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Tensor Products of Operator Algebras

EDWARD G. EFFROS*

Mathematics Department, University of Pennsylvania, Philadelphia, Pennsylvania 19174

AND

E. CHRISTOPHER LANCE*

Mathematics Department, University of Manchester, Manchester M13 9PL, Great Britain

INTRODUCTION

An early result of Murray and von Neumann [12] states that if R is a factor on a Hilbert space H, R' is its commutant, and $R \otimes R'$ is the algebraic tensor product, then the homomorphism that carries $r \otimes r'$ onto rr' is an isomorphism. Since $R \otimes R'$ may be realized as an algebra of operators on the Hilbert space completion of $H \otimes H$, and thus given the relative operator norm, it is only natural to inquire whether or not his map is isometric. Takesaki showed in [26] that this need not be the case (see Corollary 4.6 below for a definitive answer to this question).

It follows from the above discussion that if we wish to find a C^*-algebraic completion for the algebraic tensor product of two operator algebras, we will in general be faced with a multiplicity of eligible norms. In this paper we continue an investigation begun in [11] of those algebras for which the various C^*-algebraic tensor products are either unique, or have other desirable properties. We restrict our attention to algebras that have an identity.

Our task has been simplified through the use of a categorical approach. The morphisms of interest are the completely positive maps that preserve the identity. We have summarized their relevant properties in Section 1.

The various norms that can be placed on the algebraic tensor product $A \otimes B$ of two C^*-algebras A and B are discussed in Section 2. Among these there is a smallest, denoted by $\| \|_{\min}$. This norm is well behaved in that if A_1 and B_1 are C^*-algebras with $A \subseteq A_1$ and $B \subseteq B_1$, then the natural map of $A \otimes B$ into $A_1 \otimes B_1$ is isometric. Using the symbol $\otimes_{\min}$ to indicate the C^*-algebraic completion of the algebraic tensor product, we may write

$$A \otimes_{\min} B \subseteq A_1 \otimes_{\min} B_1 .$$

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At the other extreme, we have a maximal C^*-algebraic norm $\| \|_{\max}$. However, if A is a von Neumann algebra, there is an intermediate norm, the normal norm $\| \|_{\text{nor}}$, which seems to be more appropriate. $\| \|_{\min}$ and $\| \|_{\text{nor}}$ serve as the least and greatest norms for which the states normal on A are dense among all the states on the C^*-algebraic completion. Similarly, if A and B are both von Neumann algebras, there is a smaller norm, the binornal norm $\| \|_{\text{bin}}$, which respects the von Neumann structure of both A and B.

None of the norms $\| \|_{\max}$, $\| \|_{\text{nor}}$, or $\| \|_{\text{bin}}$, is well behaved in the inclusion sense mentioned above. Thus letting $\| \|_{\Gamma}$ be any of these norms, and using $\otimes_{\Gamma}$ for the corresponding completions, we are led to three questions, in which we restrict ourselves to von Neumann algebras when indicated:

1. For which algebras A do we have $A \otimes_{\Gamma} B = A \otimes_{\min} B$ for all B?

2. For which algebras A is it true that $B \subseteq B_1$ implies $A \otimes_{\Gamma} B \subseteq A \otimes_{\Gamma} B_1$?

3. For which algebras A is it true that $A \subseteq A_1$ implies $A \otimes_{\Gamma} B \subseteq A_1 \otimes_{\Gamma} B$ for all B?

In Theorem 4.1, perhaps the deepest result of the paper, we prove that each of these conditions for the binormal tensor product determines the same class of algebras, the semidiscrete von Neumann algebras. The latter are characterized by the fact that they have finite rank self-morphisms that converge in a suitable manner to the identity morphism. These algebras are introduced in Section 3, where it is shown that they include all type I algebras, as well as the product hyperfinite factors.

Turning to the normal tensor product, we prove that the von Neumann algebras which satisfy the first two conditions are again those that are semidiscrete (Theorem 4.1). The third condition corresponds to the apparently larger class of absolute retract von Neumann algebras first considered by Hakeda and Tomiyama [10] (Theorem 5.9). These algebras may also be characterized as those range von Neumann algebras for which Arveson's morphism extension theorem is valid (Theorem 5.3), hence we shall also call them injectives. Evidence that at least the finite injectives might be semidiscrete is presented in Theorem 5.11, where it is shown that the regular group von Neumann algebra of a discrete group is semidiscrete or injective if and only if the group is amenable. A negative indication for the general case is that we have been unable to prove that all type II_∞ or type III hyperfinite factors are semidiscrete, even though they are injectives. (See *Note added in proof.*)

The situation for the maximal tensor product is even more complex, since the three conditions apparently lead to three successively larger classes of C^*-algebras. The algebras satisfying the first condition are the nuclear C^*-algebras of [11], and were initially considered by Takesaki [26]. The algebras which satisfy the second condition are those C^*-algebras for which every representation generates an injective von Neumann algebra (Theorem 6.2). These are essentially the type E algebras of Tomiyama [29]. The third class of C^*-algebras are just the WEP algebras of [11], and they are further investigated in Theorem 6.3. For the regular group C^*-algebra of discrete groups, these three classes coincide, hence we conjecture that this is generally the case. (See *Note added in proof.*)

Since $\otimes_{\text{nor}}$ is asymmetrically defined, we must also consider these conditions with the second algebra fixed. For example, the first question becomes: "For which C^*-algebras A does one have that $B \otimes_{\text{nor}} A = B \otimes_{\text{min}} A$ for all von Neumann algebras B?" We find that the first two conditions determine just the nuclear algebras (Theorem 6.1), and those satisfying the third are the WEP algebras (Theorem 6.3).

As indicated above, we consider only $*$-algebras with an identity. In particular, subalgebras will always be assumed selfadjoint and to have the same identity as the containing algebra. This will be an implicit assumption whenever we write $A \subseteq B$ for two algebras A and B. Similarly, all homomorphisms will preserve both the $*$-operation and the identity.

If E, F are linear spaces, then $\mathscr{L}(E, F)$ is the space of all linear mappings from E to F, and E^d is the dual space $\mathscr{L}(E, \mathbb{C})$. If E, F are normed spaces, then $\mathscr{B}(E, F)$ is the space of all bounded mappings in $\mathscr{L}(E, F)$ and $E^* = \mathscr{B}(E, \mathbb{C})$ is the Banach dual. $\mathscr{B}(E, E)$ is abbreviated to $\mathscr{B}(E)$, and j_E denotes the canonical mapping from E to E^{**}. When convenient, we suppress the notation j_E and regard E as a subspace of E^{**}. Given linear spaces E, F, $E \otimes F$ is the algebraic tensor product of E and F.

We assume familiarity with the theory of C^*-algebras as contained in [5] or [17]. In particular, we make frequent use of the theory of second duals of C^*-algebras [5, Chap. 12]. If A is a C^*-algebra, then A_h, A^+ denote the sets of selfadjoint and positive elements of A. The state space will be denoted $S(A)$. If $\rho \in A^*$, $\rho \geqslant 0$, then π_ρ is the representation of A on the Hilbert space H_ρ, with cyclic vector ξ_ρ, obtained by the Gelfand–Neumark–Segal construction.

If R is a von Neumann algebra, then R_* denotes the predual of R, so that $j_{R_*}(R_*)$ is the set of all normal linear functionals on R. It will sometimes be convenient to refer to the relative weak* topology on this latter set as the weak* topology on R_*. (Of course, it really corresponds under the map j_{R_*} to the weak topology on R_*.) Thus if E is a Banach space and $\theta: E^* \to R^*$ is a mapping which is continuous from the weak* topology on E^* to the weak* topology on R^* and whose range consists of normal functions, we simply describe θ as being weak* continuous from E^* to R_*. However, we never use this inaccurate terminology when R_* is known to be a dual space (for instance when $R = A^{**}$ for some C^*-algebra A). Recall that the weak* topology on R is the same as the ultraweak topology, and coincides with the weak operator topology on the unit ball.

In Section 5 we follow the terminology of Semadeni [18] as regards categorical concepts, even where this clashes with established C^*-algebra usage.

1. Morphisms

If A is a C^*-algebra, let $M_n(A)$ denote the set of all $n \times n$ matrices $\mathbf{a} = [a_{ij}]$ with entries a_{ij} in A. Then $M_n(A)$ is a $*$-algebra under the obvious matrix multiplication and the $*$-operation $[a_{ij}]^* = [a_{ji}^*]$. If A acts as an algebra of operators on the Hilbert space H then $M_n(A)$ acts as a C^*-algebra of operators on $H_n = H \oplus \cdots \oplus H$ (n copies of H), if we define

$$\mathbf{a}\mathbf{x} = \Big(\sum_j a_{1j}x_j, \ldots, \sum_j a_{nj}x_j\Big) \qquad (\mathbf{x} = (x_1, \ldots, x_n) \in H_n).$$

The norm thus defined on $M_n(A)$ does not depend on the particular space H on which A acts since an isomorphism of C^*-algebras is an isometry. An element $\mathbf{a}$ in $M_n(A)$ is positive if and only if $\langle \mathbf{a}x, x\rangle \geqslant 0$ ($x \in H_n$), i.e.,

$$\sum_{i,j} \langle a_{ij}x_j, x_i\rangle \geqslant 0 \qquad (x_1, \ldots, x_n \in H).$$

The following algebraic characterization of the order on $M_n(A)$ is proved in [11, Proposition 2.1] (see also [14]):

Lemma 1.1. *An element of $M_n(A)$ is positive if and only if it is a finite sum of matrices of the form $[a_i{}^*a_j]$ $(a_1, \ldots, a_n \in A)$.*

Let A, B be C^*-algebras, and let $\Phi: A \to B$ be a linear mapping. Define $\Phi_n: M_n(A) \to M_n(B)$ by $\Phi_n([a_{ij}]) = [\Phi(a_{ij})]$. We say that Φ is *n-positive* if Φ_n is positive, and that Φ is *completely positive* if Φ is n-positive for all n. It is easy to show that for $n \geqslant 1$, $(n + 1)$-positivity implies n-positivity, but the converse is false (see [2, Theorem 1]). The following result of Choi (proved in [3]), which we shall not subsequently use, is interesting in this connection:

Proposition 1.2. *If A, B are C^*-algebras, and $\Phi: A \to B$ is a 2-positive linear isomorphism with $\Phi(1) = 1$, then Φ is a $*$-isomorphism.*

There are several situations in which maps are automatically completely positive:

Lemma 1.3. *Suppose A, B are C^*-algebras and $\Phi: A \to B$ is a linear map with $\Phi(1) = 1$. Then Φ is completely positive in any one of the following situations*:

(i) *Φ is a homomorphism.*

(ii) *Φ is positive and A or B is commutative.*

(iii) *$B \subseteq A$ (recall the identity convention of the Introduction) and Φ is a surjection satisfying $\Phi^2 = \Phi$ and $\|\Phi\| = 1$.*

(iv) *$B = \mathscr{B}(H)$, and A contains a subalgebra A_0 such that $\Phi(A) \subseteq \overline{\Phi(A_0)}$ and the restriction of Φ to A_0 is a homomorphism.*

Proof. (i) is immediate. For (ii) with A commutative, see [19, Theorem 4], and with B commutative, see [20, Lemma 6.1]. (iii) is proved in [13, Theorem 1]. (iv) is credited to Broise and proved in [21, Theorem 3.1].

We also need to consider dual and second dual spaces. If A is a C^*-algebra we define $M_n(A^*)$ to be the set of $n \times n$ matrices $\mathbf{f} = [f_{ij}]$ with entries f_{ij} in A^*. We may identify $M_n(A^*)$ with the dual $M_n(A)^*$ of the C^*-algebra $M_n(A)$ by setting

$$\mathbf{f}(\mathbf{a}) = \sum_{i,j} f_{ij}(a_{ij}).$$

From Lemma 1.1 it follows that $\mathbf{f} \geqslant 0$ if and only if

$$\sum_{i,j} f_{ij}(a_i{}^* a_j) \geqslant 0 \qquad (a_1, \ldots, a_n \in A).$$

In the same way, the dual of $M_n(A^*)$ may be identified with $M_n(A^{**})$. Since A^{**} is a C^*-algebra, so is $M_n(A^{**})$, and its C^*-algebra structure agrees with the structure which it has as the second dual of $M_n(A)$.

