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A Certain Fractional Derivative Operator and Its Applications to a New Class of Analytic and Multivalent Functions with Negative Coefficients

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Making use of a certain operator of fractional derivatives, a new subclass $\mathcal{T}_p(\alpha, \beta, \lambda)$ of analytic and p-valent functions with negative coefficients is introduced and studied here rather systematically. Coefficient estimates, distortion theorems, and various other interesting and useful properties of this class of functions are given; some of these properties involve, for example, linear combinations and modified Hadamard products of several functions belonging to the class introduced here. © 1992 Academic Press, Inc.

1. INTRODUCTION AND DEFINITIONS

Let $\mathcal{S}_p$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in \mathbb{N} = \{1, 2, 3, ...\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disk $\mathcal{U} = \{z: |z| < 1\}$. Also let $\mathcal{T}_p$ denote the subclass of $\mathcal{S}_p$ consisting of analytic and p-valent functions which can be expressed in the form:

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (a_{p+n} \geqq 0;\ p \in \mathbb{N}). \tag{1.2}$$

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Following Owa [5], we say that a function $f(z) \in \mathscr{T}_p$ is in the class $\mathscr{T}_p(\alpha, \beta)$ if and only if

$$\left| \frac{p^{-1}z^{1-p}f'(z) - 1}{p^{-1}z^{1-p}f'(z) + 1 - 2\alpha} \right| < \beta \qquad (z \in \mathscr{U};\ 0 \leqq \alpha < 1;\ 0 < \beta \leqq 1). \tag{1.3}$$

The object of the present paper is to investigate a new class $\mathscr{T}_p(\alpha, \beta, \lambda)$ of analytic and p-valent functions $f(z)$ belonging to the class $\mathscr{T}_p$ and satisfying the *additional* condition:

$$\left| \frac{\Omega_z^{(\lambda, p)}f(z) - 1}{\Omega_z^{(\lambda, p)}f(z) + 1 - 2\alpha} \right| < \beta \qquad (z \in \mathscr{U};\ 0 \leqq \alpha < 1;\ 0 < \beta \leqq 1;\ 0 \leqq \lambda \leqq 1), \tag{1.4}$$

where, for convenience,

$$\Omega_z^{(\lambda, p)}f(z) = \frac{\Gamma(1 + p - \lambda)}{\Gamma(1 + p)} z^{\lambda - p} D_z^{\lambda} f(z) \tag{1.5}$$

in terms of the *fractional derivative operator* D_z^{λ} of order λ, defined below, with

$$D_z^0 f(z) = f(z) \qquad \text{and} \qquad D_z^1 f(z) = f'(z). \tag{1.6}$$

In view of the second relationship in (1.6), we find from (1.5) that

$$\Omega_z^{(1, p)}f(z) = p^{-1}z^{1-p}f'(z). \tag{1.7}$$

Thus the condition (1.4) reduces, when $\lambda = 1$, to the inequality (1.3), and we have

$$\mathscr{T}_p(\alpha, \beta, 1) = \mathscr{T}_p(\alpha, \beta), \tag{1.8}$$

where $\mathscr{T}_p(\alpha, \beta)$ is precisely the subclass of analytic and p-valent functions studied by Owa [5].

Furthermore, by specializing the parameters λ, α, β, and p, we obtain the following subclasses studied by various other authors:

(i) $\mathscr{T}_1(\alpha, \beta, \lambda) = \mathscr{P}_{\lambda}^*(\alpha, \beta)$ (Srivastava and Owa [11]);

(ii) $\mathscr{T}_1(\alpha, \beta, 1) = \mathscr{P}^*(\alpha, \beta)$ (Gupta and Jain [2]);

(iii) $\mathscr{T}_p(\alpha, \beta, 0) = \mathscr{T}_p^*(\alpha, \beta)$,

where $\mathscr{T}_p^*(\alpha, \beta)$ represents the class of functions $f(z) \in \mathscr{T}_p$ satisfying the condition:

$$\left| \frac{z^{-p}f(z) - 1}{z^{-p}f(z) + 1 - 2\alpha} \right| < \beta \qquad (z \in \mathscr{U};\ 0 \leqq \alpha < 1;\ 0 < \beta \leqq 1). \tag{1.9}$$

Various operators of *fractional calculus* (that is, *fractional integral* and *fractional derivative*) have been studied in the literature rather extensively (cf., e.g., [1, Chap. 13; 3; 6–8; 9, p. 28 *et seq.*; 12]). We find it to be convenient to restrict ourselves to the following definitions used recently by Owa [4] (and by Srivastava and Owa [10]).

