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# Perturbation theory for the eigenvalues of factorised symmetric matrices

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**Abstract**

We obtain eigenvalue perturbation results for a factorised Hermitian matrix $H = GJG^*$ where $J^2 = I$ and $G$ has full row rank and is perturbed into $G + \delta G$, where $\delta G$ is small with respect to $G$. This complements the earlier results on the easier case of $G$ with full column rank. Applied to square factors $G$ our results help to identify the so-called quasidefinite matrices as a natural class on which the relative perturbation theory for the eigensolution can be formulated in a way completely analogous to the one already known for positive definite matrices. © 2000 Elsevier Science Inc. All rights reserved.

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Consider a Hermitian matrix in the factorised form

$$H = GJG^*, \quad J = \mathrm{diag}(\pm 1), \tag{1}$$

which we assume as non-singular (this implies that $G^*$ has full column rank). The matrix $H$ is perturbed as

$$H + \delta H = (G + \delta G)J(G + \delta G)^* \tag{2}$$

with the elementwise estimate

$$|\delta G| \leqslant \varepsilon |G|. \tag{3}$$

The (equally ordered) eigenvalues of $H$, $H + \delta H$ are denoted by $\lambda_i$, $\lambda_i + \delta\lambda_i$, respectively. In this paper we will derive relative eigenvalue perturbation bounds, i.e. bounds for $\delta\lambda_i/\lambda_i$.

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Keeping a matrix in the factorised form may have advantages because the condition of the factor is often just the square root of the condition of the product thus alleviating error troubles. This is known to be the case for the standard singular value problem with $J = I$. A similar result holds with general $J$, if $G^*G$ is positive definite (this means $G$ with full column rank) as was shown by Veselić and Slapničar [16]. In this case the relative bound (3) implies

$$\|\delta Gx\| \leqslant \nu\|Gx\| \quad \text{for all } x \tag{4}$$

with

$$\nu = \varepsilon\frac{\sigma_{\max}(|G\varDelta^{-1}|)}{\sigma_{\min}(G\varDelta^{-1})} \tag{5}$$

for any diagonal positive definite $\varDelta$. The bound (4), in turn, implies

$$1 - \nu(2+\nu) \leqslant \frac{\lambda_i + \delta\lambda_i}{\lambda_i} \leqslant 1 + \nu(2+\nu). \tag{6}$$

A related eigenvector perturbation bound was given in [13]. The quotient on the right-hand side of (5) is called the right scaled condition number of $G$. Its appearence is typical whenever relative bounds are sought.

The main technique of [16] is to convert the eigenvalue problem for $H$ into the one for the matrix

$$T = JG^*G, \tag{7}$$

or, equivalently, for the Hermitian matrix pair

$$G^*G\ ,\ J.$$

It is remarkable that even in the indefinite case the condition number of the – not necessarily orthogonal – eigenvectors of $T$ does not enter the eigenvalue bound (6).

In this paper we study the harder, complementary case with $GG^*$ positive definite (i.e. $G^*$ has full column rank). Again, the link to the matrix $T$ will be used; this time the condition number of the eigenvectors of $T$ will be a substantial part of the obtained bounds. Moreover, it turns out that the mere requirement that $G^*$ be of full column rank does not suffice to obtain reasonable results. We must ask that $H = GJG^*$ is non-singular. This can be understood from examples where a full column-rank $G^*$ and an indefinite $J$ yield even $H = 0$. The latter effect disappears if $J = I$ or if $G$ is square.

Examples of such problems are the ones in which a Hermitian matrix is given as a difference of two positive definites, which are given by their factors [9]:

$$H = MM^* - NN^* = GJG^*$$

with

$$G = \begin{bmatrix} M & N \end{bmatrix}, \quad J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.$$

This is a common way to express downdating problems. The problem of determining the eigenvalues directly from $G$, $J$ is often called the hyperbolic singular value problem; the values $\sigma_i = \mathrm{sign}(\lambda_i)\sqrt{|\lambda_i|}$ are then called hyperbolic singular values of the pair $G, J$.[1] In this case (3) implies

$$\|\delta G^* x\| \leqslant \nu \|G^* x\| \quad \text{for all } x \tag{8}$$

with

$$\nu = \varepsilon \frac{\sigma_{\max}(|G^* \Delta^{-1}|)}{\sigma_{\min}(G^* \Delta^{-1})} \tag{9}$$

for any diagonal positive definite $\Delta$. The new estimate reads – in the simplest case of $G$ square

$$1 - \nu(2+\nu)\|F\|\|F^{-1}\| \leqslant \frac{\lambda_i + \delta\lambda_i}{\lambda_i} \leqslant 1 + \nu(2+\nu)\|F\|\|F^{-1}\|, \tag{10}$$

where $F$ is the eigenvector matrix for $T$. The difference between the two cases is nicely characterised by the two types of perturbations (4) and (8) which give different results even in the case when $G$ is square. The same elementwise estimate (3) uses the *right scaled condition* of $G$ in (5) and the *left scaled condition* of $G$ in (9). The new eigenvalue bound (10) is weaker in the sense that it contains an additional condition number, namely the condition of the eigenvectors of the matrix $T$ above. But, of course, the new bound is independent of the old one and it may well happen that (10) gives sharper estimates than (6) under the same elementwise bound (3). This asymetry is typical for the true hyperbolic singular value problem, and it disappears, if $J = I$ which is the standard singular value case. We will also give some useful estimates for this new condition number and illustrate our theory by some examples.