If E is either a C^*-algebra or the dual of a C^*-algebra, and E_0 is a linear subspace of E, we write $M_n(E_0)$ for the linear space of matrices $\mathbf{e} = [e_{ij}]$ with e_{ij} in E_0, and we let $M_n(E_0)$ have the relative order structure. If F is another such space and $\Phi: E_0 \to F$ is linear, we define $\Phi_n: M_n(E_0) \to M_n(F)$ as before and we say that Φ is *completely positive* if Φ_n is positive for all n. It is easy to see that a composition of completely positive maps is completely positive. Furthermore, if E, F are as above and $\Phi: E \to F$ is bounded, then $(\Phi^*)_n = (\Phi_n)^*$, so that if Φ is completely positive, then so is $\Phi^*: F^* \to E^*$.

We say that a bounded linear map $\Phi: A \to B$ of C^*-algebras is a *morphism* if it is completely positive and $\Phi(1) = 1$. From Lemma 1.3(ii), the states of a C^*-algebra are morphisms. A composition of morphisms is again a morphism. It is a consequence of the following proposition that a morphism has norm 1.

PROPOSITION 1.4. (Stinespring). *Let A be a C^*-algebra, H a Hilbert space, and $\Phi: A \to \mathscr{B}(H)$ a morphism. There is a Hilbert space K, a representation π of A on K, and an isometric map $v: H \to K$ such that $\Phi(x) = v^*\pi(x)v$ $(x \in A)$. Conversely, given Hilbert spaces H, K, a representation $\pi: A \to \mathscr{B}(K)$ and an isometry $V: H \to K$, $\Phi(a) = V^*\pi(a)V$ defines a morphism $\Phi: A \to \mathscr{B}(H)$. If A is a von Neumann algebra and Φ is normal, then π can be chosen normal.*

Proof. The first assertion and its converse constitute Stinespring's theorem (see [1; 19]). Inspection of the proof of the first assertion shows that if Φ is normal, then so is π.

If A is a C^*-algebra, $\rho \in A^*$, $\rho \geqslant 0$, let $[\rho] \subseteq A^*$ be the complex linear span of the cone

$$C_\rho = \{\sigma \in A^*: 0 \leqslant \sigma \leqslant \alpha\rho \text{ for some } \alpha > 0\}.$$

We define a bounded linear map $\theta_\rho: \pi_\rho(A)' \to [\rho]$ as follows:

$$\theta_\rho(r)(a) = \langle \pi_\rho(a) r\, \xi_\rho\,, \xi_\rho \rangle \qquad (r \in \pi_\rho(A)',\, a \in A).$$

LEMMA 1.5. *The map θ is a completely positive linear isomorphism of $\pi_\rho(A)'$ onto $[\rho]$ and it has a completely positive inverse.*

Proof. It is evident from [5, Proposition 2.5.1] that θ_ρ maps the positive part of $\pi_\rho(A)'$ onto C_ρ, hence θ_ρ is onto. If $\theta_\rho(r) = 0$, then for all a, b in A

$$0 = \theta_\rho(r)(b^*a) = \langle r\pi_\rho(a)\, \xi_\rho\,, \pi_\rho(b)\, \xi_\rho \rangle,$$

or since ξ_ρ is cyclic, $r = 0$. Given $r_1 ,..., r_n$ in $\pi_\rho(A)'$ and $a_1 ,..., a_n$ in A, we have

$$\begin{aligned}((\theta_\rho)_n[r_i{}^*r_j])[a_i{}^*a_j] &= \sum_{i,j} \theta_\rho(r_i{}^*r_j)(a_i{}^*a_j)\\ &= \sum_{i,j} \langle r_i{}^*r_j\pi_\rho(a_i{}^*a_j)\, \xi_\rho\,, \xi_\rho \rangle\\ &= \left\| \sum r_j\pi_\rho(a_j)\, \xi_\rho \right\|^2 \geqslant 0,\end{aligned}$$

hence, by Lemma 1.1, $(\theta_\rho)_n$ is positive. Conversely, given a positive element $\mathbf{f} = [f_{ij}]$ of $M_n([\rho])$, let

$$\mathbf{x} = (\pi_\rho(a_1)\, \xi_\rho\,,..., \pi_\rho(a_n)\, \xi_\rho) \in (H_\rho)_n\,.$$

Since $\theta_\rho^{-1}(f_{ij})$ commutes with $\pi_\rho(A)$, it follows that

$$\begin{aligned}\langle (\theta_\rho^{-1})_n(\mathbf{f})\mathbf{x}, \mathbf{x} \rangle &= \sum_{i,j} \langle \theta_\rho^{-1}(f_{ij})\, \pi_\rho(a_j)\, \xi_\rho\,, \pi_\rho(a_i)\, \xi_\rho \rangle\\ &= \sum_{i,j} \langle \theta_\rho^{-1}(f_{ij})\, \pi_\rho(a_i{}^*a_j)\, \xi_\rho\,, \xi_\rho \rangle\\ &= \sum_{i,j} f_{ij}(a_i{}^*a_j)\\ &= \mathbf{f}([a_i{}^*a_j]) \geqslant 0.\end{aligned}$$

This holds for all $a_1 ,..., a_n \in A$ and ξ_ρ is cyclic, hence $(\theta_\rho^{-1})_n(\mathbf{f}) \geqslant 0$, and so $(\theta_\rho^{-1})_n \geqslant 0$.

We note for future use that although θ_ρ^{-1} is in general not norm continuous, its composition with certain other maps is continuous. Specifically, if B is a C^*-algebra and $\Phi: B \to A^*$ is a completely positive map with $\Phi(1) = \rho$, then $\Phi(B) \subseteq [\rho]$ and $\theta_\rho^{-1}\Phi: B \to \pi_\rho(A)'$ is a completely positive map which takes the identity to the identity, and therefore is a morphism. This fact is crucial in what follows, and will be used freely without further reference.

If $\{A_\lambda\}_{\lambda\in\Lambda}$ is a family of C^*-algebras (resp., von Neumann algebras), then the direct sum $\bigoplus A_\lambda$, consisting of all bounded functions $\mathbf{a}$ from Λ to $\bigcup A_\lambda$ such that $a(\lambda) \in A_\lambda$, is again a C^*-algebra (resp., von Neumann algebra). Given linear maps and C^*-algebras (resp., normal linear maps and von Neumann algebras) $\Phi_\lambda: A_\lambda \to B_\lambda$ for which $\| \Phi_\lambda \| \leqslant 1$, we have a map (resp., normal map) $\bigoplus \Phi_\lambda: \bigoplus A_\lambda \to \bigoplus B_\lambda$ defined in the usual way. A simple application of Proposition 1.4 gives:

LEMMA 1.6. *Suppose that* A_λ, B_λ, $(\lambda \in \Lambda)$ *are C^*-algebras (resp., von Neumann algebras), and that* $\Phi_\lambda: A_\lambda \to B_\lambda$ *are morphisms (resp., normal morphisms). Then* $\bigoplus \Phi_\lambda: \bigoplus A_\lambda \to \bigoplus B_\lambda$ *is a morphism (resp., normal morphism).*

2. TENSOR PRODUCTS

Let A, B be C^*-algebras and let $A \otimes B$ denote their algebraic tensor product, so that an element of $A \otimes B$ is an expression of the form $x = \sum_i a_i \otimes b_i$ $(a_1, ..., a_n \in A, b_1, ..., b_n \in B)$. If $B = M_n$, the $n \times n$ matrices, then one may identify the involutive algebras $A \otimes B$ and $M_n(A)$. We have already seen that the latter may be provided with a unique norm in which it is a C^*-algebra. In the general case, one must complete $A \otimes B$ with respect to some norm, and as we have noted in the Introduction, the latter need not be unique. A seminorm p on $A \otimes B$ is called a *C^*-seminorm* if $p(x^*x) = p(x)^2$ $(x \in A \otimes B)$, and if p is also a norm, then we call it a *C^*-norm.*

Denote by $(A \otimes B)_h$ the set of selfadjoint elements of $A \otimes B$, and by $(A \otimes B)^+ \subseteq (A \otimes B)_h$, the cone generated by elements of the form x^*x $(x \in A \otimes B)$. The set $A^+ \otimes B^+ = \{a \otimes b : a \in A^+, b \in B^+\}$ is contained in $(A \otimes B)^+$ since

$$a^*a \otimes b^*b = (a \otimes b)^*(a \otimes b),$$

but $(A \otimes B)^+$ usually does not coincide with the cone generated by $A^+ \otimes B^+$.

LEMMA 2.1. (i) *Identifying* $A_h \otimes B_h$ *with a real subspace of* $A \otimes B$, *we have* $A_h \otimes B_h = (A \otimes B)_h$.

(ii) *Given x in* $(A \otimes B)_h$, *there is a positive number* α *with* $x \leqslant \alpha 1$ *(we denote by* 1 *the element* $1 \otimes 1$ *of* $A \otimes B$).

Proof. (i) Clearly $A_h \otimes B_h \subseteq (A \otimes B)_h$. If $x = \sum_j a_j \otimes b_j$ and $x = x^*$, then

$$\begin{aligned} x &= \tfrac{1}{2}(x + x^*) \\ &= \tfrac{1}{4} \sum_j \{(a_j + a_j^*) \otimes (b_j + b_j^*) - i(a_j - a_j^*) \otimes i(b_j - b_j^*)\}, \end{aligned}$$

hence $x \in A_h \otimes B_h$.

(ii) Observe first that if $a \in A^+$ and $b \in B^+$ then clearly $a \otimes b \leqslant \| a \| \| b \| 1$. Next suppose that $a \in A_h$, $b \in B_h$. Let $a = a_1 - a_2$ with a_i in A^+ and $\| a_i \| \leqslant \| a \|$ $(i = 1, 2)$, and similiarly $b = b_1 - b_2$. Then

$$\begin{aligned} a \otimes b &= (a_1 - a_2) \otimes (b_1 - b_2) \\ &\leqslant a_1 \otimes b_1 + a_2 \otimes b_2 \\ &\leqslant 2 \| a \| \| b \| 1. \end{aligned}$$

Finally if $x \in (A \otimes B)_h$, we may suppose by (i) that $x = \sum_j a_j \otimes b_j$ with $a_j^* = a_j$, $b_j^* = b_j$. Then $x \leqslant 2 \sum_j \| a_j \| \| b_j \| 1$.

Let $S(A \otimes B)$ denote the set of all linear functionals f on the vector space $A \otimes B$ which satisfy the conditions

$$f(x^*x) \geqslant 0 \qquad (x \in A \otimes B), \qquad f(1) = 1.$$

With each f in $S(A \otimes B)$ one can associate in the usual way a Hilbert space H_f, a homomorphism π_f of $A \otimes B$ on H_f and a cyclic unit vector ξ_f such that

$$f(x) = \langle \pi_f(x) \xi_f , \xi_f \rangle \qquad (x \in A \otimes B).$$

The only point that must be checked (see [15, p. 213]) is that $\pi_f(x)$, which is initially defined on $\pi_f(A \otimes B) \xi_f$, is bounded on the pre-Hilbert space and thus extends to an operator on H_f. The condition for this is that there should exist a positive constant α such that

$$f(y^*x^*xy) \leqslant \alpha f(y^*y) \qquad (y \in A \otimes B).$$

It suffices to choose α, by Lemma 2.1(ii), so that $x^*x \leqslant \alpha 1$, since then $y^*x^*xy \leqslant \alpha y^*y$ (the map $w \rightarrow v^*wv$ is clearly positive on $A \otimes B$). Observe that the constant α thus chosen depends only on x, not on f.

