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# Chebyshev spectral solution of nonlinear Volterra–Hammerstein integral equations

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**Abstract**

In this paper, the Chebyshev spectral (CS) method for the approximate solution of nonlinear Volterra–Hammerstein integral equations

$$Y(\tau) = F(\tau) + \int_0^\tau K(\tau,r)G(r, Y(r))\,dr, \quad \tau \in [0, T],$$

is investigated. The method is applied to approximate the solution not to the equation in its original form, but rather to an equivalent equation $z(t) = g(t, y(t))$, $t \in [-1, 1]$. The function $z$ is approximated by the $N$th degree interpolating polynomial $z^N$, with coefficients determined by discretizing $g(t, y(t))$ at the Chebyshev–Gauss Lobatto nodes. We then define the approximation to $y$ to be of the form

$$y^N(t) = f(t) + \int_{-1}^1 \hat{k}(t,s)z^N(s)\,ds, \quad t \in [-1, 1],$$

and establish that, under suitable conditions, $\lim_{N\to\infty} y^N(t) = y(t)$ uniformly in $t$. Finally, a numerical experiment for a nonlinear Volterra–Hammerstein integral equation is presented, which confirms the convergence, demonstrates the applicability and the accuracy of the Chebyshev spectral (CS) method.

*Keywords:* Spectral Chebyshev projection; Volterra–Hammerstein; Chebyshev–Gauss rule

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## 1. Introduction

It is well known that spectral projection methods provide highly-accurate approximations for the solutions of operator equations in function spaces, provided that these solutions are sufficiently

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smooth (see, [5, 16–18]). In this paper, we are concerned with the extension of the Chebyshev spectral (CS) method to the numerical solution of nonlinear Volterra–Hammerstein integral equations

$$Y(\tau) = F(\tau) + \int_0^{\tau} K(\tau, r) G(r, Y(r))\, \mathrm{d}r, \quad \tau \in [0, T], \tag{1.1}$$

where $F$, $K$ and $G$ are given continuous functions, with $G(r, Y)$ nonlinear in $Y$. Throughout this paper, we assume that (1.1) has a unique solution $Y$ to be determined. Appropriate smoothness assumptions on $F$, $K$ and $G$, to be made later, will ensure that in a suitable Banach space, the right-hand side of (1.1) defines a completely continuous operator acting on $Y$.

Several numerical methods for approximating the solution of Hammerstein integral equations are known. For Fredholm–Hammerstein integral equations, the classical method of successive approximations was introduced in [21]. A variation of the Nystrom method was presented in [15]. A collocation-type method was developed in [14]. The classical method of the degenerate kernel was obtained in [12]. In [4], Brunner applied a collocation-type method to Eq. (1.1) and integro-differential equations, and discussed its connection with the iterated collocation method. Guoqiang [11] introduced and discussed the asymptotic error expansion of a collocation-type method for Volterra–Hammerstein integral equations. The methods in [2, 14, 11] transform a given integral equation into a system of nonlinear equations, which has to be solved by some kind of iterative method.

In [2, 11] the definite integrals involved in these nonlinear equations have to be evaluated at each time step of the iteration, while in [14], only in favorable cases these definite integrals may be evaluated analytically. Moreover, since approximation by piecewise-linear functions yields, at best, $O(h^2)$ convergence (see [14]), a rather large system of nonlinear equations have to be solved to obtain reasonable accuracy.

The major difference between the analyses in [2, 14, 4] and that in the present method is the fact that for polynomial interpolation, uniform convergence of interpolants cannot be guaranteed for continuous functions, regardless of the choice of the interpolation nodes. Hence, the present analysis is based on a mean-convergence property of polynomial interpolation (see (2.8) below). The advantages of our method are:

(1) the Chebyshev spectral approximation enjoys *formal* spectral accuracy, i.e., its truncation error decays as fast as the global smoothness of the underlying solution permits;

(2) the definite integrals are calculated once by the Chebyshev–Gauss quadrature rule [5, 8];

(3) spectral convergence rate can be observed for quite low order Chebyshev spectral approximations.