Another aspect of our results is that they apply to the case of a triangular factor $G$ thus allowing new eigenvalue bounds under elementwise perturbation of the matrix $H$ itself. As we know, there are classes of matrices which allow well conditioned triangular decomposition – like the scaled diagonally dominant (s.d.d.) ones (see [1]). Another such classes are the so-called quasidefinite matrices [6,14]. As a consequence of our general Theorem 7 below the quasidefinite matrices are identified as another class, allowing a very simple measure of the ‘well-behavedness’, i.e. of the sensitivity of the relative eigenvalue bound $|\delta\lambda/\lambda|$ subjected to the elementwise error bound $|\delta H_{ij}/H_{ij}|$. The new bounds appear to be a natural extension of similar bounds for the positive definite case, obtained in [3]. More interesting still, taking a positive definite matrix and changing the sign of one of its diagonal blocks (this makes the matrix quasidefinite) appears to *decrease* its eigenvalue sensitivity – a phenomenon one would not expect at the first glance. We still do not have a full quantitative description of this phenomenon.

**Theorem 1.** *Let H from* (1) *be non-singular. Then there is F such that*

$$G^* G F = J F D^2 J_1, \quad F^* J F = J_1, \tag{11}$$

[1] Written for hyperbolic singular values, the estimate (6) naturally simplifies to $|\delta\sigma_i| \leqslant \nu|\sigma_i|$.

*where* $J_1 = \mathrm{diag}(\pm 1)$ *and* $D$ *have the size of* $H$ *and* $D$ *is diagonal positive definite. If, in addition,* (2) *and* (8) *hold, then for any such* $F$ *the eigenvalue estimate for* $H$, $H + \delta H$ *reads*

$$1 - \nu(2+\nu)\|F\|^2 \leqslant \frac{\lambda_i + \delta\lambda_i}{\lambda_i} \leqslant 1 + \nu(2+\nu)\|F\|^2. \tag{12}$$

(*Here* $\nu$ *is taken from* (9).) *This estimate is sharp.*

(Note that for a square $G$ (12) just reduces to (10) since then $F$ is square with $F^*JF = J$ which implies $\|F^{-1}\| = \|F\|$ and $\|F\|\|F^{-1}\| = \|F\|^2$.)

**Proof.** We start with the eigendecomposition of $H$

$$H = GJG^* = UD^2J_1U^* \tag{13}$$

with $U$ unitary, $J_1 = \mathrm{diag}(\pm 1)$ and $D$ diagonal and positive definite. Set

$$F = JG^*UD^{-1}J_1. \tag{14}$$

Then

$$G^*GF = G^*GJG^*UD^{-1}J_1 = JJG^*UD^{-1}J_1D^2J_1 = JFD^2J_1.$$

Also

$$F^*JF = J_1D^{-1}U^*GJJJG^*UD^{-1}J_1 = J_1D^{-1}D^2J_1D^{-1}J_1 = J_1.$$

We now prove that the spectral absolute value $|H|_{\mathrm{s}} = \sqrt{H^2}$ is equal to $GFF^*G^*$. Indeed,

$$GFF^*G^* = GJG^*UD^{-2}U^*GJG^* = H|H|_{\mathrm{s}}^{-1}H = |H|_{\mathrm{s}}. \tag{15}$$

Similarly

$$|H|_{\mathrm{s}}^{-1/2}GG^*|H|_{\mathrm{s}}^{-1/2} = UD^{-1}F^*JJFD^{-1}U^* = UF^*FU^*. \tag{16}$$

Conversely, take any $F$ satisfying (11); by $F^*G^*GF = D^2$ the matrix $V = GFD^{-1}J_1$ is unitary, and

$$HV = GJG^*FD^{-1}J_1 = GJJFD^2J_1D^{-1}J_1 = VD^2J_1$$

and

$$F = JG^*VD^{-1},$$

so, (16) holds for $F$, $V$ as well and all such $F$ have the same norms. Now we estimate

$$\begin{aligned}|x^*\delta Hx| &\leqslant |x^*\delta GJG^*x| + |x^*GJ\delta G^*| + |x^*\delta GJ\delta G^*x| \\ &\leqslant 2\|\delta G^*x\|\|G^*x\| + \|\delta G^*x\|^2 \\ &= (2+\nu)\nu x^*GG^*x = (2+\nu)\nu x^*|H|_{\mathrm{s}}^{-1/2}UF^*FU^*|H|_{\mathrm{s}}^{-1/2}x \\ &\leqslant (2+\nu)\nu\|F^*F\|x^*|H|_{\mathrm{s}}x = (2+\nu)\nu\|F\|^2x^*|H|_{\mathrm{s}}x.\end{aligned}$$

Now apply Theorem 2.1 from [16] to obtain (12).

We now prove that our estimate is sharp. Take $G$ as a one-row matrix

$$G = \begin{bmatrix} g_1 & \cdots & g_n \end{bmatrix}.$$

Then there is only one eigenvalue $\lambda = GJG^*$. We choose the perturbation as

$$\delta G = \begin{bmatrix} \delta g_1 & \cdots & \delta g_n \end{bmatrix}$$

with

$$\delta g_i = \begin{cases} \nu g_i, & i \leqslant m \\ (-1 + \sqrt{1 - 2\nu - \nu^2}) g_i, & i > m \end{cases}$$

for $0 < \nu < \sqrt{2} - 1$. Then

$$(g_i + \delta g_i)^2 = \begin{cases} g_i^2 + (2+\nu)\nu g_i^2, & i \leqslant m, \\ g_i^2 - (2+\nu)\nu g_i^2, & i > m. \end{cases}$$

