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Logarithmic residues, Rouché's theorem, and spectral regularity: The C^*-algebra case

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Dedicated to Israel Gohberg, whose seminal work on analytic Fredholm operator valued functions has been a source of inspiration for the present paper.

Abstract

Using families of irreducible Hilbert space representations as a tool, the theory of analytic Fredholm operator valued function is extended to a C^*-algebra setting. This includes a C^*-algebra version of Rouché's Theorem known from complex function theory. Also, criteria for spectral regularity of C^*-algebras are developed. One of those, involving the (generalized) Calkin algebra, is applied to C^*-algebras generated by a non-unitary isometry.

Keywords: C^*-Fredholm theory; Abstract analytic Fredholm valued function; Logarithmic residue; Spectral regularity

1. Introduction

A logarithmic residue is a contour integral of the type

$$\frac{1}{2\pi i}\int_{\partial\Delta} f'(\lambda)f(\lambda)^{-1}d\lambda, \tag{1}$$

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where the analytic function f has its values in a unital (complex) Banach algebra $\mathcal{B}$ and $\partial\Delta$ is a suitable contour in the complex plane $\mathbb{C}$, in fact the positively oriented boundary of a Cauchy domain Δ. In the scalar case when $\mathcal{B} = \mathbb{C}$, the expression (1) is equal to the number of zeros of f in Δ, multiplicities of course taken into account. Thus in that situation, the integral (1) vanishes if and only if f takes non-zero values, not only on $\partial\Delta$ which has been implicitly assumed in order to let (1) make sense, but on all of Δ. This state of affairs leads to the following question: if for a Banach algebra valued analytic function f the integral (1) vanishes, can one conclude that f takes invertible values on all of Δ?

In general the answer to this question is negative. The Banach algebra $\mathcal{L}(\ell_2)$ of all bounded linear operators on ℓ_2 is a counterexample (see [4]). So the modified question is: if for a Banach algebra valued analytic function f the integral (1) vanishes, under what additional conditions can one conclude that f takes invertible values on all of Δ? This problem has been taken up, with positive results, in a number of publications by the authors, notably [2,4,6,7,10–12]. In [5–10], another issue has been studied too, namely what kind of elements are logarithmic residues? The motivation for this comes, of course, from the fact that in the scalar case, the expression (1) determines a non-negative integer. As it turns out, sums of idempotents are the most appropriate candidates, but the picture is mixed: there are examples of logarithmic residues that are not sums of idempotents.

Of special interest for the present paper is [7]. This article deals with the case where $\mathcal{B}$ is the Banach algebra of bounded linear operators $\mathcal{L}(X)$ on a (complex) Banach space while the values of the analytic function f are Fredholm operators on X. Under those circumstances both issues raised above allow for a positive conclusion: the integral (1) determines a (finite) sum of finite rank projections on X and if it vanishes, then f takes invertible values on Δ. The latter, by the way, can be straightforwardly deduced from the results obtained in [30,31] (cf., also [37,15]).

The contribution of the present paper to the further development of the theory lies in the extension of the results for the Fredholm operator case to that where the function f takes its values in the set of Fredholm elements in a unital C^*-algebra. Here we note that for such an algebra one can in a sensible manner define finite rank elements, compact elements and Fredholm elements. Details can be found in, for instance, [1,32]. The set up of the latter best fits our purposes and will be heavily used below. A review of the material in question is given in Section 2.

This brings us to a description of the contents of the different sections to be found below. Apart from the introduction (Section 1) and the list of references, the paper consists of seven sections. As already indicated, Section 2 explains the abstract C^* Fredholm framework as developed in [32] that is needed later on. Section 3 adds material to this, not available in [32], on finite rank idempotents and traces of finite rank elements. Special attention is given to sums of idempotents. In Section 4 analytic functions having values in the set of Fredholm elements in a unital C^*-algebra are investigated. Counterparts are given here for results on finite meromorphy of inverses and factorization known in the operator case. Section 5 addresses logarithmic residues and spectral regularity in the general C^*-context. Positive results are obtained regarding the two questions posed at the outset of this introduction. Further, employing the concept of the trace mentioned above and in line with what has been achieved in [31] in the operator case, an analogue of Rouché's Theorem is obtained. Attention is also given to spectral regularity of a unital C^*-algebra, i.e., the property that for every analytic function f with values in the algebra in question, the fact that (1) vanishes (or, more generally, is quasinilpotent) implies that f takes invertible values on Δ. Criteria for spectral regularity of unital C^*-algebras are developed. One of the results is that a unital C^*-algebra is spectrally regular whenever this is the case for its Calkin algebra (i.e., the quotient algebra obtained by dividing out the ideal of the compact elements). The case when the given unital C^*-algebra is simple (i.e., has no proper non-trivial closed two-sided ideals) is dealt with in Section 6. Some straightforward examples are presented in Section 7. More sophisticated examples having to do with unital C^*-algebras generated by a non-unitary isometry are considered in Section 8.