DEFINITION 1 (Fractional Integral Operator). The *fractional integral of order* λ is defined, for a function $f(z)$, by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}}\, d\zeta \qquad (\lambda > 0), \tag{1.10}$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta>0$.

DEFINITION 2 (Fractional Derivative Operator). The *fractional derivative of order* λ is defined, for a function $f(z)$, by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}}\, d\zeta \qquad (0 \leqq \lambda < 1), \tag{1.11}$$

where $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed, as in Definition 1.

DEFINITION 3 (Extended Fractional Derivative Operator). Under the hypotheses of Definition 2, the *fractional derivative of order* $n+\lambda$ is defined, for a function $f(z)$, by

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z) \qquad (0 \leqq \lambda < 1;\ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}), \tag{1.12}$$

where, as also in (1.1) and (1.2), $\mathbb{N}$ denotes the set of natural numbers.

For functions belonging to the general class $\mathcal{T}_p(\alpha, \beta, \lambda)$, we prove a number of sharp results including, for example, coefficient and distortion theorems, and theorems involving modified Hadamard products.

2. A THEOREM ON COEFFICIENT BOUNDS

We begin by proving

THEOREM 1. *A function* $f(z)$ *defined by* (1.2) *is in the class* $\mathcal{T}_p(\alpha, \beta, \lambda)$ *if and only if*

$$\sum_{n=1}^{\infty} \frac{\Gamma(n+1+p)\, \Gamma(1+p-\lambda)}{\Gamma(1+p)\, \Gamma(n+1+p-\lambda)} (1+\beta)\, a_{n+p} \leqq 2\beta(1-\alpha). \tag{2.1}$$

The condition (2.1) *is sharp.*

Proof. Assume that the inequality (2.1) holds true and let $|z|=1$. Then we obtain

$$\begin{aligned}
&|\Omega_z^{(\lambda,p)}f(z)-1|-\beta|\Omega_z^{(\lambda,p)}f(z)+1-2\alpha| \\
&\quad=\left|-\sum_{n=1}^{\infty}\frac{\Gamma(n+1+p)\,\Gamma(1+p-\lambda)}{\Gamma(1+p)\,\Gamma(n+1+p-\lambda)}a_{n+p}z^n\right| \\
&\quad\quad-\beta\left|2(1-\alpha)-\sum_{n=1}^{\infty}\frac{\Gamma(n+1+p)\,\Gamma(1+p-\lambda)}{\Gamma(1+p)\,\Gamma(n+1+p-\lambda)}a_{n+p}z^n\right| \\
&\quad\leqq\sum_{n=1}^{\infty}\frac{\Gamma(n+1+p)\,\Gamma(1+p-\lambda)}{\Gamma(1+p)\,\Gamma(n+1+p-\lambda)}(1+\beta)\,a_{n+p}-2\beta(1-\alpha) \\
&\quad\leqq 0,
\end{aligned}$$

by our hypothesis. Hence, by the maximum modulus theorem, we have

$$f(z)\in\mathscr{T}_p(\alpha,\beta,\lambda).$$

To prove the converse, assume that $f(z)$ is defined by (1.2) and is in the class $\mathscr{T}_p(\alpha,\beta,\lambda)$, so that the condition (1.4) readily yields

$$\begin{aligned}
\left|\frac{\Omega_z^{(\lambda,p)}f(z)-1}{\Omega_z^{(\lambda,p)}f(z)+1-2\alpha}\right|&=\left|\sum_{n=1}^{\infty}\frac{\Gamma(n+1+p)\,\Gamma(1+p-\lambda)}{\Gamma(1+p)\,\Gamma(n+1+p-\lambda)}a_{n+p}z^n\right| \\
&\quad\cdot\left|2(1-\alpha)-\sum_{n=1}^{\infty}\frac{\Gamma(n+1+p)\,\Gamma(1+p-\lambda)}{\Gamma(1+p)\,\Gamma(n+1+p-\lambda)}a_{n+p}z^n\right|^{-1} \\
&<\beta\qquad(z\in\mathscr{U}).
\end{aligned}\tag{2.2}$$