Now

$$\lambda + \delta\lambda = \sum_{i \leqslant m} (g_i + \delta g_i)^2 - \sum_{i > m} (g_i + \delta g_i)^2 = (2+\nu)\nu \|G\|^2.$$

On the other hand (14) gives (note that here $U = 1$, $D = |\lambda|^{1/2} = |GJG^*|^{1/2}$)

$$F = JG^*/|GJG^*|^{1/2} \quad \text{and} \quad \|F\|^2 = \|G\|^2/|GJG^*|.$$

Thus,

$$\frac{\lambda + \delta\lambda}{\lambda} = 1 + \frac{(2+\nu)\nu\|G\|^2}{|GJG^*|} = 1 + (2+\nu)\nu\|F\|^2$$

and the right-hand side inequality in (12) goes over into an equality. This shows that (12) cannot be improved in general.[2] □

Since the basis of our proof is the estimate

$$|x^* \delta H x| \leqslant (2+\nu)\nu \|F\|^2 x^* |H|_{\mathrm{s}} x \tag{17}$$

the eigenvector perturbation bound contained in [13] can be immediately taken over. Compared with the easier 'dual' result in [13] the only novelty here is the extra factor $\|F\|^2$.

We now give some results for the important case $\|F\| = 1$.

**Theorem 2.** *Let $H = GJG^*$ be as in* (1) *with*

$$J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}. \tag{18}$$

[2] In fact, the same example was produced in [16] where it was considered "incurable". We are glad to correct here this pessimistic statement.

*Then*

$$\operatorname{Tr}(GG^*)^2 \geqslant \operatorname{Tr}(H^2), \tag{19}$$

$$\operatorname{Tr} GG^* \geqslant \operatorname{Tr}(|H|_s). \tag{20}$$

*The following are equivalent*:
(i) *any of the two inequalities above becomes an equality*;
(ii) $G^*G$ *and* $J$ *commute*;
(iii) $|H|_s = GG^*$.

**Proof.** Using the Cauchy–Schwarz (C–S) inequality for the trace scalar product on matrices with the norm $\|\cdot\|_E$ we obtain

$$\begin{aligned}
\|GJG^*\|_E^2 &= \operatorname{Tr}(GJG^*GJG^*)\\
&= \operatorname{Tr}(G^*GJG^*GJ) = \langle G^*GJ, (G^*GJ)^*\rangle\\
&\leqslant \|G^*GJ\|_E \|(G^*GJ)^*\|_E = \|G^*GJ\|_E^2\\
&= \operatorname{Tr}(G^*GJJG^*G) = \operatorname{Tr}[(G^*G)^2] = \|G^*G\|_E^2.
\end{aligned}$$

Equality in C–S means that

$$G^*GJ = \alpha JG^*G \quad \text{for some } \alpha > 0.$$

By taking norms we obtain $\alpha = 1$. Thus, $G^*G$ and $J$ commute. The proof of the second inequality is similar: Decompose

$$H = GJG^* = UDJ_1DU^*$$

with $D$ diagonal and positive definite $U^*U = I_{n-m}$ and $J_1$ a diagonal matrix of signs. Then, again by the C–S inequality

$$\begin{aligned}
\operatorname{Tr}(|H|_s) &= \operatorname{Tr}(D^2) = \operatorname{Tr}((JG^*UJ_1)^*G^*U) \leqslant \|JG^*UJ_1\|_E \|G^*U\|_E\\
&= \|G^*\|_E^2 = \operatorname{Tr}(GG^*).
\end{aligned}$$

Again, the equality holds, if and only if

$$JG^*UJ_1 = \alpha G^*U, \quad \alpha > 0$$

and by taking norms $\alpha = 1$, i.e.

$$JG^*UJ_1 = G^*U \quad \text{or} \quad JG^* = G^*UJU^*$$

hence

$$JG^*G = G^*UJU^*G$$

so, $G^*G$ and $J$ commute. In this case

$$GJG^*GJG^* = GG^*GG^*,$$

which means $|H|_s = GG^*$. Conversely, the last equality implies the equality in (20) and so $G^*G$ and $J$ commute. □

In the commutativity case we can write

$$G^*G = \begin{bmatrix} G_1^*G_1 & G_1^*G_2 \\ G_2^*G_1 & G_1^*G_2 \end{bmatrix} = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix},$$

i.e. $G_1^*G_2 = 0$. Now

$$H = GJG^* = [G_1G_2]\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}\begin{bmatrix} G_1^* \\ G_2^* \end{bmatrix} = G_1G_1^* - G_2G_2^*,$$

where the product of the two terms vanishes:

$$G_1G_1^*G_2G_2^* = G_2G_2^*G_1G_1^* = 0.$$

Thus,

$$\begin{aligned} G_1G_1^* &= H_+ \quad \text{“}+\text{” part of } H, \\ G_2G_2^* &= H_- \quad \text{“}-\text{” part of } H. \end{aligned}$$

In other words, the equality sign is attained, if and only if in $GJG^* = G_1G_1^* - G_2G_2^*$ just $\pm$ parts of $H$ appear.

If $H$ is a diagonal matrix of signs then the preceding theorem is strengthened as follows.

**Theorem 3.** *Let $F$ be an $n \times m$ matrix with*

$$F^*JF = J_1, \; J_1 = \operatorname{diag}(\pm 1). \tag{21}$$

*Then*

$$\|F^*F\|_{\mathrm{E}} \geqslant \sqrt{m}, \tag{22}$$

$$\|F\|_{\mathrm{E}} \geqslant \sqrt{m}, \tag{23}$$

$$\|F\| \geqslant 1. \tag{24}$$

*The following are equivalent*:

(i) *any of the three inequalities above becomes an equality.*

(ii) $F^*F = I$.

