



Gapless superconductivity and string theory

Sergei Khlebnikov

Department of Physics and Astronomy, Purdue University, West Lafayette, IN 47907, United States

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Abstract

Coexistence of superconducting and normal components in nanowires at currents below the critical (a “mixed” state) would have important consequences for the nature and range of potential applications of these systems. For clean samples, it represents a genuine interaction effect, not seen in the mean-field theory. Here we consider properties of such a state in the gravity dual of a strongly coupled superconductor constructed from D3 and D5 branes. We find numerically uniform gapless solutions containing both components but argue that they are unstable against phase separation, as their free energies are not convex. We speculate on the possible nature of the resulting non-uniform state (“emulsion”) and draw analogies between that state and the familiar mixed state of a type II superconductor in a magnetic field.

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1. Introduction

Many proposed applications of nanoscale superconductors require understanding of how these systems behave under currents close to the critical. For instance, in designing superconducting qubits, it is essential to know how to use current to suppress the potential barrier separating the basis states.

The best studied example of nanoscale superconductor is a point-like weak link—a Josephson junction (JJ). It can often be described by a single-degree of freedom θ —the phase difference between the leads—subject to a “tilted washboard” potential

$$V(\theta) = -V_0 \cos \theta - I\theta. \quad (1)$$

E-mail address: skhlebb@purdue.edu.

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Here I is the electric current in units of $2e$; e is the electron charge. For *static* θ (when V equals the total energy of the system), variation of (1) produces the equation $V_0 \sin \theta = I$, showing that the current I is due to a gradient of the phase, that is, I is entirely a supercurrent: $I = I_s$. For a time-dependent θ , however, the current contains both superconducting and normal components. Thus, in general

$$I = I_s + I_n. \quad (2)$$

The normal component I_n includes not only the normal current through the JJ itself (due, for instance, to thermal quasiparticles) but also currents through various external resistors (“shunts”) connected in parallel to it. This is because all such currents couple to I_s via voltage fluctuations, which are proportional to the time derivative of θ .

For any $-V_0 < I < V_0$, the potential (1) has infinitely many minima, equally spaced by 2π , and important fluctuations are those that take the system from one minimum to the next. These are known as phase slips. Each phase slip generates a voltage spike in the external circuit. If such spikes occur at a non-negligible rate, at a finite I they will produce a nonzero time-averaged voltage, i.e., a finite resistance.

Recently, a number of experimental techniques have been developed for synthesizing systems in which superconductivity is one-dimensional—superconducting (SC) nanowires. These techniques, described in the books [1,2], result in wires of uniform thickness with linear cross-sectional dimensions of a few nanometers. For such thin wires, one can assume that SC properties (e.g., the supercurrent density) depend only on the lengthwise direction, even though the electron density of states still retains the 3d character. These novel systems promise a potentially new class of devices for control of superconductivity by current.

Even though we do not expect the model (1) to apply literally to the case of wires, some of the notions discussed above do carry over. The phase of the order parameter, $\phi(x, t)$, is now a function of the coordinate x along the wire and time t . The supercurrent is proportional to the gradient of the phase: $I_s \propto \nabla \phi$. For states where ϕ is a continuous function of x , we can identify the winding number as

$$W(t) = \frac{1}{2\pi} [\phi(L, t) - \phi(0, t)] \quad (3)$$

(L is the length of the wire) and a phase slip as an event that changes W by ± 1 . There is a novel aspect to a phase slip in a wire (as opposed to the case of JJ), which has to do with the continuity of ϕ . Namely, the process now occurs locally, at some point x , where the order parameter momentarily vanishes, allowing the phase to unwind [3].

Similarly to the case of JJ, one can imagine a nanowire shunted by various external impedances, resulting in a normal current connected in parallel to the superconducting one, for the total given by the same Eq. (2). In this paper, however, we wish to consider the possibility of an *intrinsic* resistive effect, namely, a normal component that is formed in the wire itself (and, unlike thermal quasiparticles, survives in the limit $T \rightarrow 0$). Such a resistor will remain even in a wire effectively decoupled from any external dissipative environment, for example, in a SC loop operated via inductive coupling to a coil.

For a nanowire, one can consider, at least theoretically, two extreme limits. One is the clean limit—a perfectly uniform wire without disorder; the other is the dirty limit—a wire with strong disorder scattering. The second limit is presumably more realistic, but the first is simpler and,

as such, may be a useful starting point. In this paper, we consider the clean limit exclusively. We also restrict ourselves to $T = 0$, where the question of existence of an intrinsic normal component is in a sense the sharpest, although the method we propose is applicable also at $T \neq 0$.

As in a JJ, the SC and normal components in a wire are coupled via phase slips. The clean limit is momentum-conserving, so the momentum unwound by a phase slip from the supercurrent must be picked up, in its entirety, by the normal component. The latter may in principle include small oscillations of the SC density (the plasma waves [4,5]), which, similarly to waves in a waveguide, are characterized by a finite impedance, but a more detailed study shows that there is a quantum anomaly involved, and each phase slip produces, via level crossing, fermionic quasiparticles, in precisely the right number to account for conservation of momentum [6].

The requirement of quasiparticle production affects the energy balance in a phase slip: for the process to occur spontaneously, the energy unwound from the supercurrent must be enough to offset the cost of the produced fermions. In mean-field theory, the free energy unwound from a loop of length L gives directly the supercurrent:

$$I_s = \frac{1}{2\pi} \frac{\partial F}{\partial W}, \quad (4)$$

where W is the winding number (3). So, one may suppose that phase slips become more favorable at larger I_s . While that is true to a degree, a direct calculation shows that, within mean-field theory, the energy $\partial F/\partial W$ is never large enough—that is not until I_s reaches the depairing current I_{dep} .¹ The question we wish to ask is whether this conclusion is a mean-field artifact; in other words, whether a window in which SC and normal components can coexist will open (below I_{dep}) once electron–electron interactions are fully taken into account.