Since $|\text{Re}(z)|\leqq|z|$ for any z, we find from (2.2) that

$$\begin{aligned}
\text{Re}\Bigg\{&\left[\sum_{n=1}^{\infty}\frac{\Gamma(n+1+p)\,\Gamma(1+p-\lambda)}{\Gamma(1+p)\,\Gamma(n+1+p-\lambda)}a_{n+p}z^n\right] \\
&\cdot\left[2(1-\alpha)-\sum_{n=1}^{\infty}\frac{\Gamma(n+1+p)\,\Gamma(1+p-\lambda)}{\Gamma(1+p)\,\Gamma(n+1+p-\lambda)}a_{n+p}z^n\right]^{-1}\Bigg\}<\beta.
\end{aligned}\tag{2.3}$$

Choose values of z on the real axis so that $\Omega_z^{(\lambda,p)}f(z)$ is real. Upon clearing the denominator in (2.3) and letting $z\to 1-$ through real values, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{\Gamma(n+1+p)\,\Gamma(1+p-\lambda)}{\Gamma(1+p)\,\Gamma(n+1+p-\lambda)}a_{n+p} \\
&\quad\leqq 2\beta(1-\alpha)-\beta\sum_{n=1}^{\infty}\frac{\Gamma(n+1+p)\,\Gamma(1+p-\lambda)}{\Gamma(1+p)\,\Gamma(n+1+p-\lambda)}a_{n+p},
\end{aligned}\tag{2.4}$$

which gives the desired assertion (2.1).

Finally, we note that the assertion (2.1) of Theorem 1 is sharp, the extremal function being

$$f(z) = z^p - \frac{2\beta(1-\alpha)\,\Gamma(1+p)\,\Gamma(n+1+p-\lambda)}{(1+\beta)\,\Gamma(n+1+p)\,\Gamma(1+p-\lambda)}\, z^{n+p}. \tag{2.5}$$

COROLLARY 1. *Let the function $f(z)$ defined by (1.2) belong to the class $\mathcal{T}_p(\alpha, \beta, \lambda)$.*
Then

$$a_{n+p} \leqq \frac{2\beta(1-\alpha)\,\Gamma(1+p)\,\Gamma(n+1+p-\lambda)}{(1+\beta)\,\Gamma(n+1+p)\,\Gamma(1+p-\lambda)} \tag{2.6}$$

for every integer $n \in \mathbb{N}$.

3. A DISTORTION THEOREM OF THE CLASS $\mathcal{T}_p(\alpha, \beta, \lambda)$

THEOREM 2. *Let the function $f(z)$ defined by (1.2) be in the class $\mathcal{T}_p(\alpha, \beta, \lambda)$.*
Then

$$|f(z)| \geqq |z|^p - \frac{2\beta(1-\alpha)(1+p-\lambda)}{(1+\beta)(1+p)}\, |z|^{p+1} \tag{3.1}$$

and

$$|f(z)| \leqq |z|^p + \frac{2\beta(1-\alpha)(1+p-\lambda)}{(1+\beta)(1+p)}\, |z|^{p+1} \tag{3.2}$$

for $z \in \mathcal{U}$.
Furthermore

$$|D_z^\lambda f(z)| \geqq \frac{\Gamma(1+p)}{\Gamma(1+p-\lambda)}\, |z|^{p-\lambda} - \frac{2\beta(1-\alpha)\,\Gamma(1+p)}{(1+\beta)\,\Gamma(1+p-\lambda)}\, |z|^{p+1-\lambda} \tag{3.3}$$

and

$$|D_z^\lambda f(z)| \leqq \frac{\Gamma(1+p)}{\Gamma(1+p-\lambda)}\, |z|^{p-\lambda} + \frac{2\beta(1-\alpha)\,\Gamma(1+p)}{(1+\beta)\,\Gamma(1+p-\lambda)}\, |z|^{p+1-\lambda} \tag{3.4}$$

whenever $z \in \mathcal{U}$.