In this paper, we apply the Chebyshev spectral method not to Eq. (1.1) in its original form, but rather to an equivalent equation

$$z(t) = g(t, y(t)), \quad t \in [-1, 1]. \tag{1.2}$$

We therefore approximate $z$ by a polynomial $z^N$ of degree $N$, whose coefficients determined by collocating

$$z(t) = g\left(t, f(t) + T/4(t+1) \int_{-1}^{1} k(t, s) z(s)\, \mathrm{d}s\right), \quad t \in [-1, 1], \tag{1.3}$$

at the Chebyshev nodes. We then take

$$y^N(t) = f(t) + T/4(t+1)\int_{-1}^{1} k(t,s)z^N(s)\,ds, \quad t \in [-1, 1],$$

as the approximation to $y$, and establish that, under suitable conditions

$$\lim_{N\to\infty} y^N(t) = y(t), \tag{1.4}$$

uniformly in $t$. We further establish a rate for the convergence of $y^N$ to $y$ that is fast enough to yield spectrally accurate results.

This paper is organized as follows: In Section 2 we describe the Chebyshev spectral (CS) method required for our subsequent development and discuss some convergence results. In Section 3, we introduce our method, and in Section 4 we report our numerical findings and demonstrate the efficiency and accuracy of the proposed method.

## 2. The interpolation operator

Let $S_N$ denote the space of algebraic polynomials of degree $\leqslant N$, and let $(T_k(t))$, $k \geqslant 0$, $-1 \leqslant t \leqslant 1$, denote the orthogonal family of Chebyshev polynomials of the first kind in this space, with respect to the weight function $w(x) = (1-x^2)^{-1/2}$. In the most common Chebyshev spectral collocation methods, the grids in the interval $[-1, 1]$ are chosen to be the extrema

$$t_j = \cos(j\pi/N), \quad j = 0, 1, \ldots, N \tag{2.1}$$

of the $N$th-order Chebyshev polynomial $T_N(t) = \cos(N\cos^{-1} t)$.

Next, to construct the interpolant of $F(t)$ at the point $t$, we define the following Lagrange polynomials:

$$\phi_k(t) = \frac{(-1)^{k+1}(1-t^2)\dot{T}_N(t)}{C_k N^2 (t-t_k)} = \frac{2}{NC_k}\sum_{j=0}^{N}\frac{T_j(t_k)T_j(t)}{C_j}, \quad (k = 0, 1, \ldots, N), \tag{2.2}$$

with $C_0 = C_N = 2$, $C_k = 1$ for $1 \leqslant k \leqslant N-1$. Note that the grids $t_k$ denote the zeros of $(1-t^2)\dot{T}(t)$, where $\dot{T}_N(t)$ is the derivative of $T_N(t)$ with respect to $t \in [-1, 1]$. It is readily verified that

$$\phi_l(t_j) = \delta_{lj} = \begin{cases} 1 & \text{if } l = j, \\ 0 & \text{if } l \neq j. \end{cases} \tag{2.3}$$

Now, the $N$th-degree interpolating polynomial, $I_N F(t)$ to $F(t)$ is given by

$$I_N F(t) = \sum_{l=0}^{N} F(t_l)\phi_l(t). \tag{2.4}$$

Alternatively, $I_N F(t)$ can be expressed in terms of series expansion of Chebyshev polynomials

$$I_N F(t) = \sum_{l=0}^{N} \hat{F}(t_l)T_l(t), \tag{2.4a}$$

where

$$\hat{F}(t_l) = \frac{2}{N}\frac{1}{C_l}\sum_{j=0}^{N}\frac{F(t_j)T_l(t_j)}{C_j}. \tag{2.4b}$$

It should be noted that the Chebyshev spectral coefficients $\hat{F}(t_l)$ in (2.4b) can be evaluated using FFT. In fact, using $T_N(t) = \cos(N\cos^{-1} t)$ in (2.4b) gives

$$\hat{F}(t_l) = \frac{2}{N}\frac{1}{C_l}\sum_{j=0}^{N}\frac{F(t_j)}{C_j}\cos\left(\frac{j\pi l}{N}\right). \tag{2.4c}$$

It is well known that polynomial interpolation based on Chebyshev points $t_j$ is well behaved compared to that based on equally spaced points (see [8]). Clearly, $I_N$ is a linear operator on $C = C[-1, 1]$, the Banach space of continuous, real-valued functions on $[-1, 1]$, with the property $I_N^2 = I_N$. This space is equipped with the uniform norm

$$\|F\|_\infty = \sup_{-1\leqslant t\leqslant 1} |F(t)|, \quad F \in C. \tag{2.5}$$