**Proof.** Anything concerning (23) and (22) follows immediately from the preceding theorem. Also, from (ii) it directly follows $\|F\| = 1$. Conversely, let $\|F\| = 1$ hold. Without loss of generality we can assume that both $J$ and $J_1$ have the block form (18) (possibly with different block sizes). By partitioning

$$F = \begin{bmatrix} F_+ \\ F_- \end{bmatrix}, \text{ according to the partition of } J$$

(21) reads

$$x^*F_+^*F_+x = x^*x + x^*F_-^*F_-x \quad \text{for } x = J_1x,$$

$$x^*F_+^*F_+x = -x^*x + x^*F_-^*F_-x \quad \text{for } x = -J_1x.$$

Now, $\|F\| \leqslant 1$ implies $\|F_{\pm}\| \leqslant 1$ which, together with the identities above, yields

$$F_+^* F_+ x = x,\ F_- x = 0 \quad \text{for } x = J_1 x,$$

$$F_-^* F_- x = x,\ F_+ x = 0 \quad \text{for } x = -J_1 x,$$

from which (ii) follows. □

The two theorems above will enable us to single out the case of commuting $G^*G, J$ as the case with optimal constant in the eigenvalue estimate (12) namely the one with $\|F\| = 1$.

**Theorem 4.** *Let $H = GJG^*$ be non-singular and let $F$ be defined by* (14). *Then*

$$\|F\|^2 = \max_{x \neq 0} \frac{x^* G G^* x}{x^* |H|_{\mathrm{s}} x} \geqslant 1. \tag{25}$$

*The following are equivalent*:

(i) *$G^*G$ and $J$ commute.*

(ii) *The inequality* (25) *becomes an equality.*

(iii) $F^*F = I$.

**Proof.** From (16) we obtain

$$F^* F = U^* |H|_{\mathrm{s}}^{-1/2} U G G^* |H|_{\mathrm{s}}^{-1/2} U. \tag{26}$$

Thus, the quantity

$$\|F\|^2 = \max_{x \neq 0} \frac{x^* |H|_{\mathrm{s}}^{-1/2} G G^* |H|_{\mathrm{s}}^{-1/2} x}{x^* x} = \max_{x \neq 0} \frac{x^* G G^* x}{x^* |H|_{\mathrm{s}} x} \tag{27}$$

is the largest eigenvalue of the generalized eigenvalue problem

$$G G^* x = \lambda |G J G^*|_{\mathrm{s}} x.$$

So, the equality in (25) is equivalent to $\|F\| = 1$ and then (by Theorem 3) with $F^*F = I$ also.

By Theorem 2 it is also clear that (i) implies (ii). Conversely, the equality in (25) implies

$$x^* G G^* x \leqslant x^* |H|_{\mathrm{s}} x \quad \text{for all } x$$

hence

$$\mathrm{Tr}(G^* G) \leqslant \mathrm{Tr}(|H|_{\mathrm{s}}),$$

which by Theorem 2 implies $|H|_{\mathrm{s}} = GG^*$ and hence (ii). □

The value $\|F\|^2$ can be understood as a sort of condition number. If $F$ is square then $\|F^{-1}\| = \|JF^*J\| = \|F\|$ and $\|F\|^2$ coincides with the standard condition number of $F$. [3]

[3] For non-square $F$ this condition is connected with the natural biorthogonality defined by $J$.

The expression for $\|F\|$ in (25) can be easily rewritten into one with scaled matrices, in fact, it is invariant under scaling. Setting

$$G = \Delta B, \quad |H|_{\mathrm{s}} = \Delta \hat{A} \Delta$$

for any diagonal positive definite $\Delta$ gives

$$\|F\|^2 = \max_{x \neq 0} \frac{x^* B B^* x}{x^* \hat{A} x}. \tag{28}$$

For $G$ square Slapničar and Veselić [17] recently proved

$$\|F\| = \|F^{-1}\| \leqslant \min_{\Omega J = J\Omega} \sqrt{\|G\Omega^{-1}\|\|\Omega G^{-1}\|}. \tag{29}$$

If $H$ is known to be s.d.d., i.e.

$$H = \Delta(J_1 + N)\Delta, \quad \|N\| < 1$$

then, according to [16, Theorem 2.29],

$$\frac{1}{x^* \hat{A} x} \leqslant \|\hat{A}^{-1}\|/x^* x \leqslant \frac{n}{(1 - \|N\|)x^* x}$$

and

$$\|F\|^2 \leqslant \frac{n\|B\|^2}{1 - \|N\|}. \tag{30}$$

Thus, a low $\|F\|$ is obtained, if $H$ is s.d.d. and the *same scaling matrix* $\Delta$ reduces the norm of $B = \Delta^{-1} G$ as well. Quantitatively, this does not show the superiority of our estimates based on $G$, $J$ over those based on the Gramm matrix $H$ itself. To illustrate properly the power of our estimates we take an example. Set

$$G = \begin{bmatrix} 400000 & 0 & 0 & 2 & 0 & 0 \\ -400000 & 4 & 0 & 0 & 2 & 0 \\ 0 & -4 & 4 & 0 & 0 & 2 \end{bmatrix}, \quad J = \begin{bmatrix} I_3 & 0 \\ 0 & -I_3 \end{bmatrix}. \tag{31}$$

This is a realistic example, obtained by the three-point discretization of the Sturm–Liouville eigenvalue problem

$$-\frac{\mathrm{d}}{\mathrm{d}x} a(x) \frac{\mathrm{d}}{\mathrm{d}x} y - v(x) y = \lambda y, \quad y(0) = 0, \quad y'(1) = 0, \quad a, v > 0 \tag{32}$$

with strongly varying $a$.

