Solving fuzzy variational inequalities over a compact set

Cheng-Feng Hu
Department of Applied Mathematics, I-Shou University, Ta-Hsu, Kaohsiung 840, Taiwan
Received 8 February 1999; received in revised form 9 December 1999

Abstract

This paper studies the fuzzy variational inequalities over a compact set. By using the tolerance approach, we show that solving such problems can be reduced to a semi-infinite programming problem. A relaxed cutting plane algorithm is proposed. In each iteration, we solve a finite optimization problem and add one more constraint. The proposed algorithm chooses a point at which the infinite constraints are violated to a degree rather than at which the violation is maximized. The iterative process ends when an optimal solution is identified. A convergence proof, under some mild conditions, is given. An efficient implementation based on the “entropic regularization” techniques is also included. To illustrate the solution procedure, a numerical example is provided. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Fuzzy mathematical programming; Variational inequalities; Cutting plane method

1. Introduction

Traditional variational inequalities, stimulated by minimization problems or partial differential equations, have made great developments both in theory and in practice over the past years. In this paper, we consider the variational inequalities in a fuzzy environment, where the rigid requirements of strictly satisfying the inequalities are to be softened and can be described as follows:

Find \( x \) such that

(i) \( x \in V \),

(ii) \( \langle F(x), z - x \rangle \gtrless 0, \forall z \in V \),

where \( V \) is a compact subset of \( \mathbb{R}^n \), \( F \) is a mapping from \( V \) into \( \mathbb{R}^n \), \( \langle F(x), z - x \rangle \gtrless 0 \) are fuzzy inequalities, \( \forall z \in V \), and “\( \gtrless \)” denotes the fuzzified version of “\( \geq \)” with the linguistic interpretation “approximately greater than or equal to”. More specifically, given \( z \in V \), each fuzzy inequality
Fig. 1. The membership function $\mu_{\tilde{C}_z}(x)$ of the fuzzy inequality $\langle F(x), z - x \rangle \geq 0$.\\

$\langle F(x), z - x \rangle \geq 0$ actually determines a fuzzy set $\tilde{C}_z$ in $\mathbb{R}^n$, whose membership function is denoted by $\mu_{\tilde{C}_z}(\cdot)$, such that for each $x \in \mathbb{R}^n$, $\mu_{\tilde{C}_z}(x)$ is the degree to which the regular inequality $\langle F(x), z - x \rangle \geq 0$ is satisfied. To specify the membership function $\mu_{\tilde{C}_z}(\cdot)$, it is commonly assumed that $\mu_{\tilde{C}_z}(x)$ should be 0 if the regular inequality $\langle F(x), z - x \rangle \geq 0$ is strongly violated, and 1 if it is satisfied. This “tolerance approach” leads to a membership function in the following form:

$$
\mu_{\tilde{C}_z}(x) = \begin{cases} 
1 & \text{if } \langle F(x), z - x \rangle \geq 0, \\
\mu_z(\langle F(x), z - x \rangle) & \text{if } -t_z < \langle F(x), z - x \rangle \leq 0, \forall z \in V, \\
0 & \text{if } \langle F(x), z - x \rangle \leq -t_z,
\end{cases}
$$

where $t_z \geq 0$ is the tolerance level which a decision maker can tolerate in the accomplishment of the fuzzy inequality $\langle F(x), z - x \rangle \geq 0$. We usually assume that $\mu_z(\cdot) \in [0, 1]$ and is continuous and strictly increasing over $[-t_z, 0]$. Fig. 1 shows some different shapes of such membership functions.

One motivation to study such a system is related to finding “almost optimal” solutions for a general convex minimization problem. Consider the following problem:

$$
\text{min } h(x) \\
\text{s.t. } x \in K,
$$

(2)

where $h(\cdot)$ is a smooth real-valued function defined on a convex set $K \subseteq \mathbb{R}^n$. Solving this problem is equivalent to solving the following variational inequalities [6,14,16]:

Find $x$ such that

$$
(\text{I}) \ x \in K, \\
(\text{II}) \ \langle \nabla h(x), z - x \rangle \geq 0, \ \forall z \in K.
$$

(3)

To find an “almost optimal” solution for problem (2), we consider solving problem (3) with $\langle \nabla h(x), z - x \rangle$ being approximately greater than or equal to 0, $\forall z \in K$, i.e., $\langle \nabla h(x), z - x \rangle \geq 0$, $\forall z \in K$. It can be shown that a solution satisfying the corresponding fuzzy inequality system to a degree $\varepsilon$ close to 1 is a near optimal solution to problem (2) [13].
2. The model

