The number of spanning trees in directed circulant graphs with non-fixed jumps

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Abstract

Let $C_{pn}(a_1, a_2, \ldots, a_k, q_1 n, q_2 n, \ldots, q_m n)$ be a directed circulant graphs with $pn$ vertices and some non-fixed jumps, where $1 \leq a_1 \leq a_2 \leq \cdots \leq a_k (\leq n - 1), 1 \leq q_1 \leq q_2 \leq \cdots \leq q_m \leq p - 1$, are fixed integers, and an integer $n$ varies. In this paper, a formula, asymptotic behaviors and linear recurrence relations for the number of its spanning trees are obtained.

Keywords: Spanning tree; Directed circulant graph; Linear recurrence relation; Asymptotic behavior

1. Introduction

The number of spanning trees in a graph (network) is an important invariant, it is also an important measure of reliability of a network. The well-known matrix-tree theorem (see e.g., [7]) can be used to count the number of spanning trees for small graphs, but this method is not feasible for large graphs. Let $G$ be an undirected (resp. directed) graph, the number of its spanning trees (resp. out-trees) be denoted by $t(G)$. For some special classes of undirected graphs, explicit formulas for $t(G)$ have been obtained so far [2,3,5,6,8,9,11].

The circulant graphs are an important class of graphs, which can be used in the design of local area networks [1]. Let $1 \leq a_1 \leq a_2 \leq \cdots \leq a_k \leq n - 1$ be positive integers. An undirected (resp. directed) circulant graph $C_n(a_1, a_2, \ldots, a_k)$ is a regular undirected (resp. directed) graph whose set of vertices is $V = \{0, 1, \ldots, n - 1\}$ and whose set of edges (resp. arcs) is

$$E = \{(i, i + a_j \pmod{n})|i = 0, 1, \ldots, n - 1, j = 1, 2, \ldots, k\}.$$

For undirected circulant graphs $C_n(1, 2), C_n(1, 3)$ and $C_n(1, 4)$, the results on the numbers of their spanning trees can be found in [9]. For undirected $C_n(a_1, a_2, \ldots, a_k)$, where $a_k < n/2$, the ones can be found in [5,11]; for undirected $C_{2n}(a_1, a_2, \ldots, a_k, n)$, where $n$ varies, the ones were given in [6]; for directed $C_n(a_1, a_2, \ldots, a_k)$, where $a_k \leq n - 1$, the ones were given in [10]. In this paper, we consider directed circulant graphs with $pn$ vertices and some non-fixed jumps $C_{pn}(a_1, a_2, \ldots, a_k, q_1 n, q_2 n, \ldots, q_m n)$, where $1 \leq a_1 \leq a_2 \leq \cdots \leq a_k (\leq n - 1), 1 \leq q_1 \leq q_2 \leq \cdots \leq q_m \leq p - 1$, are fixed integers, and an integer $n$ varies. A formula, asymptotic behaviors and linear recurrence relations for the numbers of their spanning trees are obtained.

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Keywords: Spanning tree; Directed circulant graph; Linear recurrence relation; Asymptotic behavior

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In the following sections, we will consider only directed circulant graphs, and \( e_N \) stands for \( e^{2\pi \sqrt{-1}/N} \), where \( N \) is a positive integer. If \( A(n) \) and \( B(n) \) are integers depending on an integer \( n \) such that
\[
A(n) \to \infty, \quad B(n) \to \infty, \quad A(n)/B(n) \to 1 \quad (n \to \infty)
\]
then we write it by \( A(n) \sim B(n), n \to \infty \).

2. Some lemmas

In this section, some lemmas are given.