Proof. Since $f(z) \in \mathcal{T}_p(\alpha, \beta, \lambda)$, in view of Theorem 1, we have

$$\begin{aligned} \frac{(1+p)(1+\beta)}{1+p-\lambda} \sum_{n=1}^{\infty} a_{n+p} &\leqq \sum_{n=1}^{\infty} \frac{\Gamma(n+1+p)\,\Gamma(1+p-\lambda)}{\Gamma(1+p)\,\Gamma(n+1+p-\lambda)} (1+\beta)\, a_{n+p} \\ &\leqq 2\beta(1-\alpha), \end{aligned} \tag{3.5}$$

which evidently yields

$$\sum_{n=1}^{\infty} a_{n+p} \leqq \frac{2\beta(1-\alpha)(1+p-\lambda)}{(1+\beta)(1+p)}. \tag{3.6}$$

Consequently, we obtain

$$\begin{aligned} |f(z)| &\geqq |z|^p - |z|^{p+1} \sum_{n=1}^{\infty} a_{n+p} \\ &\geqq |z|^p - \frac{2\beta(1-\alpha)(1+p-\lambda)}{(1+\beta)(1+p)} |z|^{p+1} \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} |f(z)| &\leqq |z|^p + |z|^{p+1} \sum_{n=1}^{\infty} a_{n+p} \\ &\leqq |z|^p + \frac{2\beta(1-\alpha)(1+p-\lambda)}{(1+\beta)(1+p)} |z|^{p+1}, \end{aligned} \tag{3.8}$$

which prove the assertions (3.1) and (3.2) of Theorem 2.

Next, by using the second inequality in (3.5), we observe that

$$\begin{aligned} &|z^p \Omega_z^{(\lambda,p)} f(z)| \\ &\geqq |z|^p - \sum_{n=1}^{\infty} \frac{\Gamma(n+1+p)\,\Gamma(1+p-\lambda)}{\Gamma(1+p)\,\Gamma(n+1+p-\lambda)} a_{n+p} |z|^{n+p} \\ &\geqq |z|^p - |z|^{p+1} \sum_{n=1}^{\infty} \frac{\Gamma(n+1+p)\,\Gamma(1+p-\lambda)}{\Gamma(1+p)\,\Gamma(n+1+p-\lambda)} a_{n+p} \\ &\geqq |z|^p - \frac{2\beta(1-\alpha)}{1+\beta} |z|^{p+1} \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} &|z^p \Omega_z^{(\lambda,p)} f(z)| \\ &\leqq |z|^p + \sum_{n=1}^{\infty} \frac{\Gamma(n+1+p)\,\Gamma(1+p-\lambda)}{\Gamma(1+p)\,\Gamma(n+1+p-\lambda)} a_{n+p} |z|^{n+p} \\ &\leqq |z|^p + |z|^{p+1} \sum_{n=1}^{\infty} \frac{\Gamma(n+1+p)\,\Gamma(1+p-\lambda)}{\Gamma(1+p)\,\Gamma(n+1+p-\lambda)} a_{n+p} \\ &\leqq |z|^p + \frac{2\beta(1-\alpha)}{1+\beta} |z|^{p+1}, \end{aligned} \tag{3.10}$$

which prove the assertions (3.3) and (3.4) of Theorem 2.

COROLLARY 2. *Under the hypotheses of Theorem* 2, $f(z)$ *is included in a disk with its center at the origin and radius r given by*

$$r = 1 + \frac{2\beta(1-\alpha)(1+p-\lambda)}{(1+\beta)(1+p)}, \tag{3.11}$$

and $D_z^{\lambda} f(z)$ *is included in a disk with its center at the origin and radius R given by*

$$R = \frac{\Gamma(1+p)}{\Gamma(1+p-\lambda)}\left\{1 + \frac{2\beta(1-\alpha)}{1+\beta}\right\}. \tag{3.12}$$

4. FURTHER PROPERTIES OF THE CLASS $\mathcal{T}_p(\alpha, \beta, \lambda)$

The proof of each of the following results in this section runs parallel to that of the corresponding assertion made by Srivastava and Owa [11] in the special case $p = 1$, and we omit the details involved.