To find a solution to the fuzzy variational inequalities (1), we define a fuzzy decision $\tilde{D}$ of problem (1) as the fuzzy set resulting from the intersection of fuzzy sets $\tilde{C}_z$, $\forall z \in V$. By choosing the commonly used “minimum operator” for the fuzzy set intersections [18], we can define the membership function for $\tilde{D}$ as

$$\mu_{\tilde{D}}(x) = \min_{z \in V} \{\mu_{\tilde{C}_z}(x)\}. \quad (4)$$

Therefore, a solution, say $x$, to the fuzzy variational inequalities (1) with some degree $0 \leq \mu \leq 1$, should satisfy that the inner product $\langle F(x), z - x \rangle$ is greater than or equal to zero to some degree $0 \leq \mu \leq 1$, for all $z \in V$. In this case, the solution set of the fuzzy variational inequalities (1) is a fuzzy solution set. Assuming that we are not interested in a fuzzy solution set but in a crisp “optimal” solution we could suggest the “maximizing solution” to (1), which can be taken as the solution with the highest membership in the fuzzy decision set $\tilde{D}$ and obtained by solving the following problem [2,18]:

$$\max_{x \in V} \mu_{\tilde{D}}(x),$$

or equivalently,

$$\max_{x \in V} \min_{z \in V} \{\mu_{\tilde{C}_z}(x)\}.$$  

Introducing one new variable $z$ results in an equivalent problem:

$$\begin{align*}
\max & \quad z \\
\text{s.t.} & \quad \mu_{\tilde{C}_z}(x) \geq z, \quad \forall z \in V, \\
& \quad x \in V, \\
& \quad 0 \leq z \leq 1. 
\end{align*} \quad (5)$$

Notice that problem (5) is a semi-infinite programming problem [1,12] with finitely many variables, $x_1, x_2, \ldots, x_n, z$, and infinitely many constraints. From the above procedure, we see that a system of fuzzy variational inequalities (1) can eventually be reduced to a regular semi-infinite programming problem (5).

3. An algorithm

There are many semi-infinite programming algorithms [10–12] available for solving problem (5). The difficulty lies in how to effectively deal with the infinite number of constraints. Based on a recent review [12], the “cutting plane approach” is an effective one for such application.

Following the basic concept of the cutting plane approach, we can easily design an iterative algorithm which adds one more constraint at a time until an optimal solution is identified. To be more specific, at the $k$th iteration, given a subset $V_k = \{\tilde{z}^1, \tilde{z}^2, \ldots, \tilde{z}^k\}$ of $V$, where $k \geq 1$, we consider
the following nonlinear programming problem:

Program VI$^k$

$$\max \phi(x, z) \triangleq z$$

s.t. $\mu_{C_i}(x) \geq z, \quad \forall i = 1, 2, \ldots, k,$

$x \in V,$

$$0 \leq z \leq 1.$$  \hspace{1cm} (6)

Let $F^k$ be the feasible region of Program VI$^k$. Suppose that $(x^k, z^k)$ is an optimal solution of VI$^k$. We define the “constraint violation function”:

$$v_{k+1}(z) \triangleq z^k - \mu_{C_i}(x^k), \quad z \in V.$$  \hspace{1cm} (7)

Since $\mu_{C_i}(x^k)$ is continuous over the compact set $V$, the function $v_{k+1}(z)$ achieves its maximum over $V$. Let $z^{k+1}$ be such maximizer and consider the value of $v_{k+1}(z^{k+1})$. If the value is less than or equal to zero, then $(x^k, z^k)$ becomes a feasible solution of problem (5), and hence $(x^k, z^k)$ is optimal for the problem (5) (because the feasible region $F^k$ of program VI$^k$ is no smaller than the feasible region of problem (5)). Otherwise, we know $z^{k+1} \notin V_k$. This background provides a foundation for us to outline a cutting plane algorithm for solving the semi-infinite programming problem (5).

3.1. CPSVI algorithm

Initialization: Set $k = 1$; Choose any $z^1 \in V$; Set $V_k = \{z^1\}$.

Step 1. Solve VI$^k$ and obtain an optimal solution $(x^k, z^k)$.

Step 2. Find a maximizer $z^{k+1}$ of $v_{k+1}(z)$ over $V$ with an optimum value $v_{k+1}(z^{k+1})$.