Scaling the rows of $G$ to the unit length gives $G = \Delta B$ with

$$B = \begin{bmatrix} 1 & 0 & 0 & 5 \cdot 10^{-5} & 0 & 0 \\ -1 & 10^{-4} & 0 & 0 & 5 \cdot 10^{-5} & 0 \\ 0 & -2/3 & -2/3 & 0 & 0 & 1/3 \end{bmatrix}.$$

Here $\|F\|^2 \approx 16$, cond $B \approx 10^4$, while the scaled condition of $|H|_{\mathrm{s}}$ is about $10^{10}$, so our bound is $1.6 \cdot 10^5$ which is about the full advantage of working with factors: the half relative error.

This very favourable state of affairs does not seem to be easy to single out under general conditions. A bit easier is the case where $H$ – in spite of an indefinite $J$ – is still positive definite. We have

**Theorem 5.** *Let*

$$G = \varDelta B = \begin{bmatrix} G_1 & G_2 \end{bmatrix} = \varDelta \begin{bmatrix} B_1 & B_2 \end{bmatrix}, \quad J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix},$$

*with a square $G_1$ and any positive definite diagonal $\varDelta$. Set*

$$R = B_1^{-1} B_2 = G_1^{-1} G_2$$

*and suppose*

$$\beta = \|R\| < 1.$$

*Then $H = GJG^*$ is positive definite and for $F$ from Theorem* 1 *we have*

$$\|F\|^2 = \frac{1+\beta^2}{1-\beta^2}.$$

**Proof.** The positive definiteness of $H$ follows from

$$H = G_1(I - RR^*)G_1^*. \tag{33}$$

Set

$$F = \begin{bmatrix} I \\ -R^* \end{bmatrix} (I - RR^*)^{-1/2} U,$$

where $U$ is unitary and such that

$$U^*(I - RR^*)^{1/2} G_1^* G_1 (I - RR^*)^{1/2} U$$

is diagonal. Then one readily sees that $F$ satisfies the conditions of Theorem 1 and

$$F^*F = U^*(I + RR^*)(I - RR^*)^{-1} U$$

hence

$$\|F^*F\| = \frac{1+\beta^2}{1-\beta^2}. \qquad \square$$

Now we can compare the two condition numbers for perturbation of the eigenvalues, the first starting from the matrix $H$ itself and the second starting from the factor $G$. The first is given by (33) and [16] as

$$\||B_1(I - RR^*)B_1|\| \, \|B_1^{-*}(I - RR^*)^{-1} B_1^{-1}\| \leqslant \sqrt{n} \frac{\|B_1^{-1}\|^2 \|B_1\|^2}{1-\beta^2}. \tag{34}$$

The second condition number is, by (9) and (12),

$$\|F\|^2 \frac{\sigma_{\max}(|B|)}{\sigma_{\min}(B)} \leqslant \left(\frac{1+\beta^2}{1-\beta^2}\right)^{3/2} \sqrt{n} \|B_1^{-1}\| \|B_1\|, \tag{35}$$

so that for $\beta$ not close to one the latter is about the square root of the first.

In the case of $H$ positive definite the number $\|F\|$ has an additional geometric interpretation. According to (27) we have

$$\frac{1}{\|F\|^2} = \min_{x \neq 0} x^*(GG^*)^{-1/2} GJG^*(GG^*)^{-1/2} x$$

and this is the cosine of the greatest principal angle between the column spaces of $G^*$ and $JG^*$. Indeed, $G^*(GG^*)^{-1/2}$ is an orthonormal basis in the first subspace and $JG^*(GG^*)^{-1/2}$ in the second. For $\|F\| = 1$ these two subspaces coincide.

We now concentrate to the case of a square factor $G$ such that $B = D^{-1}G$ is well conditioned. This is the case in which our previous results are rather poor. On the other hand, square factor appears in symmetric decompositions, in particular, if in

$$H = GJG^*, \quad J = \operatorname{diag}(\pm 1) \tag{36}$$

the factor $G$ is *lower triangular*. We may always choose $G$ with positive diagonal. As is easily seen, both $G$ (if it exists) and $J$ are uniquely determined by $H$. Another canonical decomposition of $H$ (again, if it exists) is *the scaling*

$$H = DAD, \quad D \text{ diagonal}, \ A_{ii} = 1. \tag{37}$$

In the particular case when $\|A - I\| < 1$ the matrix $H$ is called s.d.d. (see [1]).
By writing

$$G = DB \tag{38}$$

we have

$$A = BJB^*, \quad B \text{ lower triangular.} \tag{39}$$

We consider perturbations of $H$ of the type

$$H \mapsto H + \delta H = D(A + \delta A)D. \tag{40}$$

Here $\delta H$ may be bounded as

$$|\delta H| \leqslant \varepsilon |H| \text{ or,} \quad \text{equivalently, } |\delta A| \leqslant \varepsilon |A| \tag{41}$$

or as

$$|\delta H_{ij}| \leqslant \varepsilon \sqrt{|H_{ii}||H_{jj}|} \text{ or,} \quad \text{equivalently, } |\delta A_{ij}| \leqslant \varepsilon. \tag{42}$$

Then

$$H + \delta H = D(BJB^* + \delta A)D = DB(J + N)B^*D \tag{43}$$

with

$$N = B^{-1}\delta A B^{-*}. \tag{44}$$

We now need a lemma controlling the triangular indefinite decomposition of $J + N$ for small $N$. The following result is akin to the results of [5] for $J = I$.