THEOREM 3. *Let* $0 \leqq \lambda \leqq 1$, $0 \leqq \alpha < 1$, *and* $0 < \beta \leqq 1$. *Then*

$$\mathcal{T}_p(\alpha, \beta, \lambda) = \mathcal{T}_p\left(\frac{1-\beta+2\alpha\beta}{1+\beta}, 1, \lambda\right). \tag{4.1}$$

More generally, if $0 \leqq \alpha' < 1$ *and* $0 < \beta' \leqq 1$, *then*

$$\mathcal{T}_p(\alpha, \beta, \lambda) = \mathcal{T}_p(\alpha', \beta', \lambda) \tag{4.2}$$

if and only if

$$\frac{\beta(1-\alpha)}{1+\beta} = \frac{\beta'(1-\alpha')}{1+\beta'}. \tag{4.3}$$

THEOREM 4. *Let* $0 \leqq \lambda \leqq 1$, $0 \leqq \alpha_1 \leqq \alpha_2 < 1$, *and* $0 < \beta \leqq 1$. *Then*

$$\mathcal{T}_p(\alpha_1, \beta, \lambda) \supset \mathcal{T}_p(\alpha_2, \beta, \lambda). \tag{4.4}$$

THEOREM 5. *Let* $0 \leqq \lambda \leqq 1$, $0 \leqq \alpha < 1$, *and* $0 < \beta_1 \leqq \beta_2 \leqq 1$. *Then*

$$\mathcal{T}_p(\alpha, \beta_1, \lambda) \subset \mathcal{T}_p(\alpha, \beta_2, \lambda). \tag{4.5}$$

COROLLARY 3. *Let* $0 \leqq \lambda \leqq 1$, $0 \leqq \alpha_1 \leqq \alpha_2 < 1$, *and* $0 < \beta_1 \leqq \beta_2 \leqq 1$. *Then*

$$\mathcal{T}_p(\alpha_2, \beta_1, \lambda) \subset \mathcal{T}_p(\alpha_1, \beta_1, \lambda) \subset \mathcal{T}_p(\alpha_1, \beta_2, \lambda). \tag{4.6}$$

5. Theorems Involving Modified Hadamard Products

Let $f(z)$ be defined by (1.2), and let

$$g(z)=z^{p}-\sum_{n=1}^{\infty} b_{n+p}z^{n+p} \qquad (b_{n+p}\geqq 0;\ p\in\mathbb{N}). \tag{5.1}$$

For the *modified* Hadamard product of $f(z)$ and $g(z)$ defined here by

$$f\star g(z)=z^{p}-\sum_{n=1}^{\infty} a_{n+p}b_{n+p}z^{n+p}, \tag{5.2}$$

we first prove

THEOREM 6. *Let the functions* $f_j(z)$ $(j=1, ..., m)$ *defined by*

$$f_j(z)=z^{p}-\sum_{n=1}^{\infty} c_{n+p,j}z^{n+p} \qquad (c_{n+p,j}\geqq 0;\ j=1, ..., m;\ p\in\mathbb{N}) \tag{5.3}$$

be in the classes $\mathscr{T}_p(\alpha_j, \beta_j, \lambda)$ $(j=1, ..., m)$, *respectively. Also let*

$$\frac{2\lambda}{1+p}+\min_{1\leqq j\leqq m}\{\beta_j\}\geqq 1. \tag{5.4}$$

Then

$$f_1\star\cdots\star f_m(z)\in\mathscr{T}_p\left(\prod_{j=1}^{m}\alpha_j, \prod_{j=1}^{m}\beta_j, \lambda\right). \tag{5.5}$$

Proof. Since $f_j(z)\in\mathscr{T}_p(\alpha_j, \beta_j, \lambda)$ $(j=1, ..., m)$, by using Theorem 1, we have

$$\sum_{n=1}^{\infty}\frac{\Gamma(n+1+p)\,\Gamma(1+p-\lambda)}{\Gamma(1+p)\,\Gamma(n+1+p-\lambda)}(1+\beta_j)\,c_{n+p,j}\leqq 2\beta_j(1-\alpha_j) \tag{5.6}$$

and

$$\sum_{n=1}^{\infty} c_{n+p,j}\leqq\frac{2\beta_j(1-\alpha_j)(1+p-\lambda)}{(1+\beta_j)(1+p)} \tag{5.7}$$

for each $j=1, ..., m$.