For f in $S(A \otimes B)$, define $p_f(x) = \| \pi_f(x) \|$ $(x \in A \otimes B)$. Then p_f is a C^*-seminorm on $A \otimes B$, and so is $p_\Gamma = \sup\{p_f : f \in \Gamma\}$ for any subset Γ of $S(A \otimes B)$; this supremum is finite since $p_f(x) \leqslant \alpha^{1/2}$, where α is chosen as in the previous paragraph. If p_Γ is a C^*-norm, then we call Γ a *separating* subset of $S(A \otimes B)$, and we write $A \otimes_\Gamma B$ for the C^*-algebra obtained by completing $A \otimes B$ with respect to p_Γ. We will occasionally write $\| \; \|_\Gamma$ in place of p_Γ. If ρ is a state on $A \otimes_\Gamma B$, its restriction to $A \otimes B$ is in $S(A \otimes B)$, and in this way we define a bijective mapping from $S(A \otimes_\Gamma B)$ onto a subset $S_\Gamma(A \otimes B)$ of $S(A \otimes B)$. Clearly $\Gamma \subseteq S_\Gamma(A \otimes B)$.

Let $A^* \otimes B^*$ denote the vector space tensor product of A^* and B^*, considered as a space of linear functions on $A \otimes B$. Define

$$\min = (A^* \otimes B^*) \cap S(A \otimes B), \qquad \max = S(A \otimes B).$$

Then min and max are separating subsets of $S(A \otimes B)$, and $p_{\min}$ and $p_{\max}$ are respectively the least and the greatest C^*-norms on $A \otimes B$ (see [11] and the references cited there for the proofs of these facts; in the literature the notations $A \otimes^{\alpha} B$ and $A \otimes^{*} B$ are used for $A \otimes_{\min} B$, and $A \otimes^{\vee} B$ for $A \otimes_{\max} B$). We call $A \otimes_{\min} B$ and $A \otimes_{\max} B$ the *minimal* and *maximal tensor products* of A and B, respectively. Since $S_{\max}(A \otimes B) = S(A \otimes B)$, we may identify $S(A \otimes_{\max} B)$ with $S(A \otimes B)$.

Suppose A, B act on Hilbert spaces H, K and denote by $H \otimes_2 K$ the Hilbert space tensor product of H and K. For a in A and b in B, the operator $a \otimes b$ on $H \otimes K$ extends uniquely to a bounded operator $a \otimes b$ on $H \otimes_2 K$. This defines a (faithful) representation of $A \otimes B$ on $H \otimes_2 K$, and the completion is naturally isomorphic to $A \otimes_{\min} B$ (see [26]). If $A \subseteq A_1$, $B \subseteq B_1$ are C^*-algebras with A_1 (resp., B_1) acting on H_1 (resp., K_1), then regarding $A \otimes B$ as a subalgebra of $A_1 \otimes B_1$, the norm on $\mathscr{B}(H_1 \otimes_2 K_1)$ will define the minimal tensor norm on both $A_1 \otimes B_1$ and $A \otimes B$. It follows that the inclusion $A \otimes B \subseteq A_1 \otimes B_1$ extends to an injection

$$A \otimes_{\min} B \subseteq A_1 \otimes_{\min} B_1 .$$

Suppose that A, B, C are C^*-algebras, and $\Phi\colon A \to C$, $\psi\colon B \to C$ are homomorphisms such that

$$\Phi(a)\,\psi(b) = \psi(b)\,\Phi(a) \qquad (a \in A,\, b \in B).$$

Then $a \otimes b \to \Phi(a)\,\psi(b)$ determines a homomorphism $\eta\colon A \otimes B \to C$ which extends to a homomorphism

$$\bar{\eta}\colon A \otimes_{\max} B \to C$$

(see [9]). In particular, if $A \subseteq A_1$, $B \subseteq B_1$ are C^*-algebras as in the preceding paragraph, the inclusion maps determine a homomorphism

$$A \otimes_{\max} B \to A_1 \otimes_{\max} B_1$$

which will generally not be an injection.

Let V, W be complex vector spaces. Given f in $(V \otimes W)^d$, define T_f in $\mathscr{L}(W, V^d)$ by

$$T_f(w)(v) = f(v \otimes w).$$

It is well known and easily verified that the map $f \to T_f$ is a linear isomorphism from $(V \otimes W)^d$ onto $\mathscr{L}(W, V^d)$.

Let A, B be C^*-algebras. As indicated above, we may identify $S(A \otimes_{\max} B)$ and $S_{\max}(A \otimes B) = S(A \otimes B)$. Since $S(A \otimes_{\max} B)$ is a bounded subset of $(A \otimes_{\max} B)^*$, it is easy to see that the weak* topology of $S(A \otimes_{\max} B)$ coincides with the weak^d topology of $S(A \otimes B)$. Letting $S(A \otimes B)$ have that topology, we have

LEMMA 2.2. *Suppose that $f \in (A \otimes B)^d$. Then $f \in S(A \otimes B)$ if and only if T_f is a completely positive map from B to A^* such that $T_f(1)$ is a state. The map $f \to T_f$ is a homeomorphism from $S(A \otimes B)$ to $\mathscr{B}(B, A^*)$ when the latter is given the topology of simple weak* convergence.*

Proof. If $f \geqslant 0$, then for a in A^+, b in B^+, we have

$$T_f(b)(a) = f(a \otimes b) \geqslant 0,$$

hence, $T_f(B^+) \subseteq (A^d)^+ = (A^*)^+$ and so $T_f(B) \subseteq A^*$. In addition, $f(1 \otimes 1) = 1$ if and only if $T_f(1)(1) = 1$, i.e., $T_f(1)$ is a state. The remaining assertions of the lemma are proved in [11, Lemma 3.2].

We say that a map T from B to A^* is a *complete state map* if it is completely positive and $T(1)$ is a state.

If R is a von Neumann algebra and B is a C^*-algebra, we define nor $=$ nor$(R \otimes B)$ by

$$\text{nor} = \{f \in S(R \otimes B) \colon T_f(B) \subseteq R_*\}.$$

This is a separating subset of $S(R \otimes B)$ since if R acts (as a von Neumann algebra) on H and B acts (as a C^*-algebra) on K then the vector state induced by any unit vector $\sum_j \xi_j \otimes \eta_j$ in $H \otimes K$ is in nor. Thus we have a tensor product $R \otimes_{\text{nor}} B$, the *normal tensor product* of R and B. Observe that even for two von Neumann algebras R and S, the definition of $R \otimes_{\text{nor}} S$ is asymmetric. There is apparently no reason to believe that the natural isomorphism

$$\sum_j r_j \otimes s_j \to \sum_j s_j \otimes r_j$$

from $R \otimes S$ onto $S \otimes R$ extends to an isomorphism from $R \otimes_{\text{nor}} S$ onto $S \otimes_{\text{nor}} R$. We turn to another tensor product which is symmetrically defined.

Suppose that R, S are von Neumann algebras, and define bin $=$ bin$(R \otimes S)$ by

$$\text{bin} = \{f \in S(R \otimes S) \colon (r, s) \to f(r \otimes s) \text{ is separately weak* continuous } (r \in R, s \in S)\}.$$

This is separating (by an argument similar to that used for nor), and so defines a tensor product $R \otimes_{\text{bin}} S$, the *binormal tensor product* of R and S. It is immediate from the definitions that

LEMMA 2.3. *If $f \in S(R \otimes S)$, then $f \in$ bin if and only if the following conditions hold:*

(i) *$T_f \colon S \to R^*$ is weak* continuous,*

(ii) $T_f(S) \subseteq R_*$.

If R, S are von Neumann algebras, it follows from (ii) above that bin $\subseteq$ nor, and so $p_{\min} \leqslant p_{\text{bin}} \leqslant p_{\text{nor}} \leqslant p_{\max}$ on $R \otimes S$.

LEMMA 2.4. *Let A, B be C^*-algebras and Γ a separating subset of $S(A \otimes B)$. Suppose that Γ is convex and that whenever $f \in \Gamma$ and $y \in A \otimes B$,*

$$f(y^*xy) = f(y^*y)\, g(x) \qquad (x \in A \otimes B)$$

for some g in Γ. Then $S_\Gamma(A \otimes B)$ is the closure of Γ in $S(A \otimes B)$, and we have

$$\| x \|_\Gamma = \sup\{f(x^*x)^{1/2} : f \in \Gamma\}.$$

Proof. Suppose that x is a selfadjoint element of $A \otimes_\Gamma B$ such that $f(x) \geqslant 0$ $(f \in \Gamma)$. For f in Γ and y in $A \otimes B$ we have

$$\langle \pi_f(x)\, \pi_f(y)\, \xi_f\,,\, \pi_f(y)\, \xi_f \rangle = f(y^*xy) \geqslant 0.$$

Since $\pi_f(A \otimes B)\, \xi_f$ is dense in H_f it follows that $\pi_f(x) \geqslant 0$. But the direct sum of all the representations $\pi_f(f \in \Gamma)$ is a faithful representation of $A \otimes_\Gamma B$, hence we must have $x \geqslant 0$, and by [5, Lemma 3.4.1], Γ is dense in $S_\Gamma(A \otimes B)$. The equality then follows from [5, Proposition 2.4.7].

The above result applies in particular when Γ is equal to min, nor, or bin, since multiplication by a fixed element is weak* continuous on a von Neumann algebra. Also, if R, S are von Neumann algebras, min $\cap$ bin is a separating subset of $S(R \otimes S)$ (for the same reason that bin is) and therefore, min $\cap$ bin is dense in min.

Given C^*-algebras (or von Neumann algebras where appropriate) $A \subseteq A_1$ and $B \subseteq B_1$, and letting Γ be any of the four classes of states that we have discussed, the natural inclusion $A \otimes B \subseteq A_1 \otimes B_1$ extends to a homomorphism

$$A \otimes_\Gamma B \to A_1 \otimes_\Gamma B_1\,.$$

We have already seen that this is the case for min and max. If A is a von Neumann algebra and $\Gamma =$ nor, it is evident that if f is in nor$(A_1 \otimes B_1)$, then its restriction $f \mid A \otimes B$ is in nor$(A \otimes B)$. Given x in $A \otimes B$, we have from Lemma 2.4 that

$$\begin{aligned} \| x \|_{\text{nor}(A \otimes B)} &= \sup\{g(x^*x)^{1/2} : g \in \text{nor}(A \otimes B)\} \\ &\geqslant \sup\{f(x^*x)^{1/2} : f \in \text{nor}(A_1 \otimes B_1)\} \\ &= \| x \|_{\text{nor}(A_1 \otimes B_1)}, \end{aligned}$$

and our assertion follows. A similar argument applies to bin.

If R and S are von Neumann algebras, we denote by $R \mathbin{\overline{\otimes}} S$ the *von Neumann algebra tensor product* of R and S. $R \mathbin{\overline{\otimes}} S$ is defined to be the weak closure of the natural representation of $R \otimes S$ on $H \otimes_2 K$, where R, S act on H, K, respectively. Of course, $R \,\overline{\otimes}\, S$ is not generally a C^*-algebra tensor product of R and S. $R \otimes_{\min} S$ may be identified with the uniform closure of $R \otimes S$ in $R \,\overline{\otimes}\, S$.