**Lemma 6.** *Let $N$ be a Hermitian matrix with $\|N\| < 1/2$. Then there exists a unique lower triangular $\Gamma$ such that*

$$J + N = (I + \Gamma)J(I + \Gamma^*), \tag{45}$$

*where*

$$\|\Gamma\|_E \leqslant \frac{\sqrt{2}}{1+\sqrt{1-2\|N\|_E}}. \tag{46}$$

**Proof.** From (45) we obtain

$$N = J\Gamma^* + \Gamma J + \Gamma J \Gamma \tag{47}$$

or, with $X = \Gamma J$,

$$X + X^* + XJX^* = N, \tag{48}$$

where upper triangular $X$ is to be determined. This equation can be brought into a fixed-point form as

$$X = \mathscr{M}(X) = -\mathscr{P}(XJX^*) + \mathscr{P}(N). \tag{49}$$

Here $\mathscr{P}$ is the linear operator mapping the real space of all Hermitians into the real space of all upper triangulars with the real diagonal, defined by

$$\mathscr{P}(X + X^*) = X.$$

The Euclidian-generated norm of $\mathscr{P}$ is

$$\|\mathscr{P}\| = \max_{X\neq 0} \frac{\|\mathscr{P}(X)\|_E}{\|X\|_E} = \frac{1}{\sqrt{2}}.$$

We will treat this equation by the Banach fixed-point theorem. It is esily seen that $\mathscr{M}$ maps the (Euclidean) ball $K(0, M)$ into itself for

$$M = \frac{\sqrt{2}\|N\|_E}{1+\sqrt{1-2\|N\|_E}} < \sqrt{2}\|N\|_E,$$

under our condition $2\|N\|_E < 1$. The contractivity of $\mathscr{M}$ follows from

$$\|\mathscr{M}(X) - \mathscr{M}(Y)\|_E \leq \frac{1}{2}\|\mathscr{P}((X+Y)J(X-Y)^* + (X-Y)J(X+Y)^*)\|_E$$

$$\leqslant \|\mathscr{P}\|_E\|X+Y\|_E\|X-Y\|_E \leqslant 2\|\mathscr{P}\|_E M\|X-Y\|_E \leqslant 2\|N\|_E\|X-Y\|_E.$$

Thus, the upper diagonal factor $I + \Gamma^*$ exists and is bounded by

$$\|\Gamma\|_E = \|X\|_E \leqslant M \leqslant \sqrt{2}\|N\|_E < 1. \qquad \square$$

**Theorem 7.** *Let $H = GJG^*$, $G = DB$ lower triangular, $D$ with non-increasing diagonals. Let H be perturbed into $H + \delta H = D(A + \delta A)D$ with*

$$2\|\delta A\| < 1\ .$$

*Then the perturbation $\delta\lambda_i$ of the eigenvalue $\lambda_i$ is bounded by* (6) *with*

$$\nu = \|B\|\|B^{-1}\|^3 \frac{\sqrt{2}\|\delta A\|_E}{1+\sqrt{1-2\|\delta A\|_E\|B^{-1}\|^2}}\ . \tag{50}$$

**Proof.** From the previous lemma, (43) and (44) it follows

$$H + \delta H = DB(I + \Gamma)J(I + \Gamma^*)B^*D,$$

and now our perturbation problem reduces to the perturbation of the matrix pair

$$BD^2B^*, \quad J$$

into

$$(I + \Gamma)BD^2B^*(I + \Gamma^*), \quad J$$

or, equivalently, of

$$\hat{B}D^2\hat{B}^*, \quad J$$

into

$$(I + \hat{\Gamma})\hat{B}\hat{B}^*(I + \hat{\Gamma}^*), \quad J,$$

where

$$\hat{B} = DBD^{-1}, \quad \hat{\Gamma} = D\Gamma D^{-1}.$$

Now, by (44)

$$\|N\|_{\mathrm{E}} \leqslant \|\delta A\|_{\mathrm{E}}\|B^{-1}\|^2$$

and by the previous lemma and the fact that $|\hat{\Gamma}| \leqslant |\Gamma|$ (note that $\Gamma$ is lower triangular and the diagonal of $D$ non-increasing),

$$\|\hat{\Gamma}\|_{\mathrm{E}} \leqslant \|\Gamma\|_{\mathrm{E}} \leqslant \frac{\sqrt{2}\|\delta A\|_{\mathrm{E}}\|B^{-1}\|^2}{1 + \sqrt{1 - 2\|\delta A\|_{\mathrm{E}}\|B^{-1}\|^2}}.$$

Now (4) is applicable with $G = \hat{G}_0 D$, $\delta G = \hat{\Gamma}\hat{G}_0 D$ and $\nu = \|\hat{\Gamma}\| \leqslant \|\hat{\Gamma}\|_{\mathrm{E}}$. Hence (6) implies (50). □

Note that in (50) the expression $\|\delta A\|_{\mathrm{E}}$ can be substituted by $\varepsilon\|A\|_{\mathrm{E}}$ for perturbation (41) and by $n\varepsilon$ for perturbation (42).