Using (5.6) for any j_0 and (5.7) for the rest, we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{\Gamma(n+1+p)\,\Gamma(1+p-\lambda)}{\Gamma(1+p)\,\Gamma(n+1+p-\lambda)}\left(1+\prod_{j=1}^{m}\beta_j\right)\prod_{j=1}^{m}c_{n+p,j}\\
&\qquad\leqq\frac{2[2(1+p-\lambda)/(1+p)]^{m-1}\prod_{j=1}^{m}\beta_j(1-\alpha_j)}{\prod_{j=1,\,j\neq j_0}^{m}(1+\beta_j)}\\
&\qquad\leqq\frac{2[2-2\lambda/(1+p)]^{m-1}\prod_{j=1}^{m}\beta_j(1-\prod_{j=1}^{m}\alpha_j)}{(1+\min_{1\leqq j\leqq m}\{\beta_j\})^{m-1}}\\
&\qquad\leqq 2\prod_{j=1}^{m}\beta_j\left(1-\prod_{j=1}^{m}\alpha_j\right),
\end{aligned}$$

since

$$0<\frac{2-2\lambda/(1+p)}{1+\min_{1\leqq j\leqq m}\{\beta_j\}}\leqq 1. \tag{5.8}$$

Consequently, we have the assertion (5.5) with the aid of Theorem 1.

For $\alpha_j=\alpha$ and $\beta_j=\beta$ $(j=1, ..., m)$, Theorem 6 yields

COROLLARY 4. *Let each of the functions* $f_j(z)$ $(j=1, ..., m)$, *defined by* (5.3), *be in the same class* $\mathcal{T}_p(\alpha, \beta, \lambda)$. *Also let*

$$\frac{2\lambda}{1+p}+\beta\geqq 1.$$

Then

$$f_1 \star \cdots \star f_m(z)\in\mathcal{T}_p(\alpha^m, \beta^m, \lambda). \tag{5.9}$$

Next we prove

THEOREM 7. *Let the functions* $f(z)$ *defined by* (1.2) *and* $g(z)$ *defined by* (5.1) *be in the classes* $\mathcal{T}_p(\alpha_1, \beta_1, \lambda)$ *and* $\mathcal{T}_p(\alpha_2, \beta_2, \lambda)$, *respectively.*

Then the modified Hadamard product $f\star g(z)$ *belongs to the class* $\mathcal{T}_p(2\alpha-\alpha^2, \beta, \lambda)$, *where*

$$\alpha=\min\{\alpha_1, \alpha_2\} \qquad \textit{and} \qquad \beta=\max\{\beta_1, \beta_2\}. \tag{5.10}$$

Proof. Since

$$f(z)\in\mathcal{T}_p(\alpha_1, \beta_1, \lambda) \qquad \text{and} \qquad g(z)\in\mathcal{T}_p(\alpha_2, \beta_2, \lambda),$$

in view of Theorem 1, we have

$$\sum_{n=1}^{\infty}\frac{\Gamma(n+1+p)\,\Gamma(1+p-\lambda)}{\Gamma(1+p)\,\Gamma(n+1+p-\lambda)}(1+\beta)\,a_{n+p}$$
$$\leqq\frac{2\beta(1-\alpha)\,\beta_0(1-\alpha)[2-2\lambda/(1+p)]}{1+\beta_0}\leqq 2\beta\{1-\alpha(2-\alpha)\}, \tag{5.11}$$

where $\beta_0=\min\{\beta_1, \beta_2\}$. Moreover,

$$0\leqq\alpha(2-\alpha)<1 \qquad (0\leqq\alpha<1).$$

Hence, by Theorem 1, the modified Hadamard product $f\star g(z)$ is in the class

$$\mathcal{T}_p(2\alpha-\alpha^2, \beta, \lambda),$$

with α and β given by (5.10).