**Lemma 1** (see, e.g., Zhang and Yong [10]).
\[
t(C_n(a_1, a_2, \ldots, a_k)) = \prod_{r=1}^{n-1} \left( k - \sum_{i=1}^{k} e_n^{a_i r} \right).
\]

**Lemma 2** (Zhang and Yong [10]). If \((a_1, a_2, \ldots, a_k) = 1\), then
\[
t(C_n(a_1, a_2, \ldots, a_k)) \sim nk^n / \sum_{i=1}^{k} a_i, \quad n \to \infty.
\]

**Lemma 3.** Let \((a_1, a_2, \ldots, a_k) = d, (d, n) = 1, \) and \( a_i = b_i d, i = 1, 2, \ldots, k. \) Then \( C_n(a_1, a_2, \ldots, a_k) \) is isomorphic to \( C_n(b_1, b_2, \ldots, b_k) \), and
\[
t(C_n(a_1, a_2, \ldots, a_k)) = t(C_n(b_1, b_2, \ldots, b_k)) \sim dnk^n / \sum_{i=1}^{k} a_i, \quad n \to \infty.
\]

**Proof.** It is clear that \((b_1, b_2, \ldots, b_k) = 1.\) Let \( \psi(h) = hd \pmod{n} \), \( h \in \{0, 1, \ldots, n-1\}. \) It is easy to show that \( \psi \) is an isomorphism from \( C_n(b_1, b_2, \ldots, b_k) \) to \( C_n(a_1, a_2, \ldots, a_k). \) By Lemma 2, the lemma follows. \( \square \)

**Lemma 4.** Let \( 1 \leq h \leq p - 1, 1 \leq a_1 \leq a_2 \leq \cdots \leq a_k, 1 \leq q_1 \leq q_2 \leq \cdots \leq q_m \leq p - 1, \) and let
\[
f_h(x) = \sum_{i=1}^{k} x^{a_i} + \sum_{j=1}^{m} e_{p}^{q_j h} - k - m
\]
be a polynomial of degree \( a_k \) with a root \( \hat{\lambda}. \) If \((q_1, q_2, \ldots, q_m, p) = 1, \) then \( |\hat{\lambda}| > 1. \)

**Proof.** Let \((q_1, q_2, \ldots, q_m) = d.\) Then \((d, p) = 1\) and
\[
(q_1 h, q_2 h, \ldots, q_m h, p) = (dh, p) = (h, p) < p.
\]
Thus there exists \( j, 1 \leq j \leq m, \) such that \( q_j h \) is not divisible by \( p. \) It is clear that \( e_p^{q_j h} \neq 1 \) and \( \sum_{j=1}^{m} e_p^{q_j h} \neq m. \) Since \( \hat{\lambda} \) is a root of \( f_h(x), \) if \(|\hat{\lambda}| \leq 1, \) then
\[
k < \left| k + m - \sum_{j=1}^{m} e_p^{q_j h} \right| = \left| \sum_{i=1}^{k} \lambda^{a_i} \right| \leq \sum_{i=1}^{k} |\hat{\lambda}|^{a_i} \leq k,
\]
a contradiction. So \(|\hat{\lambda}| > 1. \) \( \square \)

The following two lemmas can be easily proved.

**Lemma 5.** If \((p, q) = 1, \) then
\[
\{qh \pmod{p} | h = 1, 2, \ldots, p - 1 \} = \{1, 2, \ldots, p - 1\}.
\]
Lemma 6. If $x \neq 1$, then
\[
\prod_{h=1}^{p-1} \left( x - \varepsilon_p^h \right) = \frac{x^p - 1}{x - 1}.
\]

Lemma 7 (See, e.g., Brualdi [4, p. 201]). Let $x_i \neq 0, i = 1, 2, \ldots, k$, let
\[
\sigma_1 = \sum_{i=1}^{k} x_i, \quad \sigma_2 = \sum_{i<j} x_ix_j, \ldots, \quad \sigma_k = \prod_{i=1}^{k} x_i,
\]
and let
\[
p(x) = \prod_{i=1}^{k} (x - x_i) = x^k - \sigma_1 x^{k-1} + \sigma_2 x^{k-2} - \cdots + (-1)^k \sigma_k.
\]
Assume that
\[
s_n = c_1 x_1^n + c_2 x_2^n + \cdots + c_k x_k^n, \quad n = 1, 2, \ldots,
\]
where $c_1, c_2, \ldots, c_k$, are any constants. Then $s_n$ satisfy the following linear recurrence relation of order $k$ with constant coefficients:
\[
s_n - \sigma_1 s_{n-1} + \sigma_2 s_{n-2} - \cdots + (-1)^k \sigma_k s_{n-k} = 0, \quad n > k.
\]
And $p(x)$ is said to be the characteristic polynomial for $s_n$, and $x_1, x_2, \ldots, x_k$, the characteristic roots for $s_n$.