One possible way to answer this question is to construct a superconductor from strings and branes and go over to the strong-coupling (large N) limit, in which N coincident branes behave as a classical gravitating object [7]; such an alternative description of a quantum system is known as a gravity dual. A well-known example of gauge/gravity duality is the AdS/CFT correspondence [8], for which a “holographic” dictionary connecting the two sides of the duality has been established [9,10]. Calculations using a gravity dual, however, are possible even in cases where a complete dictionary is not known, as long as one concentrates on those quantities that can in fact be unambiguously defined on the gravity side. The quantity we are interested in here is the energy of the ground state, $F(P_s)$, as a function of the momentum of the SC component, P_s , at fixed total momentum P . The difference

$$P_n = P - P_s \quad (5)$$

can then be attributed to the normal component. This can be seen as a clean-limit version of the formula (2).² A ground state with both component present corresponds to a minimum of $F(P_s)$ for which both P_s and P_n are nonzero.

¹ This conclusion holds rather generally, provided one neglects corrections suppressed by the ratio of the gap to the energy scale of the band structure. It does not depend on Galilean invariance or other such special symmetries.

² In our earlier paper [11], current and momentum were spoken of largely interchangeably. Here, we aim to be more careful about the distinction.

A brane construction suitable for modeling a SC nanowire has been proposed in [11]. It is based on a system of D3 and D5 branes in type IIB string theory.³ The key aspects of it are as follows:

(i) The setup contains a large number N of coincident D3 branes and a D5 probe intersecting them over a line. This breaks all supersymmetries. The direction along the line, $x \equiv x^1$, corresponds to the direction of the wire, and the two directions transverse to all branes, $x^8 + ix^9$, to the SC order parameter. (ii) N is identified with the number of occupied channels (transverse wavefunctions) in the wire, $N = N_{ch}$.⁴ (iii) Supercurrent corresponds to the D5 winding around the D3s as one moves along x . If the branes actually intersect, the low-energy spectrum of 3/5 strings contains N species of massless (1 + 1)-dimensional fermions (both left and right movers). At low values of P , however, the intersection is unstable, and the D5 moves a finite distance away from the D3s; this corresponds to a fully gapped, supercurrent-only state.

Supercurrent-only solutions have been found in [11]. The Chern–Simons term in the D5 action has the effect that the solution with winding number W carries NW units of the D5 worldvolume charge. A phase slip corresponds to the D5 crossing the D3s, with W changing by one. Conservation of charge then implies that N fundamental strings, stretching between the D5 and the D3s, must be produced. This can be seen, on the one hand, as a version of the Hanany–Witten effect [20] in string theory (creation of branes and strings at intersections) and, on the other, as a parallel to the requirement of quasiparticle production noted earlier. This parallel allows one to identify the worldvolume charge with the supercurrent momentum P_s (which, for a supercurrent-only state, is also the total momentum), as follows:

$$P_s = NW, \quad (6)$$

where by convention P_s is in units of the Fermi momentum. Instead of W , we will often use the winding number density

$$q = 2\pi W/L, \quad (7)$$

where an extra factor 2π is added for convenience.

In the leading large- N limit, supercurrent-only solutions exist for all q , no matter how large. At q above a certain q_m , however, phase slips become energetically favorable, and the supercurrent-only state unstable. The instability has nothing to do with depairing. Indeed, $q_m \sim 1/R$, where R is the length scale of the D3 metric (the only length scale seen by classical gravity). Meanwhile, the value $q = q_{dep}$ corresponding to depairing is of order of the gap Δ , i.e., of order R in units of the string tension. One sees that the ratio q_{dep}/q_m scales to infinity in the large N limit.⁵

Instability of the supercurrent-only state at $q > q_m$ means that for

$$P > P_m = Nq_m L/2\pi \quad (8)$$

³ Our approach is different from other holographic descriptions of superconductivity that have been proposed in the literature. It is distinct from the one in [12–14] in that it does not use a bulk U(1) gauge field. (States with nonzero supercurrent in the model of [12,13] have been considered in [15–18].) And our approach is distinct from the brane construction of [19] in that it preserves the worldvolume gauge symmetry. We use the corresponding conserved charge to describe the linear momentum (quantized in units of the Fermi momentum k_F).

⁴ N_{ch} is proportional to the cross-sectional area of the wire and in practice is of order of a few thousand, for the thinnest wires available. The large value reflects the 3d character of the electron density of states; this is in contrast to SC properties, which vary only along x .

⁵ This may explain why no depairing is seen in the calculation of [11].

not all of the total momentum P in the true ground state is carried by supercurrent; some must be carried by quasiparticles. On the other hand, we find that the normal-only state is unstable for any P . We conclude that, at P exceeding the bound (8), supercurrent and quasiparticles must coexist in some mixed state.

A priori, it is not clear what the nature of the mixed state is and, in particular, if it can be described by a uniform (in x) time-independent solution of dual gravity (as the supercurrent-only state could). Nevertheless, looking for such solutions is a natural first step, and that is what we describe in this paper. We find that, for a given total P , a uniform time-independent solution exists for all P_s between the value at which the normal-only state first becomes unstable and the maximum $P_s = P$. For these solutions, the D5 crosses the D3s' horizon; we argue their existence by considering the near-horizon limit and by numerical evidence. We find, however, that the free energy of such a solution is never a convex function of P_s . This means that the uniform mixed state is unstable towards phase separation, fragmenting eventually, we believe, into an “emulsion” of quasiparticle-rich droplets in a nearly quasiparticle-free matrix.⁶ Some implications of this picture are discussed in the conclusion.

