# The Number of Perfect Matchings in a Hypercube 

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## $A B S T R A C T$

A perfect matching or a 1-factor of a graph $G$ is a spanning subgraph that is regular of degree one. Hence a perfect matching is a set of independent edges which matches all the nodes of $G$ in pairs. Thus in a hypercube parallel processor, the number of perfect matchings evaluates the number of different ways that all the processors can pairwise exchange information in parallel. Making use of matrices and their permanents one can write a straightforward formula which we evaluate for $n \leq 5$.

A perfect matching or a 1 -factor of a graph $G$ is a regular spanning subgraph of degree one. In other words a perfect matching is a set of independent edges in $E(G)$ that spans $V(G)$. Define $f_{1}(G)$ as the number of 1-factors of $G$.

The bipartite adjacency matrix (ba-matrix) $B=B(G)$ of a bipartite graph $G=(V, E)$ where $V=U \cup W,|U|=m,|W|=n$, is the $m \times n$ matrix that indicates the presence or absence of an edge between each ( $u, w$ ) pair of nodes by a one or zero, respectively. The following theorem was found independently by both Fisher [1] and Kasteleyn [3].

Theorem A. The number $f_{1}(G)$ of perfect matchings in an $n \times n$ bipartite graph $G$ is $\operatorname{per} B$, the permanent of the ba-adjacency matrix $B$ of graph $G$.

$$
\begin{equation*}
f_{1}(G)=\operatorname{per} B(G) \tag{1}
\end{equation*}
$$

Proof. The permanent of a square binary matrix is simply the number of ways of choosing exactly one 1 from each row and each column. Hence, there exists a one-to-one correspondence between perfect matchings and the unit contributions to this permanent. []

The hypercube $Q_{n}$ may be recursively defined [2,p.23] in terms of cartesian product:

$$
Q_{n}= \begin{cases}K_{2} & n=1  \tag{2}\\ Q_{n-1} \times K_{2} & n \geq 2\end{cases}
$$

Using this definition, the ba-matrix $B_{n}$ of a hypercube $Q_{n}$ may be conveñiently written recursively, with $I$ denoting the identity matrix of order $2^{n-1}$ :

$$
B_{1}=[1], \quad B_{n+1}=\left[\begin{array}{cc}
B_{n} & I  \tag{3}\\
I & B_{n}
\end{array}\right]
$$

By Theorem A the value of $\operatorname{per} B_{n}$ is the number of perfect matchings in $Q_{n}$.
A submatrix $X$ of a matrix $A$ is the matrix formed by choosing a subset of the rows of $A$ and a subset of the columns of $A$. It is convenient to give an expression for counting the perfect matchings of $Q_{n+1}$.

Theorem 1. The number of perfect matchings of $Q_{n+1}$ is given by

$$
\begin{equation*}
f_{1}\left(Q_{n+1}\right)=\operatorname{per} B_{n+1}=\sum_{X \subset B_{n}}(\operatorname{per} X)^{2} \tag{4}
\end{equation*}
$$

Proof. Consider any $k \times k$ matrix $X$ in the upper left $B_{n}$ in $B_{n+1}$ as in (1) such that per $X \neq 0$. The permanent of $X$ is the number of perfect matchings in the subgraph of $Q_{n}$ induced by the nodes corresponding to the rows and columns of $X$. Call this induced subgraph $G_{X}$. Now match all nodes of $V\left(Q_{n}\right)-V\left(G_{X}\right)$ with their neighbors in the other copy of $Q_{n}$. Clearly, the unmatched nodes in the second copy of $Q_{n}$ induce a graph isomorphic to $G_{X}$ and its permanent is per $X$. Thus, for each square submatrix $X$ in $B_{n}$ there are $(\operatorname{per} X)^{2}$ perfect matchings in $B_{n+1}$. []

$$
\begin{aligned}
& \text { For example } B_{2}=\left[\begin{array}{ll}
1 & 1 \\
1 & 1
\end{array}\right] \text { so by }(1), \quad B_{3}=\left[\begin{array}{llll}
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1
\end{array}\right] \text { and } \\
& \qquad \begin{array}{l}
f_{1}\left(Q_{3}\right)=\operatorname{per} B_{3}=1^{2}+4 \operatorname{per}[1]^{2}+\operatorname{per}\left[\begin{array}{lll}
1 & 1 \\
1 & 1
\end{array}\right]^{2}=9
\end{array}
\end{aligned}
$$

Note that the empty matrix with unit permanent is an admissible submatrix of $B_{n}$ and contributes 1 to the above sum in the $1^{2}$ term.

It is convenient to introduce some additional notation. For any square matrix $A$, let

$$
\left[\begin{array}{l}
A  \tag{5}\\
k
\end{array}\right]=\sum_{X \subset A}(\operatorname{per} X)^{2}
$$

where the summation is over all $k \times k$ submatrices $X$ of $A$. Thus (1) may be rewritten

$$
f_{1}\left(Q_{n+1}\right)=\operatorname{per} B_{n+1}=\sum_{k=0}^{2^{n-1}}\left[\begin{array}{c}
B_{n}  \tag{6}\\
k
\end{array}\right]
$$