LEMMA 2.5. (i) *If A_1, A_2, B_1, B_2 are C^*-algebras and $\Phi_1: A_1 \to B_1$ $\Phi_2: A_2 \to B_2$ are morphisms, then there is a unique morphism*

$$\Phi: A_1 \otimes_{\min} A_2 \to B_1 \otimes_{\min} B_2$$

such that

$$\Phi(a_1 \otimes a_2) = \Phi_1(a_1) \otimes \Phi_2(a_2) \qquad (a_1 \in A_1, a_2 \in A_2).$$

(ii) *If R_1, R_2, S_1, S_2 are von Neumann algebras and $\Phi_1: R_1 \to S_1$, $\Phi_2: R_2 \to S_2$ are normal morphisms, then there is a unique normal morphism $\Phi: R_1 \,\overline{\otimes}\, R_2 \to S_1 \,\overline{\otimes}\, S_2$ such that $\Phi(r_1 \otimes r_2) = \Phi_1(r_1) \otimes \Phi_2(r_2)$ $(r_1 \in R_1, r_2 \in R_2)$.*

Proof. (i) The uniqueness is obvious. For $i = 1, 2$, suppose B_i acts on H_i. By Proposition 1.4, there exists for each i a Hilbert space K_i, a representation π_i of A_i on K_i, and an isometry $v_i: H_i \to K_i$ such that $\Phi_i(a_i) = v_i^* \pi_i(a_i) v_i$. By [9, Theorem 2] there is a representation $\pi_1 \otimes_{\min} \pi_2$ of $A_1 \otimes_{\min} A_2$ on $K_1 \otimes_2 K_2$ such that

$$\pi_1 \otimes_{\min} \pi_2(a_1 \otimes a_2) = \pi_1(a_1) \otimes \pi_2(a_2).$$

Also, $v_1 \otimes v_2$ extends to an isometry $v_1 \otimes v_2: H_1 \otimes_2 H_2 \to K_1 \otimes_2 K_2$. We define

$$\Phi(x) = (v_1 \otimes v_2)^* \, \pi_1 \otimes_{\min} \pi_2(x) \, v_1 \otimes v_2 \qquad (x \in A_1 \otimes_{\min} A_2).$$

(ii) The proof is the same as that of (i), except that the last part of Proposition 1.4 is used to ensure that π_i is normal, and the representation $\pi_1 \otimes_{\min} \pi_2$ is replaced by the normal representation $\pi_1 \,\overline{\otimes}\, \pi_2$ of $R_1 \,\overline{\otimes}\, R_2$ (see [4, Section 1.4, Proposition 2]).

We write $\Phi_1 \otimes_{\min} \Phi_2$ and $\Phi_1 \,\overline{\otimes}\, \Phi_2$, respectively, for the morphisms constructed in the above lemma.

3. SEMIDISCRETE VON NEUMANN ALGEBRAS

Let R be a von Neumann algebra. We say that R is *semidiscrete* if the identity map on R can be approximated in the topology of simple weak* convergence by normal morphisms of finite rank. It is evident that if R is algebraically isomorphic to a semidiscrete von Neumann algebra, then R is itself semidiscrete.

If Φ is a normal morphism of R and $f \in R_*$, then $\Phi^* f \in R_*$, so the restriction Φ_* of Φ^* to R_* is a completely positive map from R_* to itself which takes normal states to normal states (we call such a mapping a *morphism* of R_*), and Φ_* has finite rank if Φ does. Conversely, if ψ is a morphism of R_*, then ψ^* is a normal morphism of R. If R is semidiscrete, it follows that the identity map on R_* can be approximated in the topology of simple weak convergence by morphisms of R_* with finite rank. Since the set of all such morphisms of R_* is convex, it follows from [6, Corollary VI.1.5] that the identity can be approximated in the topology of simple norm convergence by morphisms of R_* with finite rank.

PROPOSITION 3.1. *If $\{R_\lambda\}_{\lambda\in\Lambda}$ is any family of von Neumann algebras, then the direct sum $\oplus R_\lambda$ is semidiscrete if and only if each R_λ is semidiscrete.*

Proof. Let Λ_0 be any finite subset of Λ and suppose that for each λ in Λ_0, Φ_λ is a normal morphism from R_λ to itself. Denote by 1_λ the identity element of R_λ, and let ρ be any normal state of $\oplus R_\lambda$. Define $\Phi\colon \oplus R_\lambda \to \oplus R_\lambda$ by

$$\Phi = \left(\left(\bigoplus_{\lambda\in\Lambda_0} \Phi_\lambda\right) \circ \theta\right) \oplus (h \circ \rho),$$

where $\theta\colon \oplus_{\lambda\in\Lambda} R_\lambda \to \oplus_{\lambda\in\Lambda_0} R_\lambda$ is the projection morphism, and $h\colon \mathbb{C} \to \oplus_{\lambda\notin\Lambda_0} R_\lambda$ is defined by $h(\alpha) = \alpha 1$. From Lemma 1.6, Φ is a normal morphism, and it is evident that Φ has finite rank if and only if each $\Phi_\lambda(\lambda \in \Lambda_0)$ has finite rank. It is a routine matter to verify that if the identity map on each R_λ can be approximated in the topology of simple weak* convergence by morphisms Φ_λ, then the identity morphism on $\oplus R_\lambda$ can be approximated by morphisms of the form Φ.

Conversely, given a morphism $\Phi\colon \oplus R_\lambda \to \oplus R_\lambda$ and an index λ_0, define $\Phi_0\colon R_{\lambda_0} \to R_{\lambda_0}$ to be the composition of morphisms

$$R_{\lambda_0} \xrightarrow{\Delta} \oplus\, S_\lambda \xrightarrow{\oplus\theta_\lambda} \oplus\, R_\lambda \xrightarrow{\Phi} \oplus\, R_\lambda \xrightarrow{\theta} R_{\lambda_0},$$

where $S_\lambda = R_{\lambda_0}$ for all λ, and $\theta_\lambda\colon S_\lambda \to R_\lambda$ is defined by

$$\begin{aligned} \theta_\lambda(a) &= a, & \lambda &= \lambda_0, \\ &= \rho(a)1, & \lambda &\neq \lambda_0, \end{aligned}$$

with ρ as above, and Δ and θ are the diagonal and projection morphisms, respectively. If morphisms of the form Φ approximate the identity map, and are of finite rank, the same is true for the corresponding morphisms Φ_0.

COROLLARY 3.2. *If R and S are von Neumann algebras, and $\Phi\colon R \to S$ is a normal surjective homomorphism, then R is semidiscrete if and only if S and* ker Φ *are semidiscrete.*

Proof. We have an isomorphism $R \simeq \ker \Phi \oplus S$.

COROLLARY 3.3. *If R is a von Neumann algebra, and $\{\pi_\lambda\}_{\lambda\in\Lambda}$ is a separating* *family of normal representations of R, then R is semidiscrete if and only if each of the algebras $\pi_\lambda(R)$ is semidiscrete.*

Proof. Let $\{e_\alpha\}_{\alpha \in A}$ be a maximal family of orthogonal central projections such that for each $\alpha \in A$, there is a $\lambda(\alpha)$ for which $\pi_{\lambda(\alpha)}$ restricts to an isomorphism on Re_α. Then $S_\alpha = \pi_{\lambda(\alpha)}(Re_\alpha)$ is a direct summand of $\pi_{\lambda(\alpha)}(R)$, and $R \cong \bigoplus S_\alpha$. If each $\pi_\lambda(R)$ is semidiscrete, the same is true for each S_α, and thus for R. The converse follows from Corollary 3.2.

PROPOSITION 3.4. *If R, S are von Neumann algebras, then $R \mathbin{\overline{\otimes}} S$ is semidiscrete if and only if R and S are semidiscrete.*

Proof. Let Φ, ψ be normal morphisms with finite rank of R, S respectively. From Lemma 2.5, $\Phi \mathbin{\overline{\otimes}} \psi$ is a normal morphism with finite rank of $R \mathbin{\overline{\otimes}} S$. Let $f \in R_*$, $g \in S_*$, then

$$\Phi \mathbin{\overline{\otimes}} \psi(x)(f \otimes g) = x(\Phi_* f \otimes \psi_* g)$$

for x in $R \mathbin{\overline{\otimes}} S$. We wish to show that given $x_1, \ldots, x_n$ in $R \mathbin{\overline{\otimes}} S$, $h_1, \ldots, h_m$ in $(R \mathbin{\overline{\otimes}} S)_*$ and $\epsilon > 0$, we can choose Φ and ψ so that

$$| \Phi \mathbin{\overline{\otimes}} \psi(x_j)(h_i) - x_j(h_i)| < \epsilon \qquad (1 \leqslant j \leqslant n, 1 \leqslant i \leqslant m).$$

Since $R_* \otimes S_*$ is norm dense in $(R \mathbin{\overline{\otimes}} S)_*$ (see [17, p. 66]), it suffices to consider the case where $h_i = f_i \otimes g_i$ ($f_i \in R_*$, $g_i \in S_*$). Thus we require

$$| x_j(\Phi_* f_i \otimes \psi_* g_i - f_i \otimes g_i)| < \epsilon.$$

From the remarks before Lemma 3.1, we can choose Φ and ψ so that $\Phi_* f_i - f_i$ and $\psi_* g_i - g_i$ are small in norm, and the result easily follows.

Conversely, let $I: R \to R$ be the identity map, and let $\rho: S \to \mathbb{C}$ be a normal state. From Lemma 2.5(ii), $I \otimes \rho: R \otimes S \to R \otimes \mathbb{C}$ extends to a normal morphism. On the other hand, the inclusion map $J: R \mathbin{\overline{\otimes}} \mathbb{C} \to R \mathbin{\overline{\otimes}} S$ is a normal isomorphism. Identifying $R \mathbin{\overline{\otimes}} \mathbb{C}$ with R, we see that each morphism Φ: $R \mathbin{\overline{\otimes}} S \to R \mathbin{\overline{\otimes}} S$ will determine a morphism

$$(I \otimes \rho) \circ \Phi \circ J: R \to R.$$

From this, it is evident that if $R \mathbin{\overline{\otimes}} S$ is semidiscrete, the same is true for R.

The following result is similar to Lemma 3.7 of [11], except that the mappings considered there do not necessarily preserve the identity, and so may not be morphisms.

PROPOSITION 3.5. *A type* I *von Neumann algebra is semidiscrete.*

Proof. In view of Propositions 3.1 and 3.4, it follows from the structure theorem for type I von Neumann algebras [17, Theorems 2.3.2, 2.3.3] that it suffices to prove the result for the cases (i) the algebra is commutative, (ii) the algebra is $\mathscr{B}(H)$ for some Hilbert space H. Case (i) is dealt with in the proof of [11; Lemma 3.7]. For case (ii), let $\{e_\lambda\}_{\lambda\in\Lambda}$ denote the directed net of all projections in $\mathscr{B}(H)$ with finite rank, and let ρ be a normal state on $\mathscr{B}(H)$. For λ in Λ define a normal morphism Φ_λ of $\mathscr{B}(H)$ by

$$\Phi_\lambda(t) = e_\lambda t e_\lambda + \rho(t)(1 - e_\lambda) \qquad (t \in \mathscr{B}(H)).$$

Then it is easy to see that Φ_λ is a finite rank morphism, and that Φ_λ tends to the identity in the appropriate topology.

PROPOSITION 3.6. *Let R be a semidiscrete von Neumann algebra, and e be a projection in R. Then eRe is semidiscrete.*

Proof. Let ρ be a normal state of eRe and Φ be a normal morphism of R with finite rank. Define ψ: $eRe \to eRe$ by

$$\psi(ere) = e\Phi(ere + \rho(ere)(1 - e))e \qquad (r \in R);$$

then ψ is a well-defined normal morphism of eRe, and by choosing Φ close enough to the identity on R we can make ψ close to the identity on eRe.

PROPOSITION 3.7. *The commutant of a semidiscrete von Neumann algebra is semidiscrete.*

Proof. Let R be a von Neumann algebra acting on H. Suppose first that R is standard, so there is an involution $j: H \to H$ (i.e., j is a conjugate linear isometry with $j^2 = I$) for which $r \to jrj$ maps R onto R'. If Φ is a normal morphism of R, then $s \to j\Phi(jsj)j$ ($s \in R'$) defines a normal linear map ψ of R' into itself. To see that ψ is a morphism, let π be a representation of R on a Hilbert space K and $V: H \to K$ an isometry with $\Phi = V^*\pi V$. j induces an involution on $V(H)$, which may be extended to an involution J of K. We have that $Vj = JV$, hence

$$\psi(s) = V^* J\pi(jsj)\, JV \qquad (s \in R').$$

Since $s \to J\pi(jsj)J$ is a representation of R', we have from Lemma 1.4 that ψ is a morphism. It is evident that ψ has finite rank and that it is close to the identity morphism if that is the case for Φ.