2. Preliminaries

The 10-dimensional metric sourced by N coincident extremal D3 branes in type IIB supergravity is [21]

$$ds^2 = \frac{1}{\sqrt{f}}(-dt^2 + (dx^i)^2) + \sqrt{f}(d\rho^2 + \rho^2 d\Omega_3^2 + d\Delta^2 + \Delta^2 d\phi^2). \quad (9)$$

The coordinates along the D3s are t and x^i , $i = 1, 2, 3$. The transverse coordinates are x^4, \dots, x^9 , out of which we have constructed a spherical system for x^4, \dots, x^7 , with radius ρ , and a polar system for x^8, x^9 , with radius Δ . Thus, ϕ is equivalent to $\phi + 2\pi$. The metric function f depends only on $r = (\rho^2 + \Delta^2)^{1/2}$ and equals

$$f(r) = 1 + \frac{R^4}{r^4} = 1 + \frac{R^4}{(\rho^2 + \Delta^2)^2}, \quad (10)$$

where

$$R^4 = 4\pi g_s (\alpha')^2 N, \quad (11)$$

g_s is the closed string coupling, and $1/(2\pi\alpha')$ is the string tension.

The probe D5 wraps x^1 and x^4, \dots, x^7 (breaking all supersymmetries). Thus, the only spatial direction common to all branes is $x^1 \equiv x$, and the directions in which all branes have definite positions are Δ and ϕ . The complex position

$$\Psi = \Delta e^{i\phi} = x^8 + ix^9 \quad (12)$$

of the D5 relative to the D3s plays the role of a superconducting order parameter. We will also use the real position vector

$$\mathbf{X} = (x^8, x^9) = (\Delta \cos \phi, \Delta \sin \phi). \quad (13)$$

⁶ The term “mixed state” may then be quite apt, as such a state would be reminiscent of the mixed state of type II superconductors in a magnetic field, with a difference that the droplets now carry “electric” rather than magnetic fluxes.

Embedding the D5 in the geometry (9) means specifying x^2, x^3, x^8, x^9 and the worldvolume gauge field A , all as functions of the worldvolume coordinates. In this paper, we consider only embeddings that have $x^2 = x^3 = 0$ and are constant over the 3-sphere in (9). These, then, are specified by

$$\mathbf{X} = \mathbf{X}(t, x, \rho), \quad (14)$$

$$A_a = A_a(t, x, \rho), \quad (15)$$

where $a = t, x, \rho$. The normal state corresponds to

$$\mathbf{X} = 0, \quad (16)$$

$$A_t = A_t(\rho), \quad (17)$$

with all the other components of A equal to zero. We refer to this as the trivial embedding.

As we will see, the trivial embedding is unstable: the D5 develops a nontrivial profile $\Delta(x, \rho)$ with characteristic magnitude $\Delta \sim R$. As a result, the near-horizon (decoupling) limit $r \ll R$, in which the background (9) approaches the $\text{AdS}_5 \times S^5$ space, and the type IIB string theory on it becomes dual to a conformal field theory (CFT) [8–10], cannot be taken here. This means that, in the description of the SC state, one cannot replace the 3/5 strings with their ground states; the entire ladder of excited string states remains. In a superconductor, that can be interpreted as quasiparticles acquiring an internal structure. While there is nothing wrong with this in principle, in practice one faces the problem of how to define, let alone use, this theory. On the other hand, on the gravity side, the low-energy modes are described by the action of the D5 embedded in the full D3 geometry (9), and the high-energy (stringy) modes are described by strings connecting the branes. As a result, many properties of the superconductor can be computed on the gravity side even without a complete definition of the dual quantum theory. In this paper, we consider several of these properties. They are (i) the symmetry breaking pattern, (ii) the quasiparticle gap (which is given by the minimal energy of the 3/5 strings), and (iii) the free energy, computed from the action of the D5 in the geometry (9).

Our theory has two $U(1)$ symmetries: one is the phase rotation of Ψ , and the other is the gauge symmetry on the D5 worldvolume; the latter has (15) for the gauge field. The first $U(1)$ is spontaneously broken by a nonzero Δ ,⁷ but the worldvolume $U(1)$ remains exact.⁸ The corresponding conserved charge has been identified with the linear momentum (in units of the Fermi momentum) in the superconductor [11], and we will mention one motivation for that shortly. Incidentally, this identification implies that, to describe a disordered superconductors, where momentum is not conserved, one will need will need a mechanism for breaking the worldvolume $U(1)$. We will touch upon this problem in the conclusion but for the rest of the paper proceed with the clean, momentum-conserving case.

The action for the D5 consists of the DBI action and a Chern–Simons (CS) term [22]; the latter describes interaction of the D5 with the 5-form field strength sourced by the D3s. For embeddings of the form (14), (15), the action can be written concisely with the help of a fictitious metric

⁷ Quantum fluctuations that could conceivably restore this symmetry in a thin wire are not seen in the leading order of the large- N approximation.

⁸ This makes our construction quite different from that of [19], in which superconductivity is related to breaking of a worldvolume gauge symmetry.

$$h_{ab} = \text{diag}(-1/f, 1/f, 1), \quad (18)$$

where f is the function (10). Let us also define the quantities

$$B^a = \frac{1}{2} \epsilon^{abc} F_{bc}, \quad (19)$$

where $F_{ab} = \partial_a A_b - \partial_b A_a$ and ϵ^{abc} is the completely antisymmetric unit tensor ($\epsilon^{tx\rho} = 1$), and the cross product

$$\mathbf{X}_{,a} \times \mathbf{X}_{,b} = x_{,a}^8 x_{,b}^9 - x_{,a}^9 x_{,b}^8. \quad (20)$$

Subscript commas denote partial derivatives.

In what follows, we choose R as our unit of length: $R = 1$.