Since, $I_N$ is a linear operator, with $I_N^2 = I_N$, then $I_N$ is a projection operator, whose range is $S_N$, the set of all polynomials of degree $\leqslant N$. Furthermore, $I_N$ is a bounded operator on $C$ with

$$\|I_N\| = \sup_{-1\leqslant t\leqslant 1} \sum_{j=0}^{N} |\phi_j(t)|. \tag{2.6}$$

Since, $I_N$ is the interpolatory operator defined by (2.4), it follows from [20] that

$$\lim_{N\to\infty} \int_{-1}^{1} |I_N F(t) - F(t)|^p (1 - t^2)^{-1/2}\,dt = 0 \tag{2.7}$$

for every $F \in C$, and for every $p \in (0, \infty)$.

Let $w$ be the Chebyshev weight function

$$w(t) = (1 - t^2)^{-1/2} \quad -1 \leqslant t \leqslant 1,$$

and let $L_{p,w}(-1, 1)$ be the space of the measurable functions $f \in L_p$ for which the weighted $L_p$ norm defined by

$$\|f\|_{p,w} := \left(\int_{-1}^{1} |f(t)|^p w(t)\,dt\right)^{1/p}$$

is finite. In terms of this norm, (2.7) may be written as

$$\lim_{N\to\infty} \|I_N F(t) - F(t)\|_{p,w} = 0, \tag{2.8}$$

from which it follows that

$$\sup_N \|I_N F\|_{p,w} < \infty, \quad \text{for all } F \in C. \tag{2.9}$$

Thus, if $I_N$ is considered as a linear operator from the space $C$ to the space $L_{p,w}(-1, 1)$ then, from Banach–Steinhaus theorem [13, p. 203], it follows that there exists a constant $k > 0$, which depends only on $p$, such that

$$\|I_N F\|_{p,w} \leqslant k\|F\|_\infty, \quad \text{for all } F \in C. \tag{2.10}$$

For any positive integer $N$, let $I_N$ be the orthogonal projection of $L_{2,w}$ onto the space spanned by $(\phi_0, \phi_1, \ldots, \phi_N)$. Then, we have the following error estimates (see [5]):

$$\|F - I_N F\|_{2,w} \leqslant MN^{-m}\|F\|_{H^l_w(-1,1)} \tag{2.11}$$

if $F \in H^m_w$ for some $m \geqslant 1$. In higher-order Sobolev norms, one has

$$\|F - I_N F\|_{H^l_w(-1,1)} \leqslant MN^{2l-m}\|F\|_{H^m_w(-1,1)} \tag{2.12}$$

for $0 \leqslant l \leqslant m$. As a consequence, we have

$$\|F' - (I_N F)'\|_{2,w} \leqslant MN^{2-m}\|F\|_{H^m_w(-1,1)}. \tag{2.13}$$

The same estimate holds in the discrete $L_{2,w}$-norms at $t_j$ (consult [5]). Thus, if $F \in C^\infty$, then the rate of convergence of $I_N F$ to $F$ is faster than any power of $1/N$.

The next theorem shows uniform convergence for the interpolating operator $I_N$.

**Theorem 1.** *If $t_j, 1 \leqslant j \leqslant N-1$ are the zeros of $\dot{T}_N(t)$ adjusted in the interval $(-1, 1)$, if $F(z)$ has no singularities except a finite number of poles, and if for some $n$, $\frac{F(z)}{z^n} \to 0$ as $|z| \to \infty$, then $I_N F(t) \to F(t)$ uniformly on $[-1, 1]$.*

**Proof.** Let $\xi(t) = \dot{T}_N(t)$ and $\xi_j(t) = \xi(t)/(t - t_j)$, $j = 1, 2, \ldots, N-1$. By Mittag Leffler's theorem [7] there exists a function $\psi(z,t)$ which has poles at $t$, $t_j$, $1 \leqslant j \leqslant N-1$, with residue $F(t)$ at $t$ and residue $-\phi_j(t)F(t_j)$ at $t_j$ such that

$$F(t) - I_N F(t) = \frac{1}{2\pi i}\int_\alpha \psi(z, t)\,\mathrm{d}z.$$

It can be easily seen that

$$\psi(z,t) = \frac{\xi(t)F(z)(t^2-1)}{\xi(z)(z-t)(z^2-1)}$$

is the desired function, and hence

$$F(t) - I_N F(t) = \frac{1}{2\pi \mathrm{i}}\int_\alpha \frac{\xi(t)F(z)(t^2-1)}{\xi(z)(z-t)(z^2-1)}\,\mathrm{d}z,$$