COROLLARY 5. *Under the hypotheses of Theorem 7, the modified Hadamard product* $f \star g(z)$ *belongs to the class* $\mathscr{T}_p(\alpha, \beta, \lambda)$.

Proof. In view of Theorem 4, we have

$$\mathscr{T}_p(\alpha, \beta, \lambda) \supset \mathscr{T}_p(2\alpha - \alpha^2, \beta, \lambda), \tag{5.12}$$

which, in conjunction with Theorem 7, shows that

$$f \star g(z) \in \mathscr{T}_p(\alpha, \beta, \lambda),$$

where α and β are given by (5.10).

We conclude this section by proving an interesting theorem involving the modified Hadamard product defined by (5.2) together with extremal functions.

THEOREM 8. *Let the functions* $f_j(z)$ $(j = 1, 2)$ *defined by* (5.3) *be in the class* $\mathscr{T}_p(\alpha, \beta, \lambda)$.
Then

$$f_1 \star f_2(z) \in \mathscr{T}_p(\gamma(p, \alpha, \beta, \lambda), \beta, \lambda), \tag{5.13}$$

where

$$\gamma(p, \alpha, \beta, \lambda) = 1 - \frac{2\beta(1-\alpha)^2 (1+p-\lambda)}{(1+\beta)(1+p)}. \tag{5.14}$$

The result is sharp.

Proof. It suffices to prove that

$$\sum_{n=1}^{\infty} \frac{(1+\beta)\,\Gamma(n+1+p)\,\Gamma(1+p-\lambda)}{2\beta(1-\gamma)\,\Gamma(1+p)\,\Gamma(n+1+p-\lambda)} c_{n,1} c_{n,2} \leqq 1 \tag{5.15}$$

for $\gamma \leqq \gamma(p, \alpha, \beta, \lambda)$. By virtue of the Cauchy–Schwarz inequality, it follows from (2.1) that

$$\sum_{n=1}^{\infty} \frac{(1+\beta)\,\Gamma(n+1+p)\,\Gamma(1+p-\lambda)}{2\beta(1-\alpha)\,\Gamma(1+p)\,\Gamma(n+1+p-\lambda)} \sqrt{c_{n,1} c_{n,2}} \leqq 1. \tag{5.16}$$

Thus we need to find the largest γ such that

$$\sum_{n=1}^{\infty} \frac{(1+\beta)\,\Gamma(n+1+p)\,\Gamma(1+p-\lambda)}{2\beta(1-\gamma)\,\Gamma(1+p)\,\Gamma(n+1+p-\lambda)} c_{n,1} c_{n,2}$$
$$\leqq \sum_{n=1}^{\infty} \frac{(1+\beta)\,\Gamma(n+1+p)\,\Gamma(1+p-\lambda)}{2\beta(1-\alpha)\,\Gamma(1+p)\,\Gamma(n+1+p-\lambda)} \sqrt{c_{n,1} c_{n,2}}, \tag{5.17}$$

or, equivalently, that

$$\sqrt{c_{n,1}c_{n,2}} \leqq \frac{1-\gamma}{1-\alpha} \qquad (n \in \mathbb{N}). \tag{5.18}$$

In view of (5.16), it is sufficient to find the largest γ such that

$$\frac{2\beta(1-\alpha)\,\Gamma(1+p)\,\Gamma(n+1+p-\lambda)}{(1+\beta)\,\Gamma(n+1+p)\,\Gamma(1+p-\lambda)} \leqq \frac{1-\gamma}{1-\alpha}. \tag{5.19}$$

The inequality (5.19) yields

$$\gamma \leqq 1 - \frac{2\beta(1-\alpha)^2}{1+\beta}\,\Psi(n) \qquad (n \in \mathbb{N}), \tag{5.20}$$

where

$$\Psi(n) = \frac{\Gamma(1+p)\,\Gamma(n+1+p-\lambda)}{\Gamma(n+1+p)\,\Gamma(1+p-\lambda)}. \tag{5.21}$$