By lowering (raising) indices with h_{ab} (its inverse), the DBI term can be written as

$$S_{\text{DBI}} = -2\pi^2 \tau_5 \int dt dx d\rho \rho^3 \sqrt{f} D^{1/2}, \quad (21)$$

where

$$D = 1 + \mathbf{X}^a \mathbf{X}_{,a} + \frac{1}{2} (\mathbf{X}^a \times \mathbf{X}^b) (\mathbf{X}_{,a} \times \mathbf{X}_{,b}) - f [B^a B_a + (B^a \mathbf{X}_{,a})^2], \quad (22)$$

and the CS term as

$$S_{\text{CS}} = 2\pi^2 \tau_5 \int dt dx d\rho \epsilon^{abc} A_a \phi_{,b} \Pi_{,c}, \quad (23)$$

where

$$\Pi(t, x, \rho) = \frac{\rho^4}{[\rho^2 + \Delta^2(t, x, \rho)]^2}. \quad (24)$$

In (21) and (23),

$$\tau_5 = \frac{1}{(2\pi)^5 g_s (\alpha')^3} \quad (25)$$

is the brane tension.

3. Identification of the momentum components

Variation of S_{CS} with respect to $A_a/2\pi\alpha'$ is the conserved U(1) current induced on the D5 worldvolume:

$$K^a = 4\pi^3 \alpha' \tau_5 \epsilon^{abc} \phi_{,b} \Pi_{,c} = \frac{N}{2\pi} \epsilon^{abc} \phi_{,b} \Pi_{,c}. \quad (26)$$

The temporal component of this is the worldvolume charge density. We see that wound configurations of the D5 (i.e., those for which the phase gradient $\phi_{,x}$ is nonzero) carry a charge density proportional to $\phi_{,x}$. This motivates the identification of the total induced charge,

$$P_s = \int dx d\rho K^t, \quad (27)$$

with the momentum (in units of the Fermi momentum k_F) carried by the SC component. (The fact that the coefficient in the relation between P_s and the momentum is exactly unity can be

established by observing that a single 3/5 string carries unit charge and corresponds to a quasi-particle that carries $\pm k_F$ of momentum.)

It may be more familiar (see, for example, [19]) to have the variational derivative with respect to A_t correspond to the number density in the dual theory, and the derivative with respect to A_x to the current density. Here, we relate $\delta S_{CS}/\delta A_t$ to the momentum, which is more like current than charge. To see how this transmutation of charge into current comes about in the dual theory, consider the way the original electron operators are packaged into the Dirac fermions representing the ground states of the 3/5 strings. The fermions, χ_n , are $(1+1)$ -dimensional but carry a Chan–Paton index $n = 1, \dots, N$. In the dual superconductor, n corresponds to the conductance channel, i.e., the transverse wavefunction, occupied by the fermion [11] (thus, a larger N means a thicker sample). Each χ_n is two-component, and the bilinear that couples to the order parameter (12) is $\mathcal{O} = \sum_n \bar{\chi}_n (1 + \gamma^5) \chi_n$. Choose the representation where the Dirac γ^0 is the Pauli σ_1 , and γ_5 is σ_3 . Then, for the bilinear \mathcal{O} to represent the SC pairing channel, the upper component of each χ (we omit the subscript n now) must be the annihilation operator of a right-moving electron, a_R , and the lower component the creation operator of a left-moving one, b_L^\dagger . As a result, the normally ordered “charge” density $\bar{\chi} \gamma^0 \chi$, which couples to A_t , is proportional to $a_R^\dagger a_R - b_L^\dagger b_L$, i.e., the momentum density (in units of k_F) in the dual theory, and the “current” density $\bar{\chi} \gamma^1 \chi$, which couples to A_x , to the number density.

A useful expression for P_s can be obtained from the Gauss law, the temporal component of the equation of motion for A_a . The equation reads

$$-\epsilon^{abc} \partial_b (\rho^3 \sqrt{f} H_c) = \frac{2\pi}{N} K^a, \quad (28)$$

where

$$H_c = -\frac{\partial \sqrt{D}}{\partial B^c} = \frac{f}{\sqrt{D}} [B_c + (B^b \mathbf{X}_{,b}) \cdot \mathbf{X}_{,c}]. \quad (29)$$

From now on we assume that the x direction is a circle (of length L). Then, setting $a = t$ in (28) and integrating over x and ρ gives

$$\int dx d\rho K^t = P(\infty) - P(0), \quad (30)$$

where

$$P(\rho) = \frac{N}{2\pi} \int dx \rho^3 \sqrt{f} H_x. \quad (31)$$

$P(\rho)$ is the flux of the “electric” induction through a surface of constant ρ . The value $P(0)$ is nonzero only if the D5 crosses the horizon, $r = 0$, of the geometry (9), which solutions considered in this paper do. One can visualize it as an effect of 3/5 strings that have fallen through the horizon and pulled the D5 with them. In this picture, $P(0)$ is the total worldvolume charge carried by these strings.⁹ Alternatively, one could think of $P(0)$ simply as an additional variable characterizing the boundary of the D5 at $r = 0$. The interpretation of it as a charge “behind the horizon,” however, is helpful in understanding that this variable is dynamical (as we discuss in more detail in Section 7): if additional quasiparticles (3/5 strings) are produced by unwinding

⁹ We wish to stress here that, unlike in descriptions of supercurrent that use a bulk gauge field [12–14], in our case the “electric” field lives only on the D5 worldvolume.

the supercurrent and fall through the horizon, both P_s and $P(0)$ change; only the total, $P(\infty)$, is conserved. According to our interpretation of the charge as momentum, $P(\infty)$ is the total momentum of electrons in the wire. Comparing (30) with (27), we see that $P_n = P(0)$ can be identified with the momentum of the normal component: it adds to the supercurrent momentum P_s for the total of $P \equiv P(\infty)$, cf. Eq. (5).