In the general case, the theory of modular Hilbert algebras [27] ensures that there is an isomorphism Φ from R to a standard algebra $\Phi(R)$. By the structure theorem for isomorphisms of von Neumann algebras [4, Section 1.4, Theorem 3] there is a Hilbert space K and a projection e' in $\Phi(R)' \,\overline{\otimes}\, \mathscr{B}(K)$ such that R is spatially isomorphic to $e'(\Phi(R) \,\overline{\otimes}\, 1)$, hence R' is isomorphic to

$e'(\Phi(R)' \bar{\otimes} \mathscr{B}(K))\, e'$. If R is semidiscrete, then by the first part of the proof, so is $\Phi(R)'$, hence by Propositions 3.4 and 3.6, so is R'.

PROPOSITION 3.8. *If R is a semidiscrete von Neumann algebra, and Φ is a normal morphism of R onto a von Neumann subalgebra S, then S is semidiscrete.*

Proof. One need only examine the compositions $\Phi \circ \psi$ for approximating finite rank morphisms $\psi: R \to R$.

COROLLARY 3.9. *If R is a finite semidiscrete factor, then any von Neumann subalgebra of R must be semidiscrete.*

Proof. If S is a von Neumann subalgebra of R, we have from [32] and Lemma 1.3 that there exists a normal morphism Φ of R onto S.

We have been unable to determine whether or not an inductive limit of semidiscrete von Neumann algebras must be semidiscrete. In particular, we do not know what the situation is for general hyperfinite type II and type III factors. We conclude this section by showing that a large subclass of the hyperfinite factors consists of semidiscrete factors. This result was obtained for the II_1 factor by one of the authors, and then for the general case by A. Connes and M. Takesaki, and independently by the other author. Connes also informs us that there exists a hyperfinite semidiscrete algebra which does not lie in this class.

By a *matrix algebra* we shall mean a C^*-algebra isomorphic to the $n \times n$ matrices M_n for some n. A factor R is *hyperfinite* if there is an ascending sequence of matrix subalgebras $M_{n(k)}$ with $R = \overline{\cup M_{n(k)}}$ (weak* closure). If B is a subalgebra of a C^*-algebra A, we let B^c denote the relative commutant of B in A. The following result is well known. It is apparent from the proof that it may be generalized to a purely algebraic context.

LEMMA 3.10. *If A is a C^*-algebra, and M is a matrix subalgebra, then A is isomorphic to $M \otimes M^c$.*

Proof. Let e_{ij} be matrix units for M. Then for each $a \in R$,

$$\theta_{ij}(a) = \sum_k e_{ki} a e_{jk}$$

is an element of M^c since it commutes with the matrix units:

$$\theta_{ij}(a)\, e_{mn} = e_{mi} a e_{jn} = e_{mn} \theta_{ij}(a).$$

We have that $a = \sum_{ij} \theta_{ij}(a)\, e_{ij}$, and it is a simple matter to verify that $a \to \sum_{ij} e_{ij} \otimes \theta_{ij}(a)$ provides the desired isomorphism.

Let us suppose that $M \cong M_n$ is a matrix subalgebra of a C^*-algebra A. From Lemma 3.10 and the discussion at the beginning of Section 2, we may identify A with $M \otimes M^c \cong M_n(M^c)$, the $n \times n$ matrices with entries in M^c.

If ρ is a state on A, we define the *diagonalized state*, $\Delta(\rho) = \Delta_M(\rho)$, by $\Delta(\rho) = \rho \mid M \otimes \rho \mid M^c$.

If ρ is a normal state on a von Neumann algebra R, ρ is an *asymptotically product state* if for each $\epsilon > 0$ and matrix subalgebra M, there exists a matrix subalgebra N containing M such that $\| \rho - \Delta_N(\rho)\| < \epsilon$. The following result is due to Stormer [22].

THEOREM 3.11. *If R is a hyperfinite factor, then the following properties are equivalent*:

(i) *R has a normal asymptotically product state.*

(ii) *Every normal state on R is asymptotically product.*

(iii) *R arises from a tensor product state on an infinite product of matrix algebras.*

A *product factor* is a factor for which every normal state is asymptotically product. Stormer has shown that the countably generated product factors are hyperfinite [23, 24].

THEOREM 3.12. *If R is a countably decomposable product factor, then it is semidiscrete.*

Proof. Given $\rho \in R_*$ and a matrix subalgebra M of R, we may define a morphism $\Phi = \Phi_M{}^\rho$ of $R = M \otimes M^c$ onto $M = M \otimes \mathbb{C}$ by letting $\Phi = I \otimes (\rho \mid M^c)$. Then we have $\Delta(\rho) = (\rho \mid M) \circ \Phi$. Given any state σ on R, we define a pre-Hilbert norm on R by $\| r \|_\sigma = \sigma(r^*r)^{1/2}$. Φ is then norm-decreasing in the following sense:

$$\begin{aligned} \| \Phi(r)\|_\rho^2 &= \rho(\Phi(r)^* \Phi(r)) \\ &\leqslant \rho(\Phi(r^*r)) \\ &= \Delta(\rho)(r^*r) \\ &= \| r \|_{\Delta(\rho)}^2 . \end{aligned}$$

In this calculation we used the "Schwarz inequality"

$$\Phi(r)^* \Phi(r) \leqslant \Phi(r^*r),$$

the validity of which is immediate from Proposition 1.4. We note that if $r \in M$, then $\Phi(r) = r$, i.e., Φ is a projection.

By using a suitable normal representation of R, we may assume that R has a separating unit vector ξ (see [4, Chap. III, 1.5, Lemma 5]), and we let ρ be the corresponding normal state on R, i.e., $\rho(r) = \langle r\xi, \xi\rangle$. It will suffice to prove that given $r_1, ..., r_n$ in R and $\epsilon > 0$, there exists a matrix algebra N in R such that if $\Phi = \Phi_N{}^\rho$, then

$$\| \Phi(r_i) - r_i \|_\rho = \|(\Phi(r_i) - r_i)\xi \| < \epsilon \qquad (i = 1,..., n).$$

To see that this is the case, we observe that for $r' \in R'$,

$$\|(\Phi(r_i) - r_i) r'\xi \| \leqslant \| r' \| \|(\Phi(r_i) - r_i)\xi \| \qquad (i = 1,..., n),$$

hence the finite rank morphisms Φ will converge to I on a dense set of vectors. Since $\| \Phi(r_i)\| \leqslant \| r_i \|$, we will have weak* convergence.

By the Kaplansky Density Theorem, if $\epsilon' > 0$, we may select a matrix algebra M in R and elements $a_1, ..., a_n$ in M with $\| a_i \| = \| r_i \|$, and

$$\| a_i - r_i \|_\rho = \|(a_i - r_i)\xi \| < \epsilon' \qquad (i = 1,..., n).$$

Let N be a matrix algebra containing M for which $\| \rho - \Delta(\rho)\| < \epsilon'$, and let $\Phi = \Phi_N{}^\rho$. Then

$$\begin{aligned}
\| \Phi(r_i) - a_i \|_\rho^2 &= \| \Phi(r_i - a_i)\|_\rho^2 \\
&\leqslant \| r_i - a_i \|_{\Delta(\rho)}^2 \\
&\leqslant \| r_i - a_i \|_\rho^2 + |(\rho - \Delta\rho)((r_i - a_i)^*(r_i - a_i))| \\
&\leqslant \epsilon'^2 + \epsilon' \cdot 4 \| r_i \|^2 \qquad (i = 1,..., n),
\end{aligned}$$

hence

$$\begin{aligned}
\| \Phi(r_i) - r_i \|_\rho &\leqslant \| \Phi(r_i) - a_i \|_\rho + \| a_i - r_i \|_\rho \\
&\leqslant (\epsilon'^2 + 4\epsilon' \| r_i \|^2)^{1/2} + \epsilon' \qquad (i = 1,..., n).
\end{aligned}$$

Making a suitable choice for ϵ', we obtain the desired inequality.

4. Tensor Product Characterizations for Semidiscrete Algebras

The main result of this section is:

THEOREM 4.1. *Suppose that R is a von Neumann algebra. Then the following are equivalent*:

(i) *R is semidiscrete.*

(ii) *For any von Neumann algebra S,*

$$R \otimes_{\text{bin}} S = R \otimes_{\min} S.$$

(ii′) *For any von Neumann algebras* S, S_1 *with* $S \subseteq S_1$,

$$R \otimes_{\text{bin}} S \subseteq R \otimes_{\text{bin}} S_1 .$$

(ii″) *For any von Neumann algebras* R_1, S *with* $R \subseteq R_1$,

$$R \otimes_{\text{bin}} S \subseteq R_1 \otimes_{\text{bin}} S.$$

(iii) *For any* C^**-algebra* B,

$$R \otimes_{\text{nor}} B = R \otimes_{\text{min}} B.$$

(iii′) *For any von Neumann algebra* R_1 *with* $R \subseteq R_1$, *and* C^**-algebra* B,

$$R \otimes_{\text{nor}} B \subseteq R_1 \otimes_{\text{nor}} B.$$

Remark. To be more precise, by an equality of tensor products we mean that the corresponding C^*-norms on the algebraic tensor product coincide. Thus instead of (iii), we could write

$$p_{\text{nor}} = p_{\text{min}} \text{ on } R \otimes B.$$

Similarly, an inclusion signifies that the natural inclusion of algebraic tensor products is isometric with respect to the indicated norms. (iii′) thus has the interpretation

$$(R \otimes B, p_{\text{nor}}) \to (R_1 \otimes B, p_{\text{nor}}) \text{ is isometric.}$$

Proof. (i) $\Rightarrow$ (iii). To prove that $p_{\text{nor}} = p_{\text{min}}$ on $R \otimes B$, it suffices to show that regarding nor and min as subsets of $S(R \otimes B)$, nor is contained in the weak^d closure of min (see Lemma 2.4 and the subsequent discussion). Given $\rho \in$ nor, let $\theta: B \to R_*$ be the corresponding complete state map. If R is semidiscrete, we have finite rank morphisms $\Phi_\nu: R_* \to R_*$ for which $\| \Phi_\nu(\sigma) - \sigma \| \to 0$ for all $\sigma \in R_*$. The net $\Phi_\nu \circ \theta$ consists of finite rank complete state maps converging to θ in the simple weak* topology, hence from Lemma 2.2, ρ is a weak^d limit of states in min.

(iii) $\Rightarrow$ (iii′). We have a commutative diagram of natural maps:

$$\begin{array}{ccc} R \otimes_{\text{nor}} B & \xrightarrow{\varphi} & R_1 \otimes_{\text{nor}} B \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi'} \\ R \otimes_{\text{min}} B & \xrightarrow{\varphi'} & R_1 \otimes_{\text{min}} B. \end{array}$$

From (iii), ψ is isometric, and from Section 2, φ' is isometric. Thus $\psi' \circ \varphi$, and in particular φ, must be isometric.

(iii′) ⇒ (iii). Say R is a von Neumann algebra on a Hilbert space H. If $R_1 = \mathscr{B}(H)$ in (iii′), we have the commutative diagram

$$\begin{array}{ccc} R \otimes_{\text{nor}} B & \subseteq & \mathscr{B}(H) \otimes_{\text{nor}} B \\ \downarrow & & \downarrow \\ R \otimes_{\text{min}} B & \subseteq & \mathscr{B}(H) \otimes_{\text{min}} B. \end{array}$$

Since $\mathscr{B}(H)$ is semidiscrete (Proposition 3.5), we have from (i) ⇒ (iii) that the right and thus the left arrows may be replaced by equalities.

(iii) ⇒ (ii) is evident from the inequality of norms on $R \otimes S$

$$p_{\text{min}} \leqslant p_{\text{bin}} \leqslant p_{\text{nor}} .$$

(ii) ⇔ (ii″) is analogous to (iii) ⇔ (iii′). (For (ii″) ⇒ (ii) let $R_1 = \mathscr{B}(H)$ for a suitable H in (ii″), and then use (i) ⇒ (ii).

(ii) ⇒ (ii′) is trivial.