Obviously, $\left[\begin{array}{c}B_{n} \\ 0\end{array}\right]=1$ and $\left[\begin{array}{c}B_{n} \\ 2^{n-1}\end{array}\right]=\left(\operatorname{per} B_{n}\right)^{2}=f_{1}{ }^{2}\left(Q_{n}\right)$. The number of
ones in $B_{n}$ is $\left[\begin{array}{c}B_{n} \\ 1\end{array}\right]$ which is the number of edges in $Q_{n}$, so $\left[\begin{array}{c}B_{n} \\ 1\end{array}\right]=n 2^{n-1}$.
To derive a closed form for $\left[\begin{array}{c}B_{n} \\ 2\end{array}\right]$ it is convenient to identify all dissimilar pairs of columns of $B_{n}$. These correspond to all dissimilar pairs of nodes in $Q_{n}$. Any pair of nodes may be completely specified by their distance because of symmetry, and the number of pairs at distance $2 k$ is

$$
\frac{1}{2}\binom{2^{n-1}}{1}\binom{n}{2 k}=2^{n-2}\binom{n}{2 k}
$$

When $k=1$ any two nodes at distance 2 are mutually adjacent to exactly two nodes and are each individually adjacent to $n-1$ other nodes. Therefore, any pair of columns in $B_{n}$ corresponding to nodes at distance 2 consist of two rows of the form 11 and $n-1$ rows of the form 10 and $n-1$ rows of the form 01 and the other $2^{n-1}-2 n$ rows 00 .

Thus, the sum of the squares of all permanents formed by selecting two nodes at distance 2 is

$$
\operatorname{per}\left[\begin{array}{ll}
1 & 1 \\
1 & 1
\end{array}\right]^{2}+2(n-2) \operatorname{per}\left[\begin{array}{ll}
1 & 1 \\
1 & 0
\end{array}\right]^{2}+2(n-2) \operatorname{per}\left[\begin{array}{ll}
1 & 1 \\
0 & 1
\end{array}\right]^{2}+(n-2)^{2} \operatorname{per}\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]^{2}=n^{2}
$$

Any two nodes at distance greater than two obviously have disjoint neighborhoods and their corresponding pair of columns in $B_{n}$ contain $n$ copies of 10 and of 01 in their rows, the other rows consisting of 00 entries. Thus the permanents of the $2 \times 2$ matrices formed in these columns contribute a factor of $n^{2}$, giving

$$
\begin{gathered}
{\left[\begin{array}{c}
B_{n} \\
2
\end{array}\right]=n^{2} 2^{n-2}\binom{n}{2}+n^{2} 2^{n-2} \sum_{k=2}^{\lfloor n / 2}\binom{n}{2 k} \text {, but }} \\
\sum_{k=0}^{\lfloor n / 2\rfloor}\binom{n}{2 k}=2^{n-1}, \quad \text { hence }\left[\begin{array}{c}
B_{n} \\
2
\end{array}\right]=n^{2} 2^{n-2}\left(2^{n-1}-1\right) \\
\text { To compute per } B_{4}=\operatorname{per}\left[\begin{array}{llllllll}
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1
\end{array}\right]
\end{gathered}
$$

we first evaluate $\left[\begin{array}{c}B_{3} \\ 3\end{array}\right]$. Since all sets of three nodes of even weight in $Q_{3}$ are similar, any three columns of $B_{3}$ may be chosen. Within any three columns there are two dissimilar (with respect to the automorphism group of $Q_{3}$ ) $3 \times 3$ submatrices giving

$$
\begin{aligned}
& {\left[\begin{array}{c}
B_{3} \\
3
\end{array}\right] }=\binom{4}{3}\left\{\operatorname{per}\left[\begin{array}{lll}
1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 1
\end{array}\right]^{2}+3 \operatorname{per}\left[\begin{array}{lll}
1 & 1 & 0 \\
1 & 0 & 1 \\
1 & 1 & 1
\end{array}\right]^{2}\right\}=124, \text { so } \\
& f_{1}\left(Q_{4}\right)=\sum_{i=0}^{i=4}\left[\begin{array}{c}
B_{3} \\
i
\end{array}\right]=1+3.2^{2}+3^{2} \cdot 2\left(2^{2}-1\right)+124+9^{2}=272
\end{aligned}
$$

We conclude by only mentioning the result that

$$
f_{1}\left(Q_{5}\right)=589,185
$$

which was similarly calculated by a computer program.
An even more difficult unsolved problem in graphical enumeration is the exact determination of the number $f_{1}{ }^{*}\left(Q_{n}\right)$ of equivalence classes of perfect matchings in hypercube $Q_{n}$ with respect to its automorphism group $\Gamma\left(Q_{n}\right)$. It has been shown that $\Gamma\left(Q_{n}\right)=\left[S_{2}\right]^{S_{n}}$, the exponentiation group [2,p.177] of the two symmetric groups $S_{2}$ raised to the power $S_{n}$. It is also known that, in principle, the number $f_{1}{ }^{*}\left(Q_{n}\right)$ of these similarity classes can be calculated from the group of the graph $Q_{n}$ with respect to the group of the subgraph $2^{n-1} K_{2}$ (which is a perfect matching of $Q_{n}$ ). But this approach has not yet proved helpful. Obviously $f_{1}{ }^{*}\left(Q_{1}\right)=f_{1}{ }^{*}\left(Q_{2}\right)=1, f_{1}{ }^{*}\left(Q_{3}\right)=2$ and we have also found that $f_{1}{ }^{*}\left(Q_{4}\right)=8$.

## References

[1] M.E. Fisher, Statistical mechanics of dimers on a plane lattice Phys. Rev. 124 (1961) 278-286.
[2] F. Harary, Graph Theory. Addison-Wesley, Reading (1969).
[3] P.W. Kasteleyn, The statistics of dimers on a lattice I: The number of dimer arrangements on a quadratic lattice. Physica 27 (1961) 1209-1225.