Eq. (28) has a class of solutions (first integrals) of the form

$$\rho^3 \sqrt{f} H_a = \phi_{,a} (\Pi - 1) + J_a, \quad (32)$$

where J_a are integration constants. These solutions are not the most general, as one can always add a gradient to the right-hand side of (32) without affecting the curl in (28), but they will be sufficient for our purposes. In fact, we will restrict the class of solutions even further—to those for which the only nonzero constant is J_x , and we will use a special notation for it:

$$J_x \equiv J_\infty, \quad (33)$$

$$J_t = J_\rho = 0. \quad (34)$$

As we will see shortly, J_∞ corresponds to the total (SC plus normal) momentum density in the wire. One may suspect that, similarly, a nonzero J_t would describe variations in the fermion number density (that is, in the Fermi momentum) but, as (34) implies, that will not be pursued here.

Using (32), Eq. (31) can be written as

$$P(\rho) = \frac{N}{2\pi} \int dx [\phi_{,x} (\Pi - 1) + J_\infty]. \quad (35)$$

For all solutions considered in this paper $\Pi(x, \infty) = 1$. This is a consequence of the boundary condition

$$\Delta \rightarrow 0 \quad \text{at} \quad \rho \rightarrow \infty, \quad (36)$$

which means that there is no symmetry-breaking source, i.e., the U(1) that rotates the phase of Ψ is broken spontaneously rather than explicitly. Then, at large ρ , the integrand on right-hand side of (35) is simply J_∞ . According to our earlier interpretation of $P(\infty)$, this means that J_∞ is, up to a factor of $N/2\pi$, the linear density of the total momentum.¹⁰ Similarly, sending $\rho \rightarrow 0$ allows us to identify

$$J_s(x) = \phi_{,x}(x, 0) [1 - \Pi(x, 0)] \quad (37)$$

as the momentum density of the supercurrent (up to the same factor) and the difference $J_\infty - J_s$ as that of the normal component.

4. Instability of the normal state

Consider linearized theory near the trivial embedding (16), (17). To the linear order in \mathbf{X} ,

$$H_c = \frac{f_0 B_c}{(1 - f_0 B^a B_a)^{1/2}} \quad (38)$$

¹⁰ This suggests a generalization (which we do not pursue here)—a J_∞ depending arbitrarily on x . This does not affect the curl in Eq. (28), so the equation is still satisfied. An x -dependent J_∞ may describe, for instance, supplying extra current to segments of the wire by connecting them to external leads.

where

$$f_0 = 1 + \frac{1}{\rho^4}. \quad (39)$$

To this order, subject to the conditions (34), there are no sources in (32) for H_t and H_ρ , while for H_x the only source is J_∞ . As a result, B^a are unchanged from the zeroth order, namely, $B^t = B^\rho = 0$ and

$$B^x(\rho) = -F_{t\rho}(\rho) = \frac{J_\infty}{C_0^{1/2}(\rho)}, \quad (40)$$

where

$$C_0(\rho) = \rho^6 f_0(\rho) + J_\infty^2. \quad (41)$$

Upon substituting (40), the linearized equation for \mathbf{X} , written in terms of the complexified coordinate (12), reads

$$\frac{1}{\sqrt{C_0}} \frac{\partial}{\partial \rho} (\sqrt{C_0} \Psi_{,\rho}) + \frac{2\Psi}{C_0} - f_0 \ddot{\Psi} + \frac{\rho^6 f_0^2}{C_0} \Psi_{,xx} - i \frac{4J_\infty}{\rho^2 C_0} \Psi_{,x} = 0. \quad (42)$$

The general solution is a superposition of plane waves

$$\Psi(t, x, \rho) = \Delta(\rho) e^{-i\omega t + iqx}. \quad (43)$$

Here q is real but ω can be complex. For $q \neq 0$, Eq. (43) describes a D5 uniformly wound about the D3s, the total of $W = qL/2\pi$ times. As we have already noted, in our interpretation, such wound states describe supercurrent.

We are interested in unstable modes—those $\Delta(\rho)$ for which ω has a positive imaginary part. Suppose $J_\infty > 0$. Then, at small ρ , unstable modes behave as $\Delta \sim \rho \exp(i\omega/\rho)$, i.e., vanish exponentially. They also vanish at $\rho \rightarrow \infty$. The equation for Δ obtained by substituting (43) in (42) has the form of a Schrödinger equation, and the boundary conditions just established mean that the unstable modes are its bound states. Asymptotically, at small ρ , the equation is

$$\Delta_{,\rho\rho} + \frac{\omega^2}{\rho^4} \Delta + \frac{\alpha}{\rho^2} \Delta = 0, \quad (44)$$

where

$$\alpha = 4q/J_\infty - (q/J_\infty)^2. \quad (45)$$

A change of the independent variable to $z = 1/\rho$ shows that this is a “fall to the center” problem. It becomes supercritical for $\alpha > \frac{1}{4}$, which means that for these α the full Schrödinger problem has an infinite number of bound states [23]. In our case, such α exist for any $J_\infty > 0$, with the maximum $\alpha = 4$ reached at $q/J_\infty = 2$. We conclude that, for any $J_\infty > 0$, the normal state is unstable. The instability band is at least as broad as the supercritical range

$$2 - \frac{\sqrt{15}}{2} < \frac{q}{J_\infty} < 2 + \frac{\sqrt{15}}{2}, \quad (46)$$

but may actually be broader, since it will include also those q for which there is only a finite number of unstable modes.