(ii′) ⇒ (ii). Let S be a von Neumann algebra on a Hilbert space H. Letting $S_1 = \mathscr{B}(H)$ in (ii′), we have the commutative diagram

$$\begin{array}{ccc} R \otimes_{\text{bin}} S & \subseteq & R \otimes_{\text{bin}} \mathscr{B}(H) \\ \downarrow & & \| \\ R \otimes_{\text{min}} S & \subseteq & R \otimes_{\text{min}} \mathscr{B}(H), \end{array}$$

where we have again used the fact that $\mathscr{B}(H)$ is semidiscrete, and the implication (i) ⇒ (ii) (note that bin is symmetrically defined).

(ii) ⇒ (i). Suppose until further notice that R, S are von Neumann algebras such that $p_{\text{bin}} = p_{\text{min}}$ on $R \otimes S$. From Lemma 2.4 and the following remarks, bin ∩ min is weak^d dense in bin. Thus from Lemma 2.2, any weak* continuous complete state map $\theta: S \to R_*$ may be approximated in the simple weak* topology by finite rank weak* continuous complete state maps $\Phi: S \to R_*$. These maps have range in R_*, hence we have simple weak convergence. Since the set $\mathscr{F}$ of all weak* continuous complete state maps from S to R_* with finite rank is convex, it follows from [6, Corollary VI.1.5] that θ can be approximated in the topology of simple norm convergence by maps in $\mathscr{F}$. We state these results as a lemma.

LEMMA 4.2. *With R, S, $\mathscr{F}$ as above, let θ be a weak* continuous complete state map from S to R_*. Given $s_1, \ldots, s_n$ in S and $\epsilon > 0$, there is a Φ in $\mathscr{F}$ such that*

$$\| \theta(s_i) - \Phi(s_i) \| < \epsilon \qquad (1 \leqslant i \leqslant n).$$

Now let $\rho = \theta(1)$, $\sigma = \Phi(1)$. Without loss of generality we may take $s_1 = 1$ in the above lemma, so that $\| \rho - \sigma \| < \epsilon$. We wish to modify the choice of Φ so as to ensure that $\sigma = \rho$; this is done in two steps: first we "increase" Φ slightly to make $\sigma \geqslant \rho$, then we "shrink" it to get equality.

LEMMA 4.3. *Suppose R, S, θ are as in Lemma 4.2, $s_1, \ldots, s_n \in S$, and $\epsilon > 0$. There is a weak* continuous completely positive mapping Φ': $S \to R_*$ with finite rank such that $\| \Phi' \| < 1 + \epsilon$, $\Phi'(1) \geqslant \theta(1)$ and*

$$\| \theta(s_i) - \Phi(s_i)\| < \epsilon \qquad (1 \leqslant i \leqslant n).$$

Proof. Let $s_1 = 1$, and suppose without loss of generality that $\| s_i \| \leqslant 1$ $(1 \leqslant i \leqslant n)$. By Lemma 4.2 we may choose Φ in $\mathscr{F}$ with

$$\| \theta(s_i) - \Phi(s_i)\| < \epsilon/2 \qquad (1 \leqslant i \leqslant n).$$

With $\rho = \theta(1)$ and $\sigma = \Phi(1)$ we have $\| \sigma - \rho \| < \epsilon/2$. Let f^+, f^- be the positive and negative parts of $\sigma - \rho$, so that $f^+, f^- \in R_*$, $\| f^+ \| + \| f^- \| < \epsilon/2$ and $\sigma - \rho = f^+ - f^-$. Let τ be any normal state of S and define ψ: $S \to R_*$ by $\psi(s)(r) = f^-(r)\,\tau(s)$. Let $\Phi' = \Phi + \psi$. Since ψ has rank one, Φ' has finite rank. Also

$$\Phi'(1) = \sigma + f^- = \rho + f^+ \geqslant \rho,$$

and since $\| \psi \| \leqslant \epsilon/2$, the other assertions of the lemma follow easily.

LEMMA 4.4. *In Lemma 4.2, Φ can be chosen so that $\Phi(1) = \theta(1)$.*

Proof. We still assume that $\| s_i \| \leqslant 1$ and $s_1 = 1$. Let $\epsilon > 0$, and choose $\delta > 0$ so that $\delta + \delta^{1/2}(1 + (1 + \delta)^{1/2}) < \epsilon$. Choose Φ' as in Lemma 4.3 so that $\| \Phi' \| < 1 + \delta$,

$$\| \theta(s_i) - \Phi'(s_i)\| < \delta \qquad (1 \leqslant i \leqslant n),$$

and $f \geqslant \rho$, where $f = \Phi'(1)$, $\rho = \theta(1)$. By [17, Theorem 1.24.3] there is an element t of R with $0 \leqslant t \leqslant 1$ such that $\rho(r) = f(trt)$ $(r \in R)$.

Let $\eta = \pi_f(t)\,\xi_f \in H_f$. Then $\rho(r) = \langle \pi_f(r)\eta, \eta \rangle$ $(r \in R)$. Also since $t^2 \leqslant t$,

$$\begin{aligned}
\langle \eta, \xi_f \rangle &= \langle \pi_f(t)\,\xi_f, \xi_f \rangle \\
&\geqslant \langle \pi_f(t^2)\,\xi_f, \xi_f \rangle \\
&= \| \pi_f(t)\,\xi_f \|^2 \\
&= \| \eta \|^2 = \rho(1) = 1,
\end{aligned}$$

and it follows that

$$\begin{aligned}\| \xi_f - \eta \|^2 &= \langle \xi_f - \eta, \xi_f - \eta \rangle \\ &= \| \xi_f \|^2 + \| \eta \|^2 - 2\langle \eta, \xi_f \rangle \\ &\leqslant f(1) + 1 - 2 \\ &< 1 + \delta - 1 = \delta.\end{aligned}$$

Define $\Phi: S \to R_*$ by

$$\Phi(s)(r) = \Phi'(s)(trt) \qquad (s \in S, r \in R).$$

Clearly Φ has finite rank and $\Phi(1) = \rho$. If $\Phi' = T_g$ for some positive linear functional g on $R \otimes_{\text{bin}} S$ then $\Phi = T_h$ where h is the state of $R \otimes_{\text{bin}} S$ defined by

$$h(x) = g((t \otimes 1)x(t \otimes 1)) \qquad (x \in R \otimes_{\text{bin}} S).$$

This shows that $\Phi \in \mathscr{F}$. For $1 \leqslant i \leqslant n$, let $s_i' = \theta_f^{-1}\Phi'(s_i) \in \pi_f(R)'$, then $\| s_i' \| \leqslant 1$. For r in R with $\| r \| \leqslant 1$, we have

$$\begin{aligned}|(\Phi(s_i) - \Phi'(s_i))(r)| &= |\Phi'(s_i)(trt - r)| \\ &= |\langle s_i'\pi_f(trt - r)\, \xi_f, \xi_f \rangle| \\ &= |\langle s_i'\pi_f(r)\eta, \eta \rangle - \langle s_i'\pi_f(r)\, \xi_f, \xi_f \rangle| \\ &\leqslant |\langle s_i'\pi_f(r)\eta, \eta - \xi_f \rangle| + |\langle s_i'\pi_f(r)(\eta - \xi_f), \xi_f \rangle| \\ &\leqslant \| \eta - \xi_f \| (1 + \| \xi_f \|) \\ &< \delta^{1/2}(1 + (1 + \delta)^{1/2}) < \epsilon - \delta.\end{aligned}$$

Hence $\| \Phi(s_i) - \Phi'(s_i) \| < \epsilon - \delta$ and so $\| \theta(s_i) - \Phi(s_i) \| < \epsilon$.

We can now complete the proof of (ii) $\Rightarrow$ (i). Suppose that R satisfies (ii), ρ is a normal state on R, and $S = \pi_\rho(R)'$. Applying Lemma 4.4 to the map $\theta_\rho : S \to R_*$, there is a net of normal finite rank morphisms $\Phi_\nu: S \to R_*$ with $\Phi_\nu(1) = \rho$ which converges in the simple weak topology to θ_ρ. From Lemma 1.5, $\psi_\nu = \theta_\rho^{-1} \circ \Phi_\nu$ is a net of finite rank morphism from S to S. To prove Theorem 4.1, it suffices to show that each ψ_ν is normal, and that the net ψ_ν converges in the simple weak* topology to $\theta_\rho^{-1}\theta_\rho = I$, since then $\pi_\rho(R)$ will be semidiscrete for each normal state ρ (Proposition 3.7), hence the same will be true for R (Corollary 3.3).

Given an index ν, let s_γ be a net in S such that $\| s_\gamma \| \leqslant 1$ and $s_\gamma \to 0$ in the weak* topology. Given $r_1, r_2 \in R$, we have

$$\langle \psi_\nu(s_\gamma)\, \pi_\rho(r_1)\, \xi_\rho, \pi_\rho(r_2)\, \xi_\rho \rangle = \Phi_\nu(s_\gamma)(r_2{}^* r_1)$$

converges to 0. Since ξ_ρ is cyclic and $\| \psi_\nu(s_\gamma) \| \leqslant 1$ (ψ_ν is a morphism), $\psi_\nu(s_\gamma)$ converges in the weak* topology to 0. Given $s \in S$, and $r_1, r_2 \in R$, we have that

$$\langle (\psi_\nu(s) - s)\, \pi_\rho(r_1)\, \xi_\rho, \pi_\rho(r_2)\, \xi_\rho \rangle = (\Phi_\nu(s) - \theta_\rho(s))(r_2{}^* r_1)$$

converges to 0. Since ξ_ρ is cyclic and $\| \psi_\nu(s) - s \| \leqslant 2$, it follows that $\psi_\nu(s)$ converges weak* to s.

If R is a von Neumann algebra on a Hilbert space H, we have a homomorphism $\eta: R \otimes R' \to \mathscr{B}(H)$ defined by $\eta(r \otimes r') = rr'$ $(r \in R, r' \in R')$. If ξ is a unit vector in H and ω_ξ is the corresponding normal state on $\mathscr{B}(H)$, it is evident that $\omega_\xi \circ \eta \in \text{bin}$, hence if x is in $R \otimes R'$,

$$\| \eta(x)\| = \sup\{(\omega_\xi \circ \eta)(x^*x)^{\frac{1}{2}}: \| \xi \| = 1\} \leqslant \| x \|_{\text{bin}} .$$

Thus η extends to a homomorphism $\bar{\eta}$: $R \otimes_{\text{bin}} R' \to \mathscr{B}(H)$.

Proposition 4.5. *If R is a von Neumann algebra on a Hilbert space H, then the homomorphism η extends to $R \otimes_{\min} R'$ if and only if R is semidiscrete.*

Proof. If R is semidiscrete, we have $\| \eta(x)\| \leqslant \| x \|_{\text{bin}} = \| x \|_{\min}$ $(x \in R \otimes S)$, hence we will have the desired extension.

Conversely, suppose that η extends. Then for each unit vector ξ in H, $\omega_\xi \circ \eta \in S_{\min}(R \otimes R') \cap \text{bin}$. Fixing ξ, let $T: R' \to R_*$ be the corresponding complete state map, and let $\rho = T(1)$, i.e.,

$$\rho(r) = \omega_\xi \circ \eta(r \otimes 1) = \langle r\xi, \xi\rangle.$$

Letting $e' \in R'$ be the projection with range $\overline{R\xi}$, we may spatially identify $\pi_\rho(R)$ with Re'. Thus from Lemma 1.5, we have a morphism

$$\theta_\rho^{-1} \circ T: R' \to e'R'e'.$$

Given $r' \in R'$, $s' = \theta_\rho^{-1}(T(r'))$ is that unique operator in $e'R'e'$ for which

$$T(r')(r) = \langle rs'\xi, \xi\rangle \qquad (r \in R).$$

But we have that

$$T(r')(r) = \langle rr'\xi, \xi\rangle = \langle re'r'e'\xi, \xi\rangle,$$

hence

$$\theta_\rho^{-1}T(r') = e'r'e'.$$

Since min $\cap$ bin is weak^d dense in $S_{\min}(R \otimes R')$, T is the limit in the simple weak* topology of a net of finite rank weak* continuous complete state maps $T_\gamma: R' \to R_*$. Since we also have $T(R') \subseteq R_*$, we have convergence in the simple weak topology. Following the arguments of Theorem 4.1, we may assume that $T_\gamma(1) = \rho$, and again we will have that the finite rank morphisms $\theta_\rho^{-1}T_\gamma: R' \to e'R'e'$ approximate $\theta_\rho^{-1}T$ in the simple weak* topology. Letting σ

be a normal state on $e'R'e'$, we may define a net of normal morphisms $\Phi_\gamma: e'R'e' \to e'R'e'$ by

$$\Phi_\gamma(s) = \theta_\rho^{-1} T_\gamma(s + \sigma(s)(1 - e')).$$

These are of finite rank, and converge to the identity map on $e'R'e'$ in the simple weak* topology. From Proposition 3.7, Re' is semidiscrete, and from Corollary 3.3, the same is true for R.