3. A formula for $t(C_{pn}(a_1, a_2, \ldots, a_k, q_1n, q_2n, \ldots, q_mn))$

In this section, a formula for $t(C_{pn}(a_1, a_2, \ldots, a_k, q_1n, q_2n, \ldots, q_mn))$ is obtained.

Theorem 1. Let $1 \leq a_1 \leq a_2 \leq \cdots \leq a_k \leq p - 1$, $1 \leq q_1 \leq q_2 \leq \cdots \leq q_m \leq p - 1$, be integers. Then
\[
t(C_{pn}(a_1, a_2, \ldots, a_k, q_1n, q_2n, \ldots, q_mn))
\]
\[
= (-1)^{(a_k+1)(p-1)n} t(C_n(a_1, a_2, \ldots, a_k)) \prod_{h=1}^{p} \prod_{s=1}^{k} \left( \lambda_{h,s}^n - \varepsilon_p^h \right),
\]
where $\lambda_{h,s}, s = 1, 2, \ldots, a_k, 1 \leq h \leq p - 1$, are all the roots of $f_{h}(x)$ in Lemma 4.

Proof. Let $R = \{lp | l = 1, 2, \ldots, n - 1 \} \cup \{lp + h | h = 1, 2, \ldots, p - 1, l = 0, 1, \ldots, n - 1 \}$. By Lemma 1,
\[
t(C_{pn}(a_1, a_2, \ldots, a_k, q_1n, q_2n, \ldots, q_mn))
\]
\[
= \prod_{r=1}^{p} \left( k + m - \sum_{i=1}^{k} \varepsilon_{a_i r} \varepsilon_{rn} - \sum_{j=1}^{m} \varepsilon_{j n} \varepsilon_{mr} \right)
\]
\[
= \prod_{l=1}^{n} \left( k - \sum_{i=1}^{k} \varepsilon_{a_i l} \varepsilon_{ln} \right) \prod_{h=1}^{p} \prod_{l=0}^{n-1} \left( k + m - \sum_{j=1}^{m} \varepsilon_{j h} \varepsilon_{p n} - \sum_{i=1}^{k} \varepsilon_{i n} \varepsilon_{(lp+h)} \right)
\]
\[
= t(C_n(a_1, a_2, \ldots, a_k)) \prod_{h=1}^{p} \prod_{l=0}^{n-1} (-1)^{f_{h}(\varepsilon_{p n})^{lp+h}}.
\]
Since
\[ fh(x) = \prod_{s=1}^{a_k} (x - \lambda_{h,s}), \]
by Lemma 6, we have
\[ \prod_{l=0}^{n-1} (-1)^l f_h (\varepsilon_{p,n}^{l+p}) = \prod_{s=1}^{a_k} \prod_{l=0}^{n-1} \left( \varepsilon_{p,n}^{l+p} \lambda_{h,s} - \varepsilon_n^l \right) \]
\[ = (-1)^{(a_k+1)n} \prod_{s=1}^{a_k} \prod_{l=0}^{n-1} \left( \varepsilon_{p,n}^{l+p} \lambda_{h,s} - \varepsilon_n^l \right) \]
\[ = (-1)^{(a_k+1)n} \prod_{s=1}^{a_k} \left( \varepsilon_{p,n}^{l+p} \lambda_{h,s} - \varepsilon_n^l \right) \]
Hence the theorem follows.

Corollary 1. \( t(C_{pn}(1, qn)) = n \prod_{h=1}^{p-1} \left( 2 - \varepsilon_p^{q_h} \right)^n. \)

Proof. Since \( f_h(x) = x + \varepsilon_p^{q_h} - 2, \) by Theorem 1 and the fact that \( t(C_n(1)) = n, \) the corollary follows.

4. Asymptotic behaviors

In this section, asymptotic behaviors for \( t(C_{pn}(a_1, a_2, \ldots, a_k, q_1 n, q_2 n, \ldots, q_m n)) \) are considered.