A numerical solution to the full Eq. (42) produces the instability chart shown in Fig. 1. Note that, for comparatively small J_∞ , the instability band includes the value $q = 0$, even though the

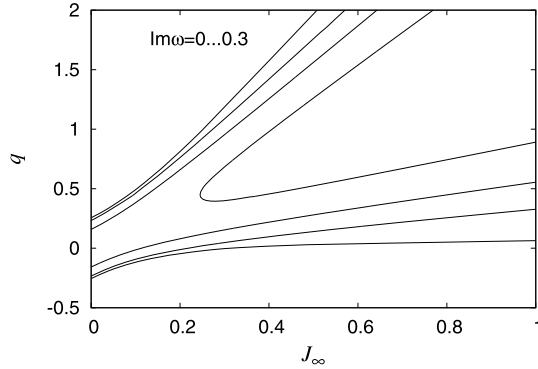


Fig. 1. Instability chart of Eq. (42) after substitution (43). The lines are the level contours of $\text{Im } \omega$ as it is increased in increments of 0.1. The outermost pair of lines corresponds to $\text{Im } \omega = 0$ and forms the boundary of the instability band.

latter is not in the supercritical range (46), but for large J_∞ , the instability occurs only for modes with q above a certain nonzero minimum: the emerging SC state necessarily has a supercurrent.¹¹

5. Uniform mixed states

The next question is what is the final state that the instability leads to. Natural first tries are the simplest configurations—those that are time-independent and uniform. By the latter we mean that Δ is independent of x , while ϕ winds along x uniformly:

$$\Delta = \Delta(\rho), \tag{47}$$

$$\phi = \phi(x) = qx. \tag{48}$$

For such configurations, the matrix

$$M_{cb} = h_{cb} + \mathbf{X}_{,c} \cdot \mathbf{X}_{,b} \tag{49}$$

multiplying B^b in (29) is diagonal. Expressing H_a from (32), subject to the conditions (34), and B^a from (29), we find

$$B^t = B^\rho = 0 \tag{50}$$

$$B^x(\rho) = -F_{t\rho}(\rho) = \frac{J(\rho)}{C^{1/2}(\rho)} (1 + \Delta^2_{,\rho})^{1/2}, \tag{51}$$

where the various functions (of ρ only) are given by

$$J(\rho) = q[\Pi(\rho) - 1] + J_\infty, \tag{52}$$

$$C(\rho) = \rho^6 f_\Delta(\rho) [1 + q^2 \Delta^2(\rho) f_\Delta(\rho)] + J^2(\rho), \tag{53}$$

$$f_\Delta(\rho) = 1 + \frac{1}{[\rho^2 + \Delta^2(\rho)]^2}. \tag{54}$$

These equations are the nonlinear counterparts to Eqs. (39), (40), (41) of the linear theory.

¹¹ For a loop of a finite length L , the values of q are quantized, and for a short loop it is possible that, at a given J_∞ , all the allowed values of q fall outside the instability band. In this case, there will be a curious reentrant behavior, when the normal state, linearly stable at that J_∞ , becomes linearly unstable again at a larger one.

The equation of motion for $\Delta(\rho)$, obtained by varying the action and substituting (50) and (51), is

$$\frac{d}{d\rho} \frac{\Delta_{,\rho} \sqrt{C}}{(1 + \Delta_{,\rho}^2)^{1/2}} = (1 + \Delta_{,\rho}^2)^{1/2} \frac{\partial \sqrt{C}}{\partial \Delta}. \quad (55)$$

The boundary condition at infinity is (36). We now proceed to establish the condition at $\rho \rightarrow 0$.

The first part of the argument is standard (for an application to a different system, see for example [24]). Recall from Section 3, Eq. (35), that $J(\rho)$ represents the flux of the worldvolume electric field through a surface of constant ρ . If $\Delta(0) \neq 0$, the D5 closes off at a finite distance from the D3s. In this case, we must have $J(0) = 0$; otherwise, the lines of the field have nowhere to end. More formally, for $J(0) \neq 0$, Eq. (51) predicts $F_{t\rho}(0) \neq 0$, meaning that the gauge field is not smooth. The only way to accommodate $J(0) \neq 0$ therefore is to have $\Delta(0) = 0$. Then, the D5 crosses the horizon, and the flux at $\rho = 0$ can be ascribed to charges behind the horizon, as discussed in Section 3. According to the interpretation of the fluxes there, for a uniform solution, $J(0)$ is the momentum density of the normal component. As we are interested here specifically in solutions for which that is nonzero, we postulate¹²

$$\Delta(0) = 0. \quad (56)$$

Eq. (56) implies that the shortest strings connecting the D5 to the D3s are of length zero, i.e., the superconductor is gapless, which is consistent with the presence of a normal component.

The second part of the argument seeks to establish the manner in which $\Delta(\rho)$ vanishes at $\rho \rightarrow 0$. The only type of solutions we have been able to find are those for which that happens slower than linearly, with Δ maintaining its sign (for definiteness, positive) at small nonzero ρ :

$$\rho^{-1} \Delta(\rho) \rightarrow \infty \quad \text{at} \quad \rho \rightarrow 0. \quad (57)$$

The precise asymptotics is discussed in Section 6.

Under the condition (57), $\Pi(0)$ in Eq. (37) is zero (and there is no longer a dependence on x as the solution is uniform), so according to that equation the momentum density of the supercurrent is

$$J_s = q. \quad (58)$$

This is the same expression as obtains for the supercurrent-only solutions, cf. Eq. (27). It is as if each electron in the wire contributes momentum $q/2$ to the superflow.¹³ The reason why this applies even in the presence of a normal component is that, under our present approximations, the number of “normal electrons” is much smaller than the total number: with the length scale R restored, the former is $P_n \sim NL/R$, while the latter is of order NLk_F . In other words, although P_s and P_n may be comparable, the first of these is due to a large number of “superconducting electrons” each contributing the small momentum $q/2$, while the second is due to a small number of “normal electrons” each carrying the large momentum k_F .

¹² The supercurrent-only solutions of [11], in contrast, have $J(0) = 0$ and $\Delta(0) \neq 0$.

¹³ The total number of electrons in a wire with N channels is Nk_FL/π (we define channels so that each contains only one projection of spin). Dividing NWk_F by this number gives $q/2$ per electron.