Step 3. If $v_{k+1}(z^{k+1}) \leq 0$, then stop with $(x^k, z^k)$ being an optimal solution of problem (5). Otherwise, set $V_{k+1} \leftarrow V_k \cup \{z^{k+1}\}$, set $k \leftarrow k + 1$, and go to step 1.

When problem (5) has at least one feasible solution, it can be shown without much difficulty that the CPSVI algorithm either terminates in a finite number of iterations with an optimal solution or generates a sequence of points $\{(x^k, z^k), k = 1, 2, \ldots\}$, which has a subsequence converging to an optimal solution $(x^*, z^*)$, under some appropriate assumptions [13]. However, for the above cutting plane algorithm, one major computation bottleneck lies in Step 2 of finding maximizers. Ideas of relaxing the requirement of finding global maximizers for different settings can be referred to [9,17]. But the required computation work could still be a bottleneck. Here we propose a simple and yet very effective relaxation scheme which chooses points at which the infinite constraints are violated to a degree rather than at which the violation are maximized. The proposed algorithm is stated as follows.

3.2. Relaxed CPSVI Algorithm

Let $\delta > 0$ be a prescribed small number.

Initialization: Set $k = 1$; Choose any $z^1 \in V$; Set $V_k = \{z^1\}$. 

Step 1. Solve VI\(^k\) and obtain an optimal solution \((x^k, z^k)\). Define \(v_{k+1}(z)\) according to (7).

Step 2. Find any \(z^{k+1} \in V\) such that \(v_{k+1}(z^{k+1}) > \delta\).

Step 3. If such \(z^{k+1}\) does not exist, then output \((x^k, z^k)\) as a solution. Otherwise, go to step 4.

Step 4. If such \(z^{k+1}\) exists, then set \(V_{k+1} \leftarrow V_k \cup \{z^{k+1}\}\).

Step 5. Set \(k \leftarrow k + 1\); go to step 1.

Note that in Step 2, since no maximizer is required, the computational work can be greatly reduced. Also note that when \(\delta\) is chosen to be sufficiently small, if the relaxed algorithm terminated in a finite number of iterations at Step 3, then an optimal solution is indeed obtained, assuming that the original problem (5) is feasible.

We now construct a convergence proof for the relaxed CPSVI algorithm.

**Theorem 1.** Given any \(\delta > 0\), assume that there is a scalar \(M > 0\) such that \(||(x, z)|| \leq M\) for each feasible solution \((x, z)\) of VI\(^1\) (bounded feasible domain assumption), then the relaxed CPSVI algorithm terminates in a finite number of iterations.

**Proof.** If the relaxed CPSVI algorithm does not terminate in a finite number of iterations, then the algorithm generates an infinite sequence \(\{(x^k, z^k)\}_{k=1}^\infty\). We have

\[
x^k - \mu \tilde{c}_k(x^k) > \delta, \quad k = 1, 2, \ldots,
\]

where \(z^{k+1}\) is generated by the relaxed CPSVI algorithm.

Due to the bounded feasible domain assumption and the compactness of \(V\), there exists a subsequence \(\{(x^i, z^i)\}_{i=1}^\infty\) of \(\{(x^k, z^k)\}_{k=1}^\infty\) such that \(\lim_{i \to \infty} (x^i, z^i) = (x^*, z^*)\), \(\lim_{i \to \infty} z^{k+1} = z^*\). Consequently, by (8), we have

\[
x^* - \mu \tilde{c}_k(x^*) \geq \delta.
\]

However, for each \(z^k\), \(k = 1, 2, \ldots\),

\[
x^j - \mu \tilde{c}_j(x^j) \leq 0, \quad \forall j \geq k.
\]

Therefore, for any fixed \(k\), as the sequence \(\{(x^k, z^k)\} \to (x^*, z^*)\), we see that

\[
x^* - \mu \tilde{c}_k(x^*) \leq 0.
\]

Since the above expression is true for all \(k\), we have

\[
x^* - \mu \tilde{c}_k(x^*) \leq 0
\]

which contradicts the fact that

\[
x^* - \mu \tilde{c}_k(x^*) \geq \delta.
\]

The theorem is proved.