One final remark to close the introduction. The expression (1) defines the *left* logarithmic residue of. There is also a *right* version obtained by replacing the left logarithmic derivative $f'(\lambda)f(\lambda)^{-1}$ by the right logarithmic derivative $f(\lambda)^{-1}f'(\lambda)$. For some special cases, the relationship between left logarithmic residues and right logarithmic residues has been investigated: see [6–8,10]. As far as the issues considered in the present paper are concerned, the results that can be obtained for the left and the right version of the logarithmic residue are analogous to each other. Therefore in what follows the qualifiers left and right will be suppressed.

2. Fredholm theory

To assist the reader, we begin by presenting an outline of the C^*-Fredholm theory as developed in [32, Chapter 6]. To serve our purposes, some simple observations not explicitly contained in [32] are added.

Let $\mathcal{B}$ be a unital C^*-algebra, unital with unit element $e_{\mathcal{B}}$. A non-zero element $r \in \mathcal{B}$ is said to be of *rank one* if for every $b \in \mathcal{B}$, there exists a complex number $\mu(b)$, necessarily unique, such that $rbr = \mu(b)r$. The function $b \mapsto \mu(b)$ is a linear functional on $\mathcal{B}$. Note that it does not vanish identically. In fact $\mu(r^*) \neq 0$, as can be seen from $\|rr^*rr^*\| = \|(rr^*)(rr^*)^*\| = \|rr^*\|^2 = \|r\|^4 \neq 0$, which implies that $rr^*r \neq 0$.

An element of $\mathcal{B}$ is of *finite rank* if it is the sum of a finite number of elements of rank one. The minimal number of such rank one elements necessary is by definition the *rank* of that element. Note that the zero element of $\mathcal{B}$ is of finite rank (empty sum); it is the unique element of $\mathcal{B}$ having rank zero.

If $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{B}$ and $a \in \mathcal{A}$ is a rank one element in $\mathcal{B}$, then obviously a is a rank one element in $\mathcal{A}$ too. The converse, however, need not be true. Leading up to an example showing this, we note that in the situation where $\mathcal{B} = \mathcal{L}(H)$ with H a Hilbert space, an element $T \in \mathcal{L}(H)$ is of finite rank k if and only if the operator T has finite range dimension k.

By way of illustration we give a first example. It will be put into a broader context in Section 8. There, and in Sections 5 and 7, other examples are presented as well.

Example 2.1. For $\mathcal{B}$ we take the C^*-algebra $\mathcal{L}(\ell_2)$. Let $V \in \mathcal{L}(\ell_2)$ be a non-unitary isometry, so $V^*V = I$ and $VV^* \neq I$, where I is the identity operator on ℓ_2. Clearly VV^* is a self-adjoint projection on ℓ_2 or, if one prefers that terminology, a self-adjoint idempotent in $\mathcal{L}(\ell_2)$. It will be shown in Section 8 that $I - VV^*$ is a rank one idempotent in the C^*-subalgebra $\mathcal{A}$ of $\mathcal{B}$ generated by the elements V, V^* and I. However, one can easily choose V in such a way that the operator $I - VV^*$ does not have range dimension one and so $I - VV^*$ is not a rank one element in the C^*-algebra $\mathcal{B} = \mathcal{L}(\ell_2)$. In fact, if S is the simple forward shift on ℓ_2 and k is a positive integer, then $V = S^k$ is a non-unitary isometry such that $I - VV^*$ has range dimension k. Letting $V : \ell_2 \to \ell_2$ be the non-unitary isometry given by

$$(Vx)_j = \begin{cases} x_{\frac{1}{2}(j+1)}, & j = 1, 3, 5, \ldots, \\ 0 & j = 2, 4, 6, \ldots, \end{cases}$$

where $x = (x_1, x_2, x_3, \ldots) \in \ell_2$, we even have that V is a non-unitary isometry for which $I - VV^*$ has infinite dimensional range. $\square$