6. Near-horizon limit and numerical solutions

Consider the limit of Eq. (55) at $\rho \rightarrow 0$. Recalling the condition (57), we can expand (55) in $\epsilon = \rho/\Delta$. For a mixed-state solution, we may assume, without loss of generality, that $J(0) > 0$ and $q \neq 0$. Then,

$$\sqrt{C} = J(\rho) + O(\epsilon^6) = J(0) + \frac{q\rho^4}{\Delta^4} + O(\epsilon^6). \quad (59)$$

Assuming that $\Delta_{,\rho}$ is of order $1/\epsilon$, we find that, to the leading order in ϵ , the limiting form of (55) is

$$j \frac{d}{d\rho} \frac{1}{\Delta_{,\rho}^2} = \frac{8\rho^4 \Delta_{,\rho}}{\Delta^5}. \quad (60)$$

The parameter

$$j \equiv J(0)/q = (J_\infty - q)/q \quad (61)$$

is the ratio of the momentum densities of the normal and SC components.

Eq. (60) is scale-invariant: if $\Delta(\rho)$ is a solution, then so is

$$\tilde{\Delta}(\rho) = c\Delta(c^{-1}\rho), \quad (62)$$

where c is any positive constant. We can think of c as a shooting parameter, which we may hope to adjust so as to obtain a solution to the full Eq. (55) with the correct asymptotics (36) at infinity. Indeed, this is precisely how we are going to search for solutions to Eq. (55) numerically.

The form of (60) suggests that it is advantageous to view ρ as a function of Δ , rather than Δ as a function of ρ . Then, (60) can be rewritten as

$$j \frac{d}{d\Delta} \rho_{,\Delta}^2 = 8 \frac{\rho^4}{\Delta^5}. \quad (63)$$

We are looking at this in the limit $\Delta \rightarrow 0$, with the boundary condition $\rho(0) = 0$. The substitution

$$\rho(\Delta) = g(z)\Delta, \quad (64)$$

where

$$z = \ln(\Delta_0/\Delta), \quad (65)$$

brings (63) to the form

$$gg' - (g')^2 - gg'' + g'g'' = -\frac{4g^4}{j}. \quad (66)$$

Δ_0 in (65) is an arbitrary constant, playing the same role as c in (62). Note that a small Δ means a large z .

Eq. (66), at large z , is suitable for an application of the WKB approximation. That amounts to ordering the terms on the left-hand side according to the number of derivatives: the more derivatives, the smaller the term. To the leading order, only the first term matters, and we find

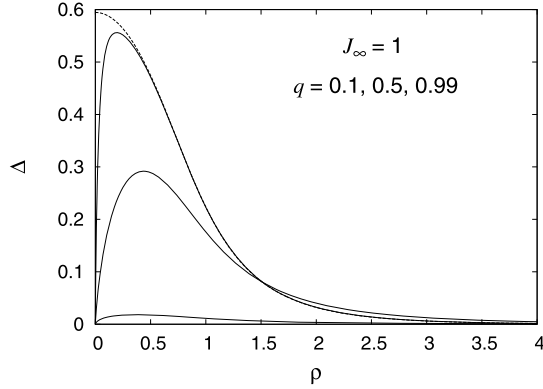


Fig. 2. D5 profiles for mixed-state solutions with total momentum density $J_\infty = 1$ and various values of the supercurrent momentum density q . Larger q correspond to larger peak values of $\Delta(\rho)$. The dashed line is the supercurrent-only solution of [11] with $q = 1$.

$$g^2(z)_{\text{LO}} = \frac{j}{8(z + z_0)}, \quad (67)$$

where z_0 is an integration constant. z_0 can be absorbed by a redefinition of Δ_0 and, in any case, is immaterial to the leading order. From (67), we conclude that a solution with the postulated asymptotics exists only for $j > 0$. Referring now to Eq. (61), we see that $j > 0$ implies that $J(0)$, q , and J_∞ are all of the same sign—which, by the assumption we have made regarding $J(0)$, is positive. In other words, we may expect to find a solution of the requisite form only for

$$0 < q < J_\infty. \quad (68)$$

This stands to reason: the condition (68) means that the SC and normal components flow in the same direction.

Numerically, given J_∞ and q , we choose a small Δ and compute ρ and $\rho_{,\Delta}$ from (64), with g given by (67) and Δ_0 a parameter. We then use this Δ and the computed $\Delta_{,\rho} = 1/\rho_{,\Delta}$ as boundary conditions for the full Eq. (55) and look for Δ_0 such that the solution satisfies also the condition (36).

Using this algorithm, we find that the upper bound in (68) is saturated, in the sense that there are solutions with q very close to J_∞ , but the lower one in general is not: a more precise condition is

$$q_{\min} < q < J_\infty, \quad (69)$$

where q_{\min} is the larger of zero and the lower instability bound of Section 4 (the lowest curve in Fig. 1). Numerically, q_{\min} departs from zero at $J_\infty = 0.32$. At the instability bound, the solution merges into the normal-state solution $\Delta \equiv 0$. This is illustrated in Fig. 2. On the other hand, as q approaches J_∞ , the solution matches, except at the smallest ρ , the gapped supercurrent-only solution of [11]. In this way—for those J_∞ for which $q_{\min} \neq 0$ —a family of mixed-state solutions with different values of q can be thought to interpolate between the normal state (in which q is undefined) and the supercurrent-only state with $q = J_\infty$.