4. **Solving program VI\(^k\)**

The relaxed CPSVI algorithm proposed in Section 3 requires an efficient algorithm for solving the optimization problem VI\(^k\) in each iteration. Notice that solving Program VI\(^k\) is equivalent to solving...
the following min–max problem:

\[
-\min_{x \in V} \mu'_F(x) \triangleq \max_{i=1,2,...,k} \{-\mu_{\tilde{C}Z_i}(x)\}.
\] (9)

One major difficulty encountered in developing solution methods for solving the min–max problem (9) is the non-differentiability of the max function \(\mu'_F(x)\). A distinct feature of the recent development centers around the idea of developing “smooth algorithms” [5,8]. Among them, a class called “regularization methods” has been developed based on approximating the max function \(\mu'_F(x)\) by certain smooth function [3,4,8]. Here we adopt the newly proposed “entropic regularization procedure” [7,15]. This procedure guarantees that, for an arbitrarily small \(\varepsilon > 0\), an \(\varepsilon\)-optimal solution of the min–max problem (9) can be obtained by solving the following problem:

\[
-\min_{x \in V} \mu_p(x) = \frac{1}{p} \ln \left\{ \sum_{i=1}^k \exp[p(-\mu_{\tilde{C}Z_i}(x))] \right\}.
\] (10)

with a sufficiently large \(p\).

It should be noted that, in practice, a sufficiently accurate approximation can be obtained by using a moderately large \(p\). Although the convergence result established in Section 3 is based on the ability to obtain the exact minimum in solving Program VI\(^k\), it remains valid with inexact minimization. Also because of the special “log-exponential” form of \(\mu_p(x)\), most over-flow problems in computation can be avoided.

5. Numerical example

In this section we use one simple example to illustrate the proposed theory and solution procedures. Let us consider the fuzzy variational inequality problem (1) with \(V, F\) and the corresponding membership functions, \(\mu_{\tilde{C}Z}(x)\), for each fuzzy variational inequality \(\langle F(x), z - x \rangle \geq 0\), specified as:

\[
V = \{ x = (x_1, x_2, x_3)^T \in \mathbb{R}^3 \mid -x_1 - x_2 + 3x_3 \geq 0, -2x_1 + x_2 - x_3 \geq 0 \},
\]

\[
F(x) = \begin{pmatrix}
2x_1 + 0.2x_1^3 - 0.5x_2 + 0.1x_3 - 3 \\
-0.5x_1 + x_2 + 0.1x_2^3 + 0.5 \\
0.5x_1 - 0.2x_2 + 2x_3 - 0.5
\end{pmatrix},
\]

and

\[
\mu_{\tilde{C}Z}(x) = \begin{cases}
1 & \text{if } \langle F(x), z - x \rangle \geq 0, \\
\frac{\langle F(x), z - x \rangle + 3}{3} & \text{if } -3 < \langle F(x), z - x \rangle \leq 0, \quad \forall z \in V. \\
0 & \text{if } \langle F(x), z - x \rangle \leq -3,
\end{cases}
\]
To solve this fuzzy variational inequality problem, we follow the “tolerance approach” to consider the following problem:

\[
 \text{max } \alpha \\
\text{s.t. } -x_1 - x_2 + 3x_3 \geq 0,
\]
\[
-2x_1 + x_2 - x_3 \geq 0,
\]
\[
\frac{1}{3}((2x_1 + 0.2x_1^3 - 0.5x_2 + 0.1x_3 - 3)z_i + (-0.5x_1 + x_2 + 0.1x_2^3 + 0.5)z_2 + (0.5x_1 - 0.2x_2 + 2x_3 - 0.5)z_3 - (0.2x_1^4 + 2x_1^2 - 3x_1)
\]
\[
+ 0.1x_2^4 + x_2^3 + 0.5x_2 + 2x_3^2 - 0.5x_3 - x_1x_2
\]
\[
+ 0.6x_1x_3 - 0.2x_2x_3)) + 3) \geq \alpha,
\]
\[
0 \leq \alpha \leq 1.
\]

Using the proposed algorithm to solve the semi-infinite programming problem (11), at the \(k\)th iteration, the following problem is considered:

**Program VI**

\[
\text{max } \alpha \\
\text{s.t. } -x_1 - x_2 + 3x_3 \geq 0,
\]
\[
-2x_1 + x_2 - x_3 \geq 0,
\]
\[
\frac{1}{3}((2x_1 + 0.2x_1^3 - 0.5x_2 + 0.1x_3 - 3)z_i^i + (-0.5x_1 + x_2 + 0.1x_2^3 + 0.5)z_2^i + (0.5x_1 - 0.2x_2 + 2x_3 - 0.5)z_3^i - (0.2x_1^4 + 2x_1^2 - 3x_1)
\]
\[
+ 0.1x_2^4 + x_2^3 + 0.5x_2 + 2x_3^2 - 0.5x_3 - x_1x_2
\]
\[
+ 0.6x_1x_3 - 0.2x_2x_3)) + 3) \geq \alpha,
\]
\[
0 \leq \alpha \leq 1,
\]