Theorem 2. Let \( (a_1, a_2, \ldots, a_k) = d, (d, n) = 1 \) and \( (q_1, q_2, \ldots, q_m, p) = 1. \) Then
\[ t(C_{pn}(a_1, a_2, \ldots, a_k, q_1 n, q_2 n, \ldots, q_m n)) \sim \frac{dn}{\sum_{i=1}^{k} a_i} \left( k \prod_{h=1}^{p-1} \left( k + m - \sum_{j=1}^{m} \varepsilon_{p}^{q_j h} \right) \right)^n, \quad n \to \infty. \]

Proof. Let \( \lambda_{h,s}, s = 1, 2, \ldots, a_k, 1 \leq h \leq p - 1, \) be all the roots of \( f_h(x) \) in Lemma 4. By Lemma 4, \( |\lambda_{h,s}| > 1, \) thus, if \( n \to \infty, \) then
\[ \prod_{h=1}^{p-1} \prod_{s=1}^{a_k} \left( \varepsilon_{h,s}^{n} - \varepsilon_p^h \right) \sim \prod_{h=1}^{p-1} \prod_{s=1}^{a_k} \lambda_{h,s}^{n} \]
\[ = \left( \prod_{h=1}^{p-1} (-1)^{a_k} \left( \sum_{j=1}^{m} \varepsilon_{p}^{h j} \right) \right)^n \]
\[ = (-1)^{(a_k+1)(p-1)n} \left( \prod_{h=1}^{p-1} \left( k + m - \sum_{j=1}^{m} \varepsilon_{p}^{h j} \right) \right)^n. \]
By Theorem 1 and Lemma 3, the theorem follows.

Corollary 2. Let \( (a_1, a_2, \ldots, a_k) = d, (d, n) = 1 \) and \( (p, q) = 1. \) Then
\[ t(C_{pn}(a_1, a_2, \ldots, a_k, qn)) \sim \frac{dn}{\sum_{i=1}^{k} a_i} ((k + 1)^n - 1)^n, \quad n \to \infty. \]
Proof. By Lemmas 5 and 6,
\[ \prod_{h=1}^{p-1} (k + 1 - \epsilon_p^h) = \prod_{h=1}^{p-1} \left( k + 1 - \epsilon_p^h \right) = \frac{(k+1)^p - 1}{k}. \]
By Theorem 2, the corollary follows. \( \square \)

Corollary 3. Let \((a_1, a_2, \ldots, a_k, q_1, q_2, \ldots, q_m) = d, (d, n) = 1\). Then
\[ t(C_{pn}(a_1, a_2, \ldots, a_k, q_1 n, q_2 n, \ldots, q_m n)) \sim \frac{dn}{\sum_{i=1}^{k} a_i} \left( k + p \right)^{n-1}, \quad n \to \infty. \]

Proof. Let \((h, p) = r, where 1 \leq h \leq p - 1, and let h = qr, p = Pr, then (P, q) = 1\). It is easy to show that
\[ \{ iq (\text{mod } P) | i = 0, 1, \ldots, P - 1 \} = \{ 0, 1, \ldots, P - 1 \} \]
and
\[ \{ j | j = 0, 1, \ldots, p - 1 \} = \{ Ps + i | i = 0, 1, \ldots, P - 1, s = 0, 1, \ldots, r - 1 \}. \]
Hence
\[ \sum_{i=0}^{P-1} \epsilon_{P r}^{(P s + i) q r} = \sum_{i=0}^{P-1} \epsilon_{P r}^{i q} = \sum_{i=0}^{P-1} \epsilon_{p r}^{i} = 0 \]
and
\[ \sum_{j=1}^{p-1} \epsilon_{p r}^{j h} = -1 + \sum_{j=0}^{p-1} \epsilon_{p r}^{j h} = -1 + \sum_{s=0}^{r-1} \sum_{i=0}^{P-1} \epsilon_{P r}^{(P s + i) q r} = -1. \]
By Theorem 2, the corollary follows. \( \square \)

5. Linear recurrence relations

In this section, we will consider linear recurrence relations and give some examples.