COROLLARY 4.6. *If R is a factor, then η is isometric with respect to $p_{\min}$ if and only if R is semidiscrete.*

Proof. Since η is an isomorphism (see the Introduction), it induces a norm p on $R \otimes R'$ with $p \geq p_{\min}$ (the latter is the minimal C^*-norm on $R \otimes R'$). η will extend to $R \otimes_{\min} R'$ if and only if $p \leq p_{\min}$, hence Corollary 4.6 follows from Proposition 4.5.

5. Injective Operator Algebras and Their Tensor Products

Suppose that A is a C^*-subalgebra of a C^*-algebra B. We say that a morphism $\Phi: B \to A$ is a *retraction* if it is a left inverse for the inclusion morphism $A \to B$. If such a map exists, we say that A is a *retract* of B. A is an *absolute retract* if there is a retraction from B onto A whenever B is a C^*-algebra containing A. This is almost identical to a notion of Hakeda and Tomiyama [10] which they call the "extension property." They did not require that their mappings preserve identities, but all of their arguments apply to the situation considered here. In particular:

THEOREM 5.1 [10, 29]. *Suppose R is a von Neumann algebra acting on H. Then the following are equivalent:*

(i) *R is an absolute retract.*

(ii) *There is a retraction from $\mathscr{B}(H)$ onto R.*

(iii) *R' is an absolute retract.*

Let A be a C^*-algebra. We say that A is *injective* if given any C^*-algebras B and B_1 with $B \subseteq B_1$ and any morphism $\theta: B \to A$, there is a morphism $\eta: B_1 \to A$ which extends θ. The following is due to Arveson, who proved a somewhat more general result [1, Theorem 1.2.3].

THEOREM 5.2. *If H is a Hilbert space, then $\mathscr{B}(H)$ is injective.*

It is a simple matter to relate these two classes of algebras:

THEOREM 5.3. *For any C^*-algebra A the following are equivalent:*

(i) *A is an absolute retract.*

(ii) *A is injective.*

(iii) *Given C^*-algebras A_1, B with $A \subseteq A_1$ and a morphism $\theta: A \to B$, there is a morphism $\eta: A_1 \to B$ which extends θ.*

Proof. (i) $\Rightarrow$ (ii). Let us suppose that A is defined on a Hilbert space H. Then given C^*-algebras B and B_1 with $B \subseteq B_1$, consider the diagram

$$\begin{array}{ccc} B_1 & & \mathscr{B}(H) \\ \cup\!\mid & & \cup\!\mid \;\big\downarrow \theta \\ B & \xrightarrow[\Phi]{} & A \end{array}$$

where θ is a retraction. Since we may regard Φ as a morphism from B into $\mathscr{B}(H)$, Theorem 5.2 provides us with an extension $\psi: B_1 \to \mathscr{B}(H)$. $\theta \circ \psi$ is then the desired extension for $\Phi: B \to A$.

(ii) $\Rightarrow$ (i). If $A \subseteq A_1$, then the identity mapping on A extends by (ii) to a morphism $\Phi: A_1 \to A$, which is the desired retraction.

(i) $\Rightarrow$ (iii). Let Φ be a retraction from A_1 to A and set $\eta = \theta \circ \Phi$.

(iii) $\Rightarrow$ (i). Take $B_1 = A$ and let θ be the identity mapping on A.

The following results are essentially due to Tomiyama (see [28–31]).

PROPOSITION 5.4. *If $\{A_\lambda\}_{\lambda \in \Lambda}$ is a family of C^*-algebras, then $\oplus A_\lambda$ is injective if and only if each A_λ is injective.*

Proof. If $A = \oplus A_\lambda$, A_λ acts on H_λ, and $\Phi_\lambda: \mathscr{B}(H_\lambda) \to A_\lambda$ is a retraction, let $H = \oplus H_\lambda$ and let j_λ be the injection of H_λ into H. Define $\Phi: \mathscr{B}(H) \to A$ by

$$\Phi(t) = (\Phi_\lambda(j_\lambda{}^* t j_\lambda)) \qquad (t \in \mathscr{B}(H)).$$

Then Φ is a retraction, and A is injective. Conversely, if A is injective, let e_λ be the projection $j_\lambda j_\lambda{}^*$, and let ρ_λ be a normal state on $\mathscr{B}(H_\lambda)$. If $\Phi: \mathscr{B}(H) \to A$ is a retraction, then

$$\Phi_\lambda(t) = j_\lambda{}^* \Phi(j_\lambda t j_\lambda{}^* + \rho_\lambda(t)(1 - e_\lambda))\, j_\lambda \qquad (t \in \mathscr{B}(H_\lambda))$$

is a retraction of $\mathscr{B}(H_\lambda)$ onto A_λ (see Lemma 1.3(iii)).

COROLLARY 5.5. *If R is a von Neumann algebra, and $\{\pi_\lambda\}_{\lambda \in \Lambda}$ is a separating family of normal representations of R, then R is injective if and only if each of the algebras $\pi_\lambda(R)$ is injective.*

Proof. See the proof of Corollary 3.3.

PROPOSITION 5.6. *If R and S are von Neumann algebras, then $R \overline{\otimes} S$ is injective if and only if R and S are injective.*

Proof. This is proved in [31]. The argument involves an unexpected subtlety, since one must extend a tensor product of nonnormal morphisms to the von Neumann algebra tensor product. This is done by means of Banach limits.

PROPOSITION 5.7. *Suppose that R is a von Neumann algebra and that $\{R_\lambda\}_{\lambda \in \Lambda}$ are von Neumann subalgebras which are directed by inclusion, such that $R = \overline{\cup R_\lambda}$ (weak* closure). If each R is injective, the same is true for R.*

Proof. See [29, Proposition 1.3].

COROLLARY 5.8. *If R is a hyperfinite factor, then it is injective.*

We next characterize injective von Neumann algebras by their tensor product properties.

THEOREM 5.9. *Suppose that R is a von Neumann algebra. Then the following are equivalent:*

(i) *R is injective.*

(ii) *For any C^*-algebras B and B_1 with $B \subseteq B_1$,*

$$R \otimes_{\mathrm{nor}} B \subseteq R \otimes_{\mathrm{nor}} B_1 .$$

Proof. We recall from the remark following the statement of Theorem 4.1, that by the inclusion in (ii) we mean that the inclusion map

$$(R \otimes B, p_{\mathrm{nor}}) \to (R \otimes B_1 , p_{\mathrm{nor}})$$

is isometric. We claim that this is the case if and only if the restriction map $\mathrm{nor}(R \otimes B_1) \to \mathrm{nor}(R \otimes B)$ is surjective.

If the restriction is surjective, we have from Lemma 2.4 that the same is true for the restriction map $S_{\mathrm{nor}}(R \otimes B_1) \to S_{\mathrm{nor}}(R \otimes B)$, hence for $x \in R \otimes B$ we have

$$\begin{aligned} \| x \|_{R \otimes_{\mathrm{nor}} B} &= \sup\{\rho(x^*x)^{1/2} \colon \rho \in S_{\mathrm{nor}}(R \otimes B)\} \\ &= \sup\{\rho(x^*x)^{1/2} \colon \rho \in S_{\mathrm{nor}}(R \otimes B_1)\} \\ &= \| x \|_{R \otimes_{\mathrm{nor}} B_1} . \end{aligned}$$

Conversely, given an isometry of one C^*-algebra into another (preserving the identity), the adjoint (i.e., the restriction map) is surjective. Thus from (ii) we have that the restriction map $S_{\mathrm{nor}}(R \otimes B_1) \to S_{\mathrm{nor}}(R \otimes B)$ must be surjective.

If $\rho \in \text{nor}(R \otimes B_1)$, it is evident that $\rho \mid R \otimes B \in \text{nor}(R \otimes B)$. On the other hand suppose that $\rho \in S_{\text{nor}}(R \otimes B_1)$ is such that $\rho \mid R \otimes B \in \text{nor}(R \otimes B)$. Letting $T_\rho: B_1 \to R^*$ be the corresponding complete state map, $T_\rho(1) \in R_*$. Given $b \in B_1$ with $0 \leqslant b \leqslant 1$, we have that $\sigma = T_\rho(b)$ is an element of R^* with $0 \leqslant \sigma \leqslant T(1)$. Since $T_\rho(1)$ is normal, the same is true for σ (this is clear from the monotonicity characterization for normal states). Thus $T_\rho(B_1) \subseteq R_*$, and $\rho \in \text{nor}(R \otimes B_1)$.

We conclude that (ii) is the case if and only if each complete state map $\theta: B \to R_*$ extends to a complete state map $\eta: B_1 \to R_*$. Letting $\rho = \theta(1)$, we have from Lemma 1.5 that each morphism $\Phi: B \to \pi_\rho(R)'$ extends to a morphism $\psi: B_1 \to \pi_\rho(R)'$, hence that the $\pi_\rho(R)'$ are injective. Equivalently, one has (ii) if and only if the algebras $\pi_\rho(R)$ $(\rho \in R_*)$ are injective, i.e., from Corollary 5.5, R is injective.

COROLLARY 5.10. *If R is a semidiscrete von Neumann algebra, then it is injective.*

Proof. We have that (iii) of Theorem 4.1 implies (ii) of Theorem 5.9.

Given a discrete group G, the ***regular group von Neumann algebra*** $R(G)$ of G is the von Neumann algebra generated by $\lambda(G)$, where λ is the unitary representation of G on $\ell^2(G)$ defined by

$$(\lambda(s)f)(t) = f(s^{-1}t) \qquad (f \in \ell^2(G),\ s, t \in G).$$

The equivalence (ii) $\Leftrightarrow$ (iii) in the following theorem is due to Tomiyama [30].

THEOREM 5.11. *Suppose that G is a discrete group and that $R(G)$ is its regular group von Neumann algebra. Then the following are equivalent:*

(i) *$R(G)$ is semidiscrete.*

(ii) *$R(G)$ is injective.*

(iii) *G is amenable.*

Proof. Eymard [7] has shown that $R(G)_*$ may be identified with the functions on G of the form

$$f(s) = \langle \lambda(s)\xi, \eta \rangle \qquad (\xi, \eta \in \ell^2(G)).$$

Given such a function f, the action of f is defined first on $r \in R(G)$ of the form $r = \sum_{i=1}^n \alpha_i \lambda(s_i)$, by

$$\langle f, r \rangle = \sum \alpha_i f(s_i),$$

and then it is shown that this function has a unique extension to an element of $R(G)_*$. $R(G)_*$ is a commutative Banach algebra under pointwise multiplication on G.

If G is amenable, there exists a net of functions ξ_ν on G vanishing off finite subsets of G such that $\| \xi_\nu \|_2 = 1$ and $\rho_\nu(s) = \langle \lambda(s)\, \xi_\nu , \xi_\nu \rangle$ converges to 1 for each $s \in G$ (see [8, Theorem 3.5.2; 5, Proposition 18.3.6]). A simple calculation shows that the ρ_ν correspond to states in $R(G)_*$ (i.e., they are positive definite on G). Given $f \in R(G)_*$, we have for each $t \in G$,

$$\langle \rho_\nu f - f, \lambda(t) \rangle = \rho_\nu(t) f(t) - f(t)$$

converges to 0, hence $\rho_\nu f - f$ converges weakly to 0. Thus the multiplication operators $M(\rho_\nu)f = \rho_\nu f\,(f \in R(g)_*)$ converge weakly to the identity operator. As in the proof of [11, Proposition 4.1] these operators are morphisms. They are of finite rank since if ξ_ν vanishes off $\{t_1 ,..., t_n\}$, then ρ_ν vanishes off $\{t_i t_j : i, j = 1,..., n\}$. Thus $R(G)$ is semidiscrete.