which is equivalent to the following min–max problem:

\[
-\min_{x \in F} \max_{i=1,2,\ldots,k} \left\{ -\frac{1}{3}((2x_1 + 0.2x_1^3 - 0.5x_2 + 0.1x_3 - 3)z_i^i + (-0.5x_1 + x_2 + 0.1x_2^3 + 0.5)z_2^i
\]
\[
+ (0.5x_1 - 0.2x_2 + 2x_3 - 0.5)z_3^i
\]
\[
-(0.2x_1^4 + 2x_1^2 - 3x_1 + 0.1x_2^4 + x_2^3 + 0.5x_2 + 2x_3^2 - 0.5x_3 - x_2x_3
\]
\[
+ 0.6x_1x_3 - 0.2x_2x_3)) + 3) \right\}.
\]
A near-optimal solution of the min–max problem can be obtained by solving the following problem:

\[-\min_{x \in V} \frac{1}{p} \ln \left\{ \sum_{i=1}^{k} \exp\left[ p \left( -\frac{1}{3} \left( \left( 2x_1 + 0.2x_3^3 - 0.5x_2 + 0.1x_3 - 3 \right)z_i^1 \right) 
+ \left( -0.5x_1 + x_2 + 0.1x_3^2 + 0.5 \right)z_i^2 
+ \left( 0.5x_1 - 0.2x_2 + 2x_3 - 0.5 \right)z_i^3 
- (0.2x_1^4 + 2x_1^2 - 3x_1 + 0.1x_2^4 + x_2^2 + 0.5x_2 + 2x_3^2 - 0.5x_3 - x_1x_2 
+ 0.6x_1x_3 - 0.2x_2x_3 \right) + 3 \right) \right] \right\},\]

with a sufficiently large \( p \).

In our implementation, we use a fixed \( p = 1000 \) at each iteration for solving Program \( VI_k \). The algorithm terminated after 11 iterations at the point \( x^* = (0.0259, 0.3651, 0.3133)^T \) with \( z^* = 0.9155 \). Computational results for this problem are listed in Table 1.

### Table 1
Computational results of the relaxed CPSVI algorithm

<table>
<thead>
<tr>
<th>( k )</th>
<th>( (x^k, z^k) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>((0, 0.000 ))</td>
</tr>
<tr>
<td>2</td>
<td>((0.0864, 0.3534, 0.1807, 1.0000))</td>
</tr>
<tr>
<td>3</td>
<td>((0.0216, 0.3578, 0.3125, 0.9151))</td>
</tr>
<tr>
<td>4</td>
<td>((2.6636e-007, 9.3225e-007, 3.9954e-007, 1.0000))</td>
</tr>
<tr>
<td>5</td>
<td>((0.1319, 0.4616, 0.1978, 0.9988))</td>
</tr>
<tr>
<td>6</td>
<td>((1.4974e-006, 5.2407e-006, 2.2460e-006, 1.0000))</td>
</tr>
<tr>
<td>7</td>
<td>((0.0157, 0.3424, 0.3109, 0.9143))</td>
</tr>
<tr>
<td>8</td>
<td>((0.1324, 0.4635, 0.1987, 0.9984))</td>
</tr>
<tr>
<td>9</td>
<td>((7.6347e-009, 2.6722e-008, 1.1452e-008, 1.0000))</td>
</tr>
<tr>
<td>10</td>
<td>((0.0039, 0.2824, 0.2745, 0.9309))</td>
</tr>
<tr>
<td>11</td>
<td>((0.0259, 0.3651, 0.3133, 0.9155))</td>
</tr>
</tbody>
</table>

6. Conclusion

In this paper, a fuzzy variational inequality problem is studied. By using the “tolerance approach,” we have shown that solving such problems can be reduced to a semi-infinite programming problem. A relaxed cutting plane algorithm is proposed for solving the fuzzy variational inequalities over a compact set. One obvious advantage of the proposed cutting plane algorithm is that only those constraints which tend to be binding are generated. This leads to efficiency in terms of both cpu and memory requirements, especially for solving large-scale problems. Moreover, an “entropic regularization” technique is applied to solve the nonlinear Program \( VI_k \) required by the proposed cutting plane algorithm in each iteration. This method essentially provides a smooth and uniform approximation for solving the min–max problem.
Acknowledgements

The author would like to thank professor Shu-Cherng Fang for his very constructive and valuable suggestions.

References