Theorem 3. For \(n = 1, 2, \ldots\), let
\[ s_n := t(C_{pn}(a_1, a_2, \ldots, a_k, q_1 n, q_2 n, \ldots, q_m n)) / \left( (-1)^{(a_k+1)(p-1)n} t(C_n(a_1, a_2, \ldots, a_k)) \right). \]
Then \(s_n\) satisfy a linear recurrence relation of order \(2^{a_k(p-1)}\) with constant coefficients.

Proof. By Theorem 1, we have
\[ s_n = \prod_{h=1}^{p-1} \prod_{s=1}^{a_k} \left( \epsilon_{h,s}^{n} - \epsilon_{p}^{h} \right), \quad n = 1, 2, \ldots, \]
by Lemma 7, the theorem follows. \( \square \)

Corollary 4. For \(n = 1, 2, \ldots\), let
\[ s_{q,n} := t(C_{pn}(a_1, a_2, \ldots, a_k, q n)) / \left( (-1)^{(a_k+1)(p-1)n} t(C_n(a_1, a_2, \ldots, a_k)) \right). \]
If \((p, q) = 1\), then the characteristic polynomial for \(s_{q,n}\) is the same as the one for \(s_{1,n}\).
Proof. Let $\lambda_{q,h,s}, s = 1, 2, \ldots, a_k, 1 \leq h \leq p - 1$, be all the roots of the polynomial

$$f_{q,h}(x) = \sum_{i=1}^{k} x^{a_i} + \epsilon_q^h p - k - 1,$$

and let

$$S_q = \{f_{q,h}(x)| h = 1, 2, \ldots, p - 1\}$$

and

$$E_q = \{\lambda_{q,h,s}| s = 1, 2, \ldots, a_k, h = 1, 2, \ldots, p - 1\}.$$

By Theorem 3, $s_{q,n}$ satisfy a linear recurrence relation of order $2a_k(p - 1)$ with constant coefficients. Since

$$s_{q,n} = \prod_{h=1}^{p-1} \frac{\lambda_{q,h,s} - \epsilon_h^n}{a_k}, \quad n = 1, 2, \ldots,$$

it is clear that one characteristic root for $s_{q,n}$ is 1, and the other ones are the products of the elements in the set $E_q$. By Lemma 5, $S_q = S_1$, hence $E_q = E_1$. By Lemma 7, the corollary follows. □

Now we give some examples. By Theorems 1 and 3 and Corollaries 1 and 4, one can verify the examples.

Example 1. $t(C_{2n}(1, n)) = n(3^n + 1)$.

Example 2. $t(C_{2n}(2, n)) = n(3^n - 1)$, where $n$ is odd.

Example 3. $t(C_{2n}(1, 2, n)) = t(C_n(1, 2))(4^n + (-1)^n + A_n)$, where $t(C_n(1, 2)) = n(2^n + (-1)^{n-1})/3$ (see [10]), and $A_n$ satisfy a linear recurrence relations of order 2:

$$A_n = A_{n-1} + 4A_{n-2}, \quad A_1 = 1, \quad A_2 = 9.$$

Example 4. $t(C_{3n}(1, n)) = n(7^n + 1 + A_n)$, where

$$A_n = -\sum_{h=1}^{2} \epsilon_3^h (2 - \epsilon_3^{2h})^n$$

satisfy a linear recurrence relation of order 2:

$$A_n = 5A_{n-1} - 7A_{n-2}, \quad A_1 = 4, \quad A_2 = 13.$$

Example 5. $t(C_{3n}(1, 2n)) = n(7^n + 1 + A_n)$, where

$$A_n = -\sum_{h=1}^{2} \epsilon_3^h (2 - \epsilon_3^{2h})^n$$

satisfy a linear recurrence relation of order 2:

$$A_n = 5A_{n-1} - 7A_{n-2}, \quad A_1 = 1, \quad A_2 = -2.$$

Example 6. $t(C_{4n}(1, n)) = n(3^n + 1)(5^n + 1 + A_n)$, where

$$A_n = \sqrt{1}(2 - \sqrt{-1})^n - \sqrt{-1}(2 + \sqrt{-1})^n$$

satisfy a linear recurrence relation of order 2:

$$A_n = 4A_{n-1} - 5A_{n-2}, \quad A_1 = 2, \quad A_2 = 8.$$
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References