(i) $\Rightarrow$ (ii) follows from Corollary 5.10.

(ii) $\Rightarrow$ (iii) is a consequence of the proof of [11, Theorem 4.2].

6. Tensor Products of C^*-Algebras

If A and B are C^*-algebras, we have one-to-one correspondences between the following sets:

(i) $S_{\max}(B \otimes A)$,

(ii) the complete state maps $\Phi: A \to B^*$,

(iii) $\operatorname{nor}(B^{**} \otimes A)$.

It should be emphasized that the correspondence between (i) and (iii) is not a homeomorphism (the latter is not generally weak^d compact). Nonetheless we may exploit this information in the proof of

THEOREM 6.1. *Suppose that A is a C^*-algebra. Then the following are equivalent*:

(i) *For any C^*-algebra B,*

$$B \otimes_{\max} A = B \otimes_{\min} A.$$

(ii) *For any von Neumann algebra R,*

$$R \otimes_{\mathrm{nor}} A = R \otimes_{\min} A.$$

(iii) *If R and R_1 are von Neumann algebras with $R \subseteq R_1$,*

$$R \otimes_{\mathrm{nor}} A \subseteq R_1 \otimes_{\mathrm{nor}} A.$$

Proof. (i) $\Rightarrow$ (ii). Applying (i) to $B = R$, we have

$$p_{\min} \leqslant p_{\text{nor}} \leqslant p_{\max} \leqslant p_{\min}$$

on $R \otimes A$.

(ii) $\Rightarrow$ (i). Let H and K be Hilbert spaces on which A and B^{**} act as a C^*-algebra and a von Neumann algebra, respectively. Suppose that ρ is a state in $S_{\max}(B \otimes A) = \text{nor}(B^{**} \otimes A)$. From (ii) there is a net of states ρ_ν which converges on each element of $B^{**} \otimes A$ to ρ, and each ρ_ν is a finite convex combination of states of the form $\langle \cdot\, \xi, \xi \rangle$, ξ a unit vector in $K \otimes H$. Since ρ_ν and ρ are weak* continuous in their first variables, the corresponding complete state maps Φ_ν and Φ map A into B^*. Given $a \in A$, the net $\Phi_\nu(a)$ converges to $\Phi(a)$ on elements of B^{**}, i.e., $\Phi_\nu(a)$ converges weakly, and a fortiori, weakly* to $\Phi(a)$. Since the Φ_ν are of finite rank, we conclude that ρ is in $S_{\min}(B \otimes A)$, i.e., $S_{\max}(B \otimes A) = S_{\min}(B \otimes A)$.

(ii) $\Leftrightarrow$ (iii). This argument is virtually the same as that for (ii) $\Leftrightarrow$ (ii′) in Theorem 4.1, since we may let R be a von Neumann algebra on K and $R_1 = \mathscr{B}(K)$.

A C^*-algebra which satisfies condition (i) of the above theorem is said to be *nuclear* (see [11]; the condition was first studied by Takesaki [23], who called it Property T).

It is evident from (i) of Theorem 6.1 that if A is nuclear, then given C^*-algebras B and B_1 with $B \subseteq B_1$, we have

$$A \otimes_{\max} B \subseteq A \otimes_{\max} B_1 .$$

We have been unable to establish a converse. The algebras A for which this inclusion always holds are characterized by:

THEOREM 6.2. *Suppose that A is a C^*-algebra. Then the following are equivalent*:

(i) *A^{**} is injective.*

(i′) *If π is a representation of A, then $\overline{\pi(A)}$ is injective.*

(ii) *If B and B_1 are C^*-algebras with $B \subseteq B_1$, then*

$$A \otimes_{\max} B \subseteq A \otimes_{\max} B_1 .$$

Proof. (i) $\Leftrightarrow$ (i′) is immediate from Corollary 5.5.

(i) $\Leftrightarrow$ (ii). We will have (ii) if and only if the restriction map $S_{\max}(A \otimes B_1) \to S_{\max}(A \otimes B)$ is surjective (see the proof of Theorem 5.9), i.e., if and only if the restriction map

$$\text{nor}(A^{**} \otimes B_1) \to \text{nor}(A^{**} \otimes B)$$

is surjective. From the proof of Theorem 5.9, this is equivalent to the inclusion

$$A^{**} \otimes_{\mathrm{nor}} B \subseteq A^{**} \otimes_{\mathrm{nor}} B_1 ,$$

and thus from that theorem to the injectivity of A^{**}.

The algebras satisfying (i) of Theorem 6.2 have been studied by Tomiyama [29]. We turn next to what is apparently a still more general class of C^*-algebras introduced in [11].

Let A, B be C^*-algebras with $A \subseteq B$ and denote the inclusion mapping by $i: A \to B$, so that $i^*: B^* \to A^*$ is the restriction mapping. We say that a linear mapping $d: A^* \to B^*$ is a *dilation* if it is a morphism (see the beginning of Section 3) such that i^*d is the identity on A^*.

THEOREM 6.3. *Suppose that A is a C^*-algebra. Then the following are equivalent*:

(i) *For any C^*-algebra A_1 with $A \subseteq A_1$, there is a dilation $d: A^* \to A_1^*$.*

(ii) *For any C^*-algebras A_1, B with $A \subseteq A_1$,*

$$A \otimes_{\max} B \subseteq A_1 \otimes_{\max} B.$$

(iii) *For any C^*-algebra A_1 with $A \subseteq A_1$ and von Neumann algebra R,*

$$R \otimes_{\mathrm{nor}} A \subseteq R \otimes_{\mathrm{nor}} A_1 .$$

(iv) *Given a C^*-algebra A_1 with $A \subseteq A_1$, a von Neumann algebra R, and a morphism $\theta: A \to R$, there is a morphism $\eta: A_1 \to R$ which extends θ.*

(v) *If A^{**} acts as a von Neumann algebra on H, there is a morphism $\Phi: \mathscr{B}(H) \to A^{**}$ such that $\Phi j_A(a) = j_A(a)$ $(a \in A)$ (see the introduction for this notation).*

(v′) *If π is a faithful representation of A on a Hilbert space H, there is a morphism $\Phi: \mathscr{B}(H) \to \overline{\pi(A)}$ such that $\Phi(\pi(a)) = \pi(a)$ $(a \in A)$.*

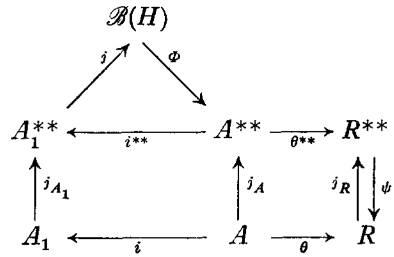
Proof. (i) ⇒ (ii). Condition (ii) holds if and only if each complete state map $\theta: B \to A^*$ lifts to a complete state map $\eta: B \to A_1^*$. If $d: A^* \to A_1^*$ is a dilation, it suffices to take $\eta = d \circ \theta$.

(ii) ⇒ (v′). This is just Theorem 3.3 of [11].

(v′) ⇒ (v) is trivial.

(v) ⇒ (iv). Suppose that (v) holds, and let A, A_1, R, θ be as in (iv). Let $i: A \to A_1$ be the inclusion mapping. Suppose that A_1^{**} acts as a von Neumann algebra on H, and let $j: A_1^{**} \to \mathscr{B}(H)$ be the inclusion mapping. Since i^{**} is a normal isomorphism of A^{**} onto the weak* closure of $j_{A_1}(A)$, we may regard

A^{**} as a von Neumann algebra on H. We select a morphism $\Phi: \mathscr{B}(H) \to A^{**}$ so that (v) holds. Let $\psi = (j_{R_*})^*: R^{**} \to R$. ψ is a left inverse for j_R. The diagram



is commutative except in two respects. First, at the right-hand edge, $j_R \circ \psi$ is not the identity on R^{**}. Second, in the upper triangle, $\Phi j i^{**}$ is not the identity on A^{**}. However, its retriction to $j_A(A)$ is the identity, and this is enough to ensure that $\eta = \psi\theta^{**}\Phi j j_{A_1}$ is the resuired extension of θ.

(iv) $\Rightarrow$ (v′). Let $R = \overline{\pi(A)}$, $A_1 = \mathscr{B}(H)$, and $\theta = \pi$ in (iv).

(v′) $\Rightarrow$ (i). Suppose that $A \subseteq A_1$, and that A_1^{**} acts as a von Neumann algebra on H. Choose a morphism from $\mathscr{B}(H)$ to A^{**} as in (v), and let Φ denote its restriction to A_1. Then $d = \Phi^* j_{A*}$ is easily seen to be a dilation from A^* to $A_1{}^*$.

Summarizing the relations between the various classes of C^*-algebras considered in [11] and this section, we have:

THEOREM 6.4. *Let A be a C^*-algebra, and consider the following properties for A:*

(i) *A^{**} is semidiscrete.*

(ii) *The identity mapping on A^* can be approximated in the topology of simple norm convergence by morphisms of A^* with finite rank.*

(iii) *The identity mapping on A^* can be approximated in the topology of simple weak* convergence by morphisms of A^* with finite rank (this is the "completely positive approximation property" of* [11]).

(iv) *For any C^*-algebra B, $A \otimes_{\max} B = A \otimes_{\min} B$ (i.e., A is nuclear).*

(v) *A^{**} is injective.*

(vi) *A satisfies any one (hence all) of the conditions of Theorem* 6.3 *(these are the "WEP" algebras of* [11]).

Then we have

$$\text{(i)} \Leftrightarrow \text{(ii)} \Rightarrow \text{(iii)} \Rightarrow \text{(iv)} \Rightarrow \text{(v)} \Rightarrow \text{(vi)}.$$

Proof. (i) $\Leftrightarrow$ (ii). See the remarks at the beginning of Section 3.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (iv) is Theorem 3.6 of [11].

(iv) $\Rightarrow$ (v). See the remarks preceding Theorem 6.2.

(v) $\Rightarrow$ (vi). It is evident that (v) implies (v) of Theorem 6.3, since we can choose $\Phi: \mathscr{B}(H) \rightarrow A^{**}$ to be a retraction.

It seems to be quite possible that all of the conditions of Theorem 6.4 are equivalent. If A is the regular group C^*-algebra of a discrete group G, (iii)–(vi) are each equivalent to the amenability of G (see [11, Proposition 4.1, Theorem 4.2]). Tomiyama has observed that if (iv) and (v) are equivalent, then the quotient of a nuclear C^*-algebra is again nuclear. The latter assertion is still unproved. In any event, the implications

$$A^{**} \text{ semidiscrete} \Rightarrow A \text{ nuclear} \Rightarrow A^{**} \text{ injective}$$

have led us to conjecture that the question of whether or not A is nuclear depends only on the structure of the von Neumann algebra A^{**}.

Note added in proof. Remarkable progress has been made since this paper was submitted for publication. In the intervening period of more than three years, the following results have been proved. In many cases, the arguments use theorems from [11] and this paper.

1. If R is a von Neumann algebra on a separable Hilbert space, then R is injective if and only if it is semidiscrete. If R is a factor, it is equivalent to assume that it is hyperfinite [38], [35].
2. Properties (i)–(v) of Theorem 6.4 are equivalent [36].
3. $\mathscr{B}$(H) satisfies (vi) but not (v) of Theorem 6.4 (H a separable Hilbert space) [41].
4. The Brown-Douglas-Fillmore Theorem stating that *Ext* is a group is valid for all separable nuclear C^*-algebras [33, 37].
5. R. I. Loebl has independently observed the relationship between the Arveson-Hahn-Banach Theorem and the Hakeda-Tomiyama "extension property" [40].

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