Returning to the general C^*-framework, and motivated by the situation in the case $\mathcal{L}(H)$ with H a Hilbert space, we proceed as follows. Let $\mathcal{C}_0(\mathcal{B})$ denote the set of all finite rank elements in $\mathcal{B}$. Then $\mathcal{C}_0(\mathcal{B})$ is a two-sided ideal in $\mathcal{B}$, possibly non-closed. We write $\mathcal{C}(\mathcal{B})$ for the closure of $\mathcal{C}_0(\mathcal{B})$. The elements of $\mathcal{C}(\mathcal{B})$ are called *compact*. So the compact elements form a closed two-sided ideal in $\mathcal{B}$. Both $\mathcal{C}_0(\mathcal{B})$ and $\mathcal{C}(\mathcal{B})$ are closed under the *-operation.

Let κ be the canonical *-homomorphism from $\mathcal{B}$ onto what we shall call the *Calkin algebra* of $\mathcal{B}$, that is the quotient algebra $\mathcal{B}/\mathcal{C}(\mathcal{B})$. An element $a \in \mathcal{B}$ is said to be *Fredholm* or a *Fredholm element* if $\kappa(a)$ is invertible in $\mathcal{B}/\mathcal{C}(\mathcal{B})$, in other words, if a is invertible modulo the closed ideal $\mathcal{C}(\mathcal{B})$. The set of all Fredholm elements in $\mathcal{B}$ is denoted by $\mathcal{F}(\mathcal{B})$. It is an open subset of $\mathcal{B}$ containing the unit element of $\mathcal{B}$, and it is closed under the *-operation. Further $\mathcal{F}(\mathcal{B})$ is closed under taking products, and $\mathcal{F}(\mathcal{B}) + \mathcal{C}(\mathcal{B})$ is contained in $\mathcal{F}(\mathcal{B})$, i.e., the sum of a Fredholm element and a compact element is again Fredholm.

If $a \in \mathcal{B}$ is a Fredholm element then it is invertible modulo the possibly non-closed ideal $\mathcal{C}_0(\mathcal{B})$. Indeed, if g is a left inverse of a modulo $\mathcal{C}(\mathcal{B})$, so that $ga - e_{\mathcal{B}}$ is compact, and $h \in \mathcal{C}_0(\mathcal{B})$ is such that $\|ga - e_{\mathcal{B}} - h\| < 1$, then $ga - h$ is invertible in $\mathcal{B}$ while $(ga - h)^{-1}g$ is a left inverse of a modulo $\mathcal{C}_0(\mathcal{B})$. The argument for the right invertibility is analogous.

Next we bring in the concept of an irreducible representation. In what follows, H stands for a Hilbert space. A unital *-homomorphism $\psi : \mathcal{B} \to \mathcal{L}(H)$ is called a *representation* of $\mathcal{B}$. It is said to be *irreducible* if the subalgebra $\psi[\mathcal{B}]$ of $\mathcal{L}(H)$ does not have a non-trivial (closed) invariant subspace. Notice that in the C^*-setting, having a non-trivial invariant subspace is equivalent to having a non-trivial closed invariant subspace (see Corollary 2.8.4 in [20]). Moreover, from general C^*-theory it is known that $\psi[\mathcal{B}]$ is closed in $\mathcal{L}(H)$. The closed two-sided ideal of the compact operators on H is denoted by $\mathcal{K}(H)$.

Theorem 2.2. *Let $\mathcal{B}$ be a unital C^*-algebra, and let $\psi : \mathcal{B} \to \mathcal{L}(H)$ be an irreducible representation of $\mathcal{B}$. Then either $\psi[\mathcal{C}(\mathcal{B})] = \{0\}$ or $\psi[\mathcal{C}(\mathcal{B})] = \mathcal{K}(H)$.*

This result is obtained by combining Theorem 6.37 in [32] and Theorem 5.39 in [21]; see [32, p. 289]. A simple consequence of Theorem 2.2, is that an irreducible representation $\psi : \mathcal{B} \to \mathcal{L}(H)$ maps a Fredholm element in $\mathcal{B}$ into a Fredholm operator on H, in other words $\psi[\mathcal{F}(\mathcal{B})] \subset \mathcal{F}\big(\mathcal{L}(H)\big)$.

For every rank one element $r \in \mathcal{B}$ we denote by $\mathcal{J}(r)$ the smallest closed ideal in $\mathcal{B}$ which contains r.

Theorem 2.3 (*[32, Theorem 6.39]*). *Let $\mathcal{B}$ be