where the contour $\alpha$ has been chosen to encircle not only $t$ and $t_j$ but also $z_k$. If $F(z)$ has poles at $z_k$, $1 \leqslant k \leqslant s$ within the contour $\alpha$ none of which lie on $[-1, 1]$, then we have the following error:

$$F(t) - I_N F(t) + \sum_{k=1}^{s} R_k \frac{\xi(t)(t^2-1)}{\xi(z_k(z_k-t)(z_k^2-1)} = \frac{1}{2\pi \mathrm{i}}\int_\alpha \frac{\xi(t)F(z)(t^2-1)}{\xi(z)(z-t)(z^2-1)}\,\mathrm{d}z,$$

where $R_k$ is the residue of $F(z)$ at the pole $z_k$. The condition $|F(z)/z^n| \to 0$ ensures that the integral tends to zero as the contour expands to infinity, if $N$ is sufficiently large, and we thus obtain a new form of the error,

$$F(t) - I_N F(t) = -\sum_{k=1}^{s} R_k \frac{\xi(t)(t^2-1)}{\xi(z_k)(z_k - t)(z_k^2 - 1)}.$$

From the well-known properties of Chebyshev polynomials

$$|T_N(t)| \leqslant 1 \quad \text{for all } t \in [-1, 1]$$

and

$$(1-t^2)\dot{T}_N(t) = N[T_{N-1}(t) - tT_N(t)].$$

We may obtain the following error estimate:

$$|F(t) - I_N F(t)| \leqslant \sum_{k=0}^{s} \frac{2|R_k|}{|T_{N-1}(z_k) - z_k T_N(z_k)||z_k - t|}.$$

Since, $T_{N-1}(z) - zT_N(z)$ is a polynomial of degree $(N+1)$, we can easily see that $|T_{N-1}(z) - zT_N(z)| \to \infty$ as $N \to \infty$ for any $z$ outside $[-1, 1]$. Since, no $z_k$ lies in $[-1, 1]$ we have therefore proved that

$$|F(t) - I_N F(t)| \to 0 \quad \text{as } N \to \infty.$$

## 3. The Chebyshev spectral discretization

We approximate $y$ by a polynomial $y^N$ of degree $N$ with coefficients determined by discretizing the equation

$$y(t) = f(t) + \frac{T}{4}(t+1)\int_{-1}^{1} k(t,s)g(s, y(s))\,\mathrm{d}s, \quad t \in [-1, 1], \tag{3.1}$$

at the Chebyshev nodes $t_k$, $k = 0, 1, \ldots, N$ as follows: Define

$$z(t) = g(t, y(t)), \quad t \in [-1, 1]. \tag{3.2}$$

Using (3.1), we get

$$z(t) = g(t, f(t) + \int_{-1}^{1} \hat{k}(t,s)z(s)\,\mathrm{d}s), \quad -1 \leqslant t \leqslant 1, \tag{3.3}$$

where $\hat{k}(t, s) = (T/4(t+1))k(t,s)$. The Chebyshev spectral interpolant $z^N$ of $z$ at the point $t$ is defined by

$$z^N(t) = \sum_{l=0}^{N} a_l \phi_l(t). \tag{3.4}$$

The Chebyshev spectral coefficients $z^N(t_k) = a_k$ for $0 \leqslant k \leqslant N$ are determined by collocating (3.3) at the Chebyshev nodes $t_k$:

$$z^N(t_k) = g(t_k,\ f(t_k) + \int_{-1}^{1} \hat{k}(t_k,s) z^N(s)\,\mathrm{d}s). \tag{3.5}$$

The required approximation to the solution $y$ of Eq. (3.1) is

$$\begin{aligned} y^N(t) &= f(t) + \int_{-1}^{1} \hat{k}(t,s) z^N(s)\,\mathrm{d}s \\ &= f(t) + \sum_{l=0}^{N} a_l \int_{-1}^{1} \hat{k}(t,s)\phi_l(s)\,\mathrm{d}s, \end{aligned} \tag{3.6}$$