Let us compare the new estimate with two earlier ones. The first is the case with $J = I$ ($H$ positive definite) where a very simple calculation in the spirit of [3] [13] gives $\nu = \|\delta A\|\|A^{-1}\|$ such that our estimate has essentially an extra factor $\|B\|\|B^{-1}\|$. Another related estimate is the one for s.d.d. matrices from [1]. There perturbation (3) implies

$$1 - \frac{n\varepsilon}{1 - \|N\|} \leqslant \frac{\lambda_i + \delta\lambda_i}{\lambda_i} \leqslant 1 - \frac{n\varepsilon}{1 - \|N\|},$$

where $N$ is the off-diagonal part of $A$. This estimate is not strictly comparable with ours but it has about the same force: for $\|N\|$ very small this yields $|\delta\lambda_i/\lambda_i| \leqslant n\varepsilon$ whereas ours yields $|\delta\lambda_i/\lambda_i| \leqslant \sqrt{2}n\varepsilon$.

On the other hand, our estimate covers much more than just s.d.d. matrices. Take an example:

$$A = \begin{bmatrix} 1 & 0 & z \\ 0 & 1 & z \\ z & z & 1 \end{bmatrix}, \quad z \text{ real.}$$

Then

$$B = \begin{bmatrix} 1 & & \\ & 1 & \\ z & z & \sqrt{1-2z^2} \end{bmatrix}, \quad J = I$$

and for $z < 1/\sqrt{2}$ $A$ is positive definite and also s.d.d. At the boundary $z = 1/\sqrt{2}$ all existing estimates necessarily become void.

Now change $A_{33}$ into $-1$. Then

$$B = \begin{bmatrix} 1 & & \\ & 1 & \\ z & z & \sqrt{1+z^2} \end{bmatrix}, \quad J = \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix},$$

while here the s.d.d – based estimate stops at the singularity $z = 1/\sqrt{2}$, the new estimate (50) yields useful bounds, except when $z$ itself is extremly large. This is seen from

$$B^{-1} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & (1+2z^2)^{-1/2} \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ -z & -z & 1 \end{bmatrix}.$$

It appears that taking a positive definite matrix and changing the sign of one of its diagonal blocks makes the matrix better behaved (a full quantitative formulation of this phenomenon is still wanted).

This suggests one to consider the class of Hermitian matrices which – up to a simultaneous permutation of rows and columns – has the form

$$H = \begin{bmatrix} \hat{H}_{11} & \hat{H}_{12} \\ \hat{H}_{12}^* & -\hat{H}_{22} \end{bmatrix}, \quad \hat{H}_{11}, \hat{H}_{22} \text{ positive definite.} \tag{51}$$

Such matrices are called *quasidefinite*. The set of quasidefinites is obviously scaling invariant. Another remarkable property of these matrices is that they always allow decomposition (36) with $G$ lower triangular. Moreover, the diagonals of $H$ and $J$ have the same signs (cf. [14,6]).

We will now derive the eigenvalue bounds for an elementwise perturbed quasidefinite matrix $H$.

**Theorem 8.** *Let $H = DAD$ be quasidefinite and let $A$ be partitionied according to* (51). *Then the bound* (50) *in Theorem* 7 *holds with*

$$\|B\|_{\mathrm{E}} \leqslant \sqrt{n}\max(\|\hat{A}_{11} + \hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{12}^*\|, \|\hat{A}_{22} + \hat{A}_{12}^*\hat{A}_{11}^{-1}\hat{A}_{12}\|),$$
$$\|B^{-1}\|_{\mathrm{E}} \leqslant \sqrt{n}\max(\|\hat{A}_{11}^{-1}\|, \|\hat{A}_{22}^{-1}\|).$$

**Proof.** Let $H$, $A$, $D$, $B$ be as in (36)– (39). Then

$$A = P\begin{bmatrix}\hat{A}_{11} & 0\\ 0 & -\hat{A}_{22}\end{bmatrix}P^{\mathrm{T}} + P\begin{bmatrix}0 & \hat{A}_{12}\\ \hat{A}_{12}^{*} & 0\end{bmatrix}P^{\mathrm{T}} = BJB^{*}$$

with $\hat{A}_{11}$, $\hat{A}_{22}$ positive definite, $B$ lower triangular, $P$ a permutation and $J = \operatorname{sign}(\operatorname{diag}(H_{11}, \ldots, H_{nn}))$. Then

$$P^{\mathrm{T}}JP = \begin{bmatrix}I & 0\\ 0 & -I\end{bmatrix}$$

and

$$JA = T + S = L\Delta M^{*} \tag{52}$$

with

$$T = P\begin{bmatrix}\hat{A}_{11} & 0\\ 0 & \hat{A}_{22}\end{bmatrix}P^{\mathrm{T}}, \quad \text{positive definite,} \tag{53}$$

$$S = P\begin{bmatrix}0 & \hat{A}_{12}\\ -\hat{A}_{12}^{*} & 0\end{bmatrix}P^{\mathrm{T}}, \quad \text{skew-Hermitian,} \tag{54}$$

$L$, $M$ lower triangular with unit diagonal and $\Delta$ diagonal with positive diagonal elements, actually,

$$B^{*} = \Delta^{1/2}M^{*}, \quad B = JLJ\Delta^{1/2}.$$

Now our considerations will closely follow the proof of the main theorem in Section 2 of [8]. In contrast to [8] our matrices may be complex, but their structure (53) and (54) allows the basic relation (52).[4] As in [8] we rewrite (52) as

$$\Delta M^{*}L^{-*} = L^{-1}CC^{*}L^{-*} + L^{-1}SL^{-*}$$

with $T = CC^{*}$. Hence

$$\Delta_{ii} = (\Delta M^{*}L^{-*})_{ii} = (L^{-1}CC^{*}L^{-*})_{ii} + (L^{-1}SL^{-*})_{ii}. \tag{55}$$

