2+h/2-it)\,\Gamma(n/2+1/2+h/2+it) \times (h/2-1/2-n/2-it)\,F(t+i)\,\bar{G}(t)\,dt$$
$$= \int_{-\infty}^{+\infty} \Gamma(n/2+h/2-it)\,\Gamma(n/2+1+h/2+it) + (h/2-1-n/2-it)\,F(t+\tfrac{1}{2}i)\,\bar{G}(t+\tfrac{1}{2}i)\,dt$$

and

$$\int_{-\infty}^{+\infty} \Gamma(n/2+1/2+h/2-it)\,\Gamma(n/2+1/2+h/2+it) \times (h/2-1/2-n/2+it)\,F(t)\,\bar{G}(t+i)\,dt$$
$$= \int_{-\infty}^{+\infty} \Gamma(n/2+1+h/2-it)\,\Gamma(n/2+h/2+it) \times (h/2-1-n/2+it)\,F(t+\tfrac{1}{2}i)\,\bar{G}(t+\tfrac{1}{2}i)\,dt.$$

The desired identity now follows from the recurrence relation $\Gamma(z+1) = z\Gamma(z)$ for the gamma function since

$$(n/2+h/2+it)(n/2+1-h/2+it) - (n/2+h/2-it)(n/1+1-h/2-it) = 2(n+1)it$$

for all real t.

Proof of Theorem 8. Regularity of the norm-determining measure for the space $\mathcal{N}(h)$ is verified by a straightforward calculation using the identity of Theorem 7. The measure has total mass one and is supported in the half-plane $z+\bar{z} \geqslant -1$ if $h \geqslant 1$. The measure with mass one concentrated at the origin is obviously regular.

Consider an extreme point μ of the set of regular nonnegative measures of total mass one which are supported in the half-plane $z+\bar{z} \geqslant -1$. By Theorem 2, a number h, $0 \leqslant h \leqslant 1$, exists such that μ is supported in the union of the vertical lines which intersect the real axis in points which are congruent to $-h/2$ modulo one-half. By Theorem 3, h can be chosen so that the vertical line through the point $-h/2$ contains a point of the support of μ. The mass of μ is concentrated at the origin if h is zero and if the support of μ contains no point of the vertical line which intersects the real axis in the point $-\frac{1}{2}$. Otherwise h can be chosen positive.

For each nonnegative integer n, let μ_n be the nonnegative measure on the

Borel subsets of the real line such that, for every Borel set E, the μ_n-measure of E is the μ-measure of the set of point $n/2 - h/2 + it$ with t in E. By Theorem 3, a positive number p exists such that

$$\mu_0(E) = p \int_E \Gamma(h/2 - it)\,\Gamma(h/2 + it)\,dt$$

for every real Borel set E. The identity of Theorem 1 implies that the identity

$$\int (h/2 - n/2 - it)\,F(t+i)\,\bar{G}(t)\,d\mu_n(t)$$

$$- \int (h/2 - n/2 + it)\,F(t)\,\bar{G}(t+i)\,d\mu_n(t)$$

$$= -2i \int tF(t + \tfrac{1}{2}i)\,\bar{G}(t + \tfrac{1}{2}i)\,d\mu_{n-1}(t)$$

holds for all polynomials $F(z)$ and $G(z)$ if n is positive.

By Theorems 6 and 7, the identity

$$n!\,\mu_n(E) = p \int_E \Gamma(n/2 + h/2 - it)\,\Gamma(n/2 + h/2 + it)\,dt$$

holds for every Borel set E when $h < 1$ or $n > 1$. When $n = 1$ and $h = 1$, an additional nonnegative measure with mass concentrated at the origin might appear as an added term in μ_1, but it is zero because μ is assumed to be an extreme point. It follows that μ is a positive multiple of the norm-determining measure for $\mathcal{N}(h)$. Since μ has total mass one, it is the norm-determining measure for $\mathcal{N}(h)$.

By the Krein–Milman theorem, the set of regular nonnegative measures with total mass one which are supported in the half-plane $z + \bar{z} \geqslant -1$ is the closed convex span of its extreme points. It follows that the norm-determining measure for $\mathcal{N}(h)$ is an extreme point of the set when $0 < h \leqslant 1$. The measure with mass one concentrated at the origin is obviously an extreme point.

Proof of Theorem 9. The set of regular nonnegative measures of total mass one which are supported in the half-plane $z + \bar{z} \geqslant -1$ is a compact convex set. By the Krein–Milman theorem, the set is the closed convex span of its extreme points. By Theorem 8, the extreme points are the norm-determining measures for $\mathcal{N}(h)$ when $0 < h \leqslant 1$ and the measure with mass one concentrated at the origin. The theorem now follows from a computation of moment-generating functions of these extremal measures. The moment-generating function of the norm-determining measure for $\mathcal{N}(h)$ is $(1 - z)^h$.

The moment-generating function of the measure with mass one concentrated at the origin is 1.

Proof of Theorem 10. The theorem is a consequence of the identity

$$S_n(z+w)=\sum S_k(z)\,S_{n-k}(w),$$

where $S_n(z)$ is the nth Newton polynomial. Summation is from $k=0$ to $k=n$.

Proof of Theorem 11. The convolution μ of the norm-determining measure for $\mathcal{N}(h_1)$ and the norm-determining measure for $\mathcal{N}(h_2)$ is supported in the half-plane $z+\bar{z}\geqslant -h_1-h_2$. By Theorem 10, μ is a regular measure whose moment-generating function is the product $(1-z)^{-h_1-h_2}$ of the moment-generating functions $(1-z)^{-h_1}$ of the norm-determining measure for $\mathcal{N}(h_1)$ and the moment-generating function $(1-z)^{-h_2}$ of the norm-determining measure for $\mathcal{N}(h_2)$. So the moment-generating function of μ is equal to the moment-generating function of the norm-determining measure of $\mathcal{N}(h_1+h_2)$. By the proof of Theorem 1, two regular measures which have the same moment-generating functions are equal because they have the same Fourier transforms.

Proof of Theorem 12. It is clearly sufficient to give a proof in the case that μ_- is the norm-determining measure for $\mathcal{N}(h)$ for some positive number h. Then σ_- is the measure with mass one concentrated at h. Choose μ_+ to be h times the norm-determining measure for $\mathcal{N}(h+1)$. The theorem is proved by verifying that the identity

$$\langle zF(z-1),G(z)\rangle_{\mathcal{N}(h)}=h\langle F(z),G(z+1)-G(z)\rangle_{\mathcal{N}(h+1)}$$

holds for all polynomials $F(z)$ and $G(z)$. By linearity it is sufficient to verify the identity in the case that $F(z)$ is the mth Newton polynomial and $G(z)$ is the nth Newton polynomial for some nonnegative integers m and n. Each side of the identity is zero unless $n=m+1$, in which case $zF(z-1)=-nG(z)$ and $G(z+1)-G(e)=-F(z)$. The desired identity holds because

$$n\,\|G(z)\|^2_{\mathcal{N}(h)}=\frac{h(h+1)\cdots(h+n)}{1\cdots m}=h\,\|F(z)\|^2_{\mathcal{N}(h+1)}.$$

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