The integrals in (3.5) and (3.6) can be calculated, very accurately, using Chebyshev–Gauss integration rule, stating that there exists weights (Chebyshev weights) $w_j$

$$w_j = \begin{cases} \pi/2N & \text{if } j = 0, N, \\ \pi/N & \text{if } l \leqslant j \leqslant N-1, \end{cases} \tag{3.7}$$

such that for all $h \in S_{2N-1}$, we have (see, eg., [5, 8])

$$\int_{-1}^{1} (1-x^2)^{-1/2} h(x)\,\mathrm{d}x = \sum_{j=0}^{N} w_j h(t_j). \tag{3.8}$$

For computational reasons with $z^N(t_k) = a_k$ we write Eq. (3.5) as

$$\begin{aligned} a_k &= g(t_k,\ f(t_k) + \int_{-1}^{1} \hat{k}(t_k,s) z^N(s)\,\mathrm{d}s) \\ &\approx g(t_k,\ f(t_k) + \sum_{l=0}^{N} (1-s_l^2)^{1/2}\, \hat{k}(t_k,s_l)\, a_l\, w_l), \end{aligned} \tag{3.9}$$

if $k(t, s)$ is smooth enough in $s$. Thus,

$$y^N(t) \approx f(t) + \sum_{l=0}^{N} (1-s_l^2)^{1/2}\, \hat{k}(t,s_l)\, a_l\, w_l, \tag{3.10}$$

where $s_l = \cos(\pi l/N)$, $l = 0, 1, \ldots, N$.

Note that at the Chebyshev nodes $t_k$, $k = 0, 1, \ldots, N$, Eq. (3.10) is a system of nonlinear algebraic equations for the unknowns $y^N(t_0)$, $y^N(t_1)$, $\ldots$, $y^N(t_N)$.

## 4. Convergence results

The theoretical analysis of (3.1) will be carried out in the space $C = C([-1, 1])$. It is also convenient to make the following assumptions on $f$, $k$ and $g$ in Eq. (3.1):

A.1: $f \in C$;

A.2: $\sup_{-1\leqslant t\leqslant 1}\int_{-1}^{1}|\hat{k}(t,s)|^{q}\,\mathrm{d}s<\infty$, and $\lim_{t\to\hat{t}}\int_{-1}^{1}|\hat{k}(t,s)-\hat{k}(\hat{t},s)|^{q}\,\mathrm{d}t=0$, for $t\in[-1,1]$ and $q>1$;

A.3: the function $g(t,y)$ is defined and continuous on $D=[-1,1]\times\mathbb{R}$:

A.4: the partial derivative $g_y(t,\ y)$ exists and is continuous on $D$.

With $q$ as in assumption A.2, let $p$ be the number given by

$$\frac{1}{p}+\frac{1}{q}=1. \tag{4.1}$$

Note that it follows from the results of [10], under assumption A.2, the linear integral operator $\hat{K}$, defined by

$$(\hat{K}v)(t)=\int_{-1}^{1}\hat{k}(t,s)v(s)\,\mathrm{d}s,\quad t\in[-1,\ 1],\ f\in L_p \tag{4.2}$$

is a compact operator from $L_p$ to $C$, and hence it is necessarily completely continuous (see [13, p. 244]). Next we define a nonlinear completely continuous operator $T:L_p\to C$ by

$$T(v)(t)=f(t)+K(v)(t),\quad t\in[-1,\ 1],\ v\in L_p, \tag{4.3}$$

and a continuous, bounded operator $G$ by

$$G(v)(t)=g(t,\ v(t)),\quad t\in[-1,\ 1]\quad\text{and}\quad v\in C. \tag{4.4}$$

With the above notation, Eqs. (3.1) and (3.3) can be written in operator form as

$$y=TG(y),\quad y\in C, \tag{4.5}$$

$$z=GT(z),\quad z\in Lp, \tag{4.6}$$

respectively. Eqs. (4.5) and (4.6) are equivalent in the sense of one-to-one correspondence [14]. The approximation $z^N$ in operator form is

$$z^N=I_N GT(z^N). \tag{4.7}$$

Moreover, under assumptions A.1–A.4, the operator $GT$ is continuously Frechet differentiable on $L_p$. Its Frechet derivative at $z\in L_p$ is completely continuous linear operator $(GT)_t(z)$ given by

$$[(GT)_t(z)v](t)=g_y(t,f(t)+(Kz)(t))(Kv)(t),\quad t\in[-1,\ 1], \tag{4.8}$$

Below we give results analogous to those in [14].

**Theorem 2.** *Let $\hat{y}$ be a geometrical isolated solution of* (4.5), *and let $\hat{z}$ be the corresponding solution of* (4.6). *Suppose A.1–A.3 hold.*

(i) *If $\hat{y}$ has a nonzero index, then there exists an $N_0$ such that for $N_0\leqslant N$,* (4.6) *has a solution $z^N\in S_N$ satisfying*

$$\|\hat{z}-z^N\|_{p,w}\to 0\quad as\quad N\to\infty.$$

(ii) *Suppose* A.4 *holds, and that* 1 *is not an eigenvalue of the linear operator* $(GT)_t(\hat{z})$. *Then, there exists a neighborhood of* $\hat{z}$ *and an* $N_1$ *such that for* $N \geqslant N_1$ *a solution* $z^N$ *of* (4.7) *is unique in that neighborhood, and*

$$c_1 \|\hat{z} - I_N\hat{z}\|_{p,w} \leqslant \|\hat{z} - z^N\|_{p,w} \leqslant c_2 \|\hat{z} - I_N\hat{z}\|_{p,w},$$

*where* $c_1$ *and* $c_2$ *are constant independent of* $N$.

**Corollary 3.** *Under the conditions in Theorem* 2 (ii) *there exists a constant* $\alpha > 0$ *such that*

$$\|\hat{z} - z^N\|_{p,w} \leqslant \alpha \inf_{\psi \in S_N} \|\hat{z} - \psi\|_\infty.$$

**Proof.** For any $\psi \in S_N$, we have

$$\begin{aligned} \|\hat{z} - I_N\hat{z}\|_{p,w} &= \|(\hat{z} - \psi) - I_N(\hat{z} - \psi)\|_{p,w} \\ &\leqslant \|\hat{z} - \psi\|_{p,w} + \|I_N(\hat{z} - \psi)\|_{p,w} \\ &\leqslant \left[\int_{-1}^{1} |\hat{z}(s) - \psi(s)|^p w(s)\,\mathrm{d}s\right]^{1/p} + c_3\|\hat{z} - \psi\|_\infty \\ &\leqslant \left(\left[\int_{-1}^{1} w(s)\,\mathrm{d}s\right]^{1/p} + c_3\right) \|\hat{z} - \psi\|_\infty, \end{aligned}$$

where the second step follows from the fact that the operator $I_N$ is bounded from $C$ to $L_p$.

Note that

$$\inf_{\psi \in S_N} \|\hat{z} - \psi\|_\infty \to 0 \quad \text{as} \quad N \to \infty$$

follows from the Weierstrass approximation theorem [8], and is bounded above by the use of the Jackson theorems [6, p. 147].

**Theorem 4.** *Let* $\hat{y} \in C$ *be geometrically isolated solution of* (4.5), *and let* $\hat{z}$ *be the corresponding solution of* (4.6). *Suppose* A.1 *and* A.3 *hold.* (i) *If* $\hat{y}$ *has a nonzero index, then with* $z^N$ *as in Theorem* 2 (i), *and* $N \geqslant N_0$, (3.8) *defines an approximation* $y^N \in C$ *satisfying*

$$\|\hat{y} - y^N\|_\infty \to 0 \text{ as } N \to \infty.$$

(ii) *Suppose* A.4 *holds, and that 1 is not an eigenvalue of the linear operator* $(GT)_t(\hat{z})$. *Then, for* $N \geqslant N_1$, *the approximation* $y^N$ *given by* (3.8), *with* $z^N$ *as in Theorem* 2 (ii), *satisfies*

$$\|\hat{y} - y^N\|_\infty \leqslant \beta \inf_{\psi \in S_N} \|\hat{z} - \psi\|_\infty,$$

*where* $\beta$ *is a positive constant independent of* $N$.

**Proof.** (i) Since $T : L_{p,w} \to C$ and $y^N = T(z^N)$, it follows from that $y^N \in C$.

$$|\hat{y}(t) - y^N(t)| = \left| \int_{-1}^{1} \hat{k}(t,s)[\hat{z}(s) - z^N(s)]\,\mathrm{d}s \right|$$
$$\leqslant \int_{-1}^{1} \left| \frac{\hat{k}(t,s)}{w(s)} \right| \left| \hat{z}(s) - z^N(s) \right| w(s)\,\mathrm{d}s$$
$$\leqslant \left[ \int_{-1}^{1} \left| \frac{\hat{k}(t,s)}{w(s)} \right|^p w(s)\,\mathrm{d}s \right]^{1/p} \times \left[ \int_{-1}^{1} |\hat{z}(s) - z^N(s)|^q w(s)\,\mathrm{d}s \right]^{1/q},$$

where in the last step Holder inequality for the distribution $w(s)\,\mathrm{d}s$ has been used. Now for $1 \leqslant p$,

$$\sup_{-1 \leqslant t \leqslant 1} [w(t)]^{1-p} = 1.$$

Therefore,

$$|\hat{y}(t) - y^N(t)| \leqslant \left[ \int_{-1}^{1} |\hat{k}(t,s)|^p\,\mathrm{d}s \right]^{1/p} \|\hat{z} - z^N\|_{p,w},$$

and hence,

$$\|\hat{y} - y^N\|_\infty \leqslant \gamma \|\hat{z} - z^N\|_{p,w} \to 0 \quad \text{as} \quad N \to \infty,$$

where

$$\gamma = \sup_{-1 \leqslant t \leqslant 1} \left[ \int_{-1}^{1} |\hat{k}(t,s)|^p\,\mathrm{d}s \right]^{1/p} < \infty.$$

(ii) This follows similarly, with the aid of Corollary 3. Thus, $y^N$ converges *uniformly* to $\hat{y}$.

## 5. An illustrative example

Consider the nonlinear Volterra integral equations

$$Y(\tau) = F(\tau) + \int_0^\tau K(\tau, r) Y(r)^2\,\mathrm{d}r, \quad \tau \in [0, 1], \tag{5.1}$$

where $F(\tau) = -1/4\tau^5 - 2/3\tau^4 - 5/6\tau^3 - \tau^2 + 1$ and $K(\tau, r) = \tau r + 1$. This problem has a unique solution $Y(\tau) = \tau + 1$, and hence it serves as a test problems. The time transformations $\tau = 1/2(t+1)$ and $r = \tau/2(s+1)$ are used to transform the intervals $\tau \in [0, 1]$ into $t \in [-1, 1]$ and $r \in [0, \tau]$ into $s \in [-1, 1]$. Thus, the integral Eq. (5.1) can be replaced by

$$y(t) = f(t) + \int_{-1}^{1} \hat{k}(t, s) y(s)^2\,\mathrm{d}s, \tag{5.2}$$

where $f(t) = -\frac{1}{4}(\frac{1}{2}(t+1))^5 - \frac{2}{3}(\frac{1}{2}(t+1))^4 - \frac{5}{6}(\frac{1}{2}(t+1))^3 - (\frac{1}{2}(t+1))^2 + 1$ and $\hat{k}(t, s) = \frac{1}{4}(t+1)k(t, s)$. The Chebyshev spectral method was applied to approximate the solution of Eq. (5.2), and the

Table 1
Results for the example

| Methods | $\|\hat{z} - z^N\|_\infty$ | $\|\hat{y} - y^N\|_\infty$ | Exec. time (s) |
|---|---|---|---|
| Chebyshev spectral | | | |
| $N = 4$ | $< 10^{-5}$ | $< 10^{-6}$ | 0.93 |
| $N = 6$ | $< 10^{-8}$ | $< 10^{-9}$ | 1.41 |
| $N = 8$ | $< 10^{-10}$ | $< 10^{-11}$ | 1.79 |
| $N = 10$ | $< 10^{-13}$ | $< 10^{-14}$ | 2.00 |

resulting nonlinear system of equations (3.10) was solved by More and Cosnard's Algorithm [19]. All computations were carried out with very high precision on a Sun-SPARCII workstation. In Table 1, we display the Chebyshev spectral approximate solutions of many orders. The error estimates $\|\hat{z} - z^N\|_\infty$ and $\|\hat{y} - y^N\|_\infty$ are calculated by taking the largest of the errors at the Chebyshev nodes $t_k$, $k = 0, 1, \ldots, N$. As seen from the results of Table 1, the Chebyshev spectral approximation enjoys *formal* spectral accuracy, i.e its truncation error decays as fast as the global smoothness of the underlying solution permits.

## 6. Conclusions

In this paper, the Chebyshev spectral method has been used for the solution of operator equations, such as the nonlinear Volterra–Hammerstein integral equations. With the availability of this methodology, it will now be possible to investigate the spectral solution of nonlinear physical problems, particularly of the nonlinear initial-value problems.

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