Although the skew-Hermitian matrix $L^{-1}SL^{*}$ may be non-real its diagonal must vanish (in the real case this is trivial). Indeed, $(L^{-1}SL^{-*})_{ii}$ is purely imaginary, whereas the other two terms in (55) are real. So, $(L^{-1}SL^{-*})_{ii} = 0$. We obtain

$$\Delta_{ii} = \|CL^{-*}e_i\|^2$$

or

$$\|CL^{-*}\Delta^{-1/2}e_i\| = 1$$

[4] Matrices $JA$ would be called non-Hermitian positive definite in the terminology of [8] which treats real matrices.

and

$$\|C^* L^{-*} \varDelta^{-1/2}\|_{\mathrm{E}} = \sqrt{n}. \tag{56}$$

Similarly,

$$\|C^* M^{-*} \varDelta^{-1/2} e_i\| = 1$$

and

$$\|C^* M^{-*} \varDelta^{-1/2}\|_{\mathrm{E}} = \sqrt{n}. \tag{57}$$

We do the same for $(JA)^{-1}$:

$$(JA)^{-1} = M^{-*} \varDelta^{-1} L^{-1} = C^{-*}(I + \hat{S})^{-1} C^{-1}$$

with $\hat{S} = C^{-1} S C^{-*}$ skew-Hermitian and

$$(I + \hat{S})^{-1} = (I - \hat{S})(I - \hat{S}^2)^{-1}$$

with $-\hat{S}^2$ positive definite. Thus,

$$\varDelta L^{-1} M = M^* C^{-*}(I - \hat{S}^2)^{-1} C^{-1} M - M^* C^{-*} \hat{S}(I - \hat{S}^2)^{-1} C^{-1} M.$$

Here again the rightmost term is skew-Hermitian with vanishing diagonal elements and

$$\frac{1}{\varDelta_{ii}} = \|(I - \hat{S}^2)^{-1/2} C^{-1} M e_i\|^2$$

or

$$\|(I - \hat{S}^2)^{-1/2} C^{-1} M \varDelta^{1/2} e_i\| = 1$$

and hence

$$\|(I - \hat{S}^2)^{-1/2} C^{-1} M \varDelta^{1/2}\|_{\mathrm{E}} = \sqrt{n}. \tag{58}$$

Similarly,

$$\|(I - \hat{S}^2)^{-1/2} C^{-1} L \varDelta^{1/2} e_i\| = 1$$

and

$$\|(I - \hat{S}^2)^{-1/2} C^{-1} L \varDelta^{1/2}\|_{\mathrm{E}} = \sqrt{n}. \tag{59}$$

We need the following norms:

$$\begin{aligned}
\|B\|_{\mathrm{E}} &= \|L \varDelta^{1/2}\|_{\mathrm{E}} \leqslant \sqrt{n}\|C(I - \hat{S}^2)^{1/2}\| \\
&= \sqrt{n}\|T - ST^{-1}S\| = \sqrt{n}\|C + SC^{-*}\|, \\
\|B^{-1}\|_{\mathrm{E}} &= \|L^{-*} \varDelta^{1/2}\|_{\mathrm{E}} \leqslant \sqrt{n}\|C^{-1}\|.
\end{aligned}$$

Since the norm is permutation invariant,

$$\begin{aligned}
&\|T - ST^{-1}S\| \\
&\quad = \left\| \begin{bmatrix} \hat{A}_{11} + \hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{12}^* & 0 \\ 0 & \hat{A}_{22} + \hat{A}_{12}^*\hat{A}_{11}^{-1}\hat{A}_{12} \end{bmatrix} \right\| \\
&\quad = \max(\|\hat{A}_{11} + \hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{12}^*\|, \|\hat{A}_{22} + \hat{A}_{12}^*\hat{A}_{11}^{-1}\hat{A}_{12}\|)
\end{aligned}$$

and also

$$\|C^{-1}\| = \max(\|\hat{A}_{11}^{-1}\|, \|\hat{A}_{22}^{-1}\|).$$

This gives the assertion. $\square$

The foregoing result essentially enlarges our knowledge on the class of well-behaved indefinite Hermitian matrices. While positive definite matrices are completely classified in this respect, the indefinite case appeared as more difficult (see [3,16]). Moreover, our result shows that a quasidefinite matrix behaves, in a sense, better than a positive definite one. While the latter needs a reasonable norm of the full scaled matrix inverse $A^{-1}$, the former needs the same only for the diagonal blocks of $A$ – the off-block diagonal of $A$ should just not be too large. So, the new estimates may carry improvements even in the case of an s.d.d. matrix: not all off-diagonals are equally dangerous. The only case, where our estimates carry no improvement at all, are positive definite matrices.

For $H$ tridiagonal the decomposition $H - \lambda I = GJG^{\mathrm{T}}$ with $G$ bidiagonal has been deeply studied in the recent paper [10] showing that $G$ may be reliably used for accurate eigensolution in spite of the absence of pivoting.

Finally, let us mention some open problems, connected with our present results. A drawback of our last theorem is that it gives estimates in terms of the Euclidian norm, whereas direct spectral norm estimates would be more desirable. They would also allow a more qualitative comparison with the positive definite case treated in [3]. This would urge us to improve the technique of [8] correspondingly – a task to be made in the future. Another subject for future work is to construct an algorithm for accurate computing of the eigensolution of such matrices. A general method was recommended in [15,11] and analysed in [12]; it begins by the universal block indefinite symmetric decomposition based on [2] with complete pivoting (this algorithm was analysed in [12]) and continues by a one-sided hyperbolic Jacobi algorithm on the so-obtained factor. Now, our analysis suggests that for quasidefinites a simple $LDL^*$ decomposition with previous sorting of the diagonal should be enough. The comparison of the two is in order.

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