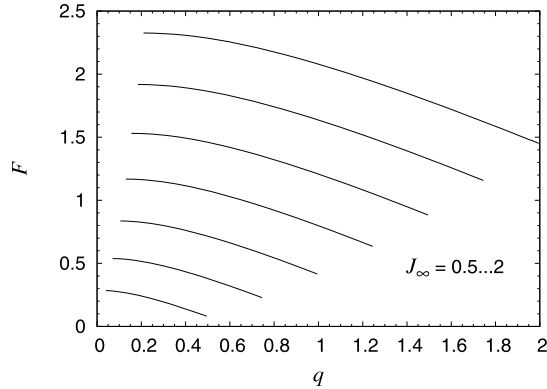


Fig. 3. Free energy of a uniform mixed state as a function of q . J_∞ increases bottom to top in increments of 0.25.

7. Non-convexity of the free energy

Eq. (55) is the condition of local extremum, with respect to $\Delta(\rho)$, for the functional

$$F = \int_0^\infty d\rho [\sqrt{C}(1 + \Delta_{,\rho}^2)^{1/2} - \rho^3 \sqrt{f_0}], \quad (70)$$

where f_0 is given by (39). This identifies F , up to an overall normalization, as the free energy density. The last term under the integral does not depend on Δ and so does not contribute to (55); its role is to make the integral convergent at the upper limit.

In a clean (disorder-free) conductor, a phase slip changes the momentum carried by the supercurrent without changing the total momentum. In our case, the former is represented by q and the latter by J_∞ , so it makes sense to consider F as a function of q at fixed J_∞ . The minimum of this function will be a candidate ground state—not necessarily the true one, as the procedure applies to uniform states only.

Numerically computed $F(q)$ curves for several values of J_∞ are shown in Fig. 3. The ends of each curve correspond to the endpoints of the interval (69). At the right end, $q \rightarrow J_\infty$, the free energy is the same as for the gapped solution to which the mixed-state solution converges pointwise (cf. Fig. 2).

We see that—among uniform states with a given J_∞ —the gapped, supercurrent-only state has the lowest free energy. Based on that, one might suppose that this is the state that the system will always evolve to. With the aid of Fig. 2, one could visualize such an evolution as the D5 peeling itself off the horizon, to form the state represented by the dashed line. From the earlier work [11] we know, however, that above a certain value of J_∞ ,

$$J_\infty > J_\infty^{(m)}, \quad (71)$$

the gapped state is unstable to decay by phase slips, which is accompanied by production of quasiparticles.¹⁴ What we learn here, then, is not that the gapped state is always stable, but rather that the quasiparticles that are produced by its decay cannot be described by a uniform solution.

¹⁴ For production of N well separated quasiparticles in a long wire, $J_\infty^{(m)}$ is about 0.57 [11]. The threshold may be lower for production of a bound state.

This conclusion is supported by the observation that none of the curves in Fig. 3 is convex. That means that, in a long wire, a uniform state with $q < J_\infty$ is unstable with respect to phase separation—fragmentation into regions with different winding number densities.¹⁵ Unlike the instability of the gapped state, this one does not rely on phase slips: it occurs even at fixed total winding, equal to that of the initial uniform state. The form of the free-energy curves (with the absolute minimum reached at $q = J_\infty$) suggests that, upon phase separation, there will be regions with $q \approx J_\infty$, which are almost quasiparticle-free, while quasiparticles are concentrated in droplets dispersed among these regions—an “emulsion.”

We find plausible the following hypothesis about the nature of the quasiparticle-rich droplets: they are “baryons,” each made of N D3/D5 strings. Recall that these strings carry color with respect to the $SU(N)$ gauge group that lives on the D3s’ worldvolume. The baryon is colorless.¹⁶ The complete antisymmetry of its wavefunction with respect to color means that there is a quasiparticle in each of the N conductance channels of the wire.

8. Conclusion

In the present work, we have aimed to understand the nature of the mixed SC-normal state that forms in a strongly coupled thin superconductor at currents above a certain I_m but well below the depairing current I_{dep} . This has been done here in the context of the same D3/D5 system as we used in [11]. The momentum carried by the supercurrent is represented by the flux of the world-volume gauge field induced on the D5, as the latter winds around the D3s, and the momentum of the normal component by the flux due to charges behind the horizon. Natural first guesses for the mixed state are uniform solutions, in which both these fluxes are uniform in x (the coordinate along the wire) and, in the leading large N limit, entirely classical. We have argued that such solutions exist but are unstable against fragmentation, leading eventually, we believe, to a non-uniform ground state—an “emulsion” of quasiparticle-rich droplets in a nearly quasiparticle-free matrix.

The non-uniform mixed state hypothesized here is similar to the mixed state of type II superconductors, with the droplets seen as “electric” analogs of the magnetic flux lines, and the winding density q_m , at which production of quasiparticles becomes energetically favorable, as the counterpart of the lower critical field H_{c1} . This analogy leads us to speculate further on the properties of such a state, in particular, on the role of disorder. All quasiparticles produced by phase slips carry the same momentum (either k_F or $-k_F$). One may wonder, then, if in the presence of disorder, when momentum is no longer conserved, the quasiparticles will not simply disappear, and the wire will not revert to the purely SC state. The analogy with a type II superconductor makes us think that this is unlikely. In that case, the presence of flux lines in the sample is not a result of any conservation law (they can enter and leave the sample through the boundary) but a consequence of the energetics: the difference $H - H_{c1}$ (in our case, $q - q_m$) plays the role of a chemical potential for a flux line (in our case, a quasiparticle).

¹⁵ This instability is distinct from the ones leading to spatially modulated currents [14,18] in models using a bulk $U(1)$ field. In our case, the system has fewer translational directions, and the total current remains uniform. It is the partition of the total into superconducting and normal components that becomes x -dependent.

¹⁶ It is similar to the baryon vertex described in [25,26], except that in our case the N strings connect the D5 to the D3s, rather than to the boundary of an AdS space.

Guided by the same analogy, one may contemplate a periodic array of droplets—an analog of the Abrikosov flux lattice. One may wonder if there are classical solutions of dual gravity capable of describing such an array.

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