Since $\Psi(n)$ defined by (5.21) is a decreasing function of n ($n \in \mathbb{N}$) for fixed λ, we have

$$\gamma \leqq \gamma(p, \alpha, \beta, \lambda) = 1 - \frac{2\beta(1-\alpha)^2\,\Gamma(1+p)\,\Gamma(2+p-\lambda)}{(1+\beta)\,\Gamma(2+p)\,\Gamma(1+p-\lambda)}, \tag{5.22}$$

that is,

$$\gamma \leqq \gamma(p, \alpha, \beta, \lambda) = 1 - \frac{2\beta(1-\alpha)^2\,(1+p-\lambda)}{(1+\beta)(1+p)}, \tag{5.23}$$

which evidently proves the assertion (5.13) under the constraint (5.14).

Finally, by taking the functions

$$f_j(z) = z^p - \frac{2\beta(1-\alpha)(1+p-\lambda)}{(1+\beta)(1+p)}\,z^{p+1} \qquad (j = 1, 2), \tag{5.24}$$

we can see that the result in Theorem 8 is sharp.

COROLLARY 6. *Let the functions* $f_j(z)$ $(j = 1, 2)$ *defined by* (5.3) *be in the class* $\mathcal{T}_p(\alpha, \beta, 1)$.

Then

$$f_1 \star f_2(z) \in \mathcal{T}_p(\nu(p, \alpha, \beta), \beta, 1), \tag{5.25}$$

where

$$\nu(p, \alpha, \beta) = 1 - \frac{2\beta(1-\alpha)^2\,p}{(1+\beta)(1+p)}. \tag{5.26}$$

The result is sharp for the functions

$$f_j(z) = z^p - \frac{2\beta(1-\alpha)^2 p}{(1+\beta)(1+p)} z^{p+1} \qquad (j = 1, 2). \tag{5.27}$$

Setting $\lambda = 0$ in Theorem 8, we have

COROLLARY 7. *Let the functions* $f_j(z)$ $(j = 1, 2)$ *defined by* (5.3) *be in the class* $\mathscr{T}_p(\alpha, \beta, 0)$.
Then

$$f_1 \star f_2(z) \in \mathscr{T}_p(\delta(\alpha, \beta), \beta, 0), \tag{5.28}$$

where

$$\delta(\alpha, \beta) = 1 - \frac{2\beta(1-\alpha)^2}{1+\beta}. \tag{5.29}$$

The result is sharp for the functions

$$f_j(z) = z^p - \frac{2\beta(1-\alpha)^2}{1+\beta} z^{p+1} \qquad (j = 1, 2). \tag{5.30}$$

6. LINEAR COMBINATION OF FUNCTIONS IN THE CLASS $\mathscr{T}_p(\alpha, \beta, \lambda)$

Finally, we prove

THEOREM 9. *Let each of the functions* $f_j(z)$ $(j = 1, ..., m)$ *defined by* (5.3) *be in the same class* $\mathscr{T}_p(\alpha, \beta, \lambda)$.
Then the function $h(z)$ *given by*

$$h(z) = \frac{1}{m} \sum_{j=1}^{m} f_j(z) \tag{6.1}$$

is also in the class $\mathscr{T}_p(\alpha, \beta, \lambda)$.

Proof. Substituting for $f_j(z)$ from (5.3) into the definition (6.1) of $h(z)$, we have the expression:

$$h(z) = z^p - \sum_{n=1}^{\infty} \left(\frac{1}{m} \sum_{j=1}^{m} c_{n+p,j} \right) z^{n+p}. \tag{6.2}$$

Since

$$f_j(z) \in \mathscr{T}_p(\alpha, \beta, \lambda) \qquad (j = 1, ..., m),$$

by using Theorem 1, we obtain

$$\sum_{n=1}^{\infty} \frac{\Gamma(n+1+p)\,\Gamma(1+p-\lambda)}{\Gamma(1+p)\,\Gamma(n+1+p-\lambda)} (1+\beta)\left(\frac{1}{m}\sum_{j=1}^{m} c_{n+p,j}\right) \leqq 2\beta(1-\alpha), \qquad (6.3)$$

which, in view of Theorem 1, yields Theorem 9.

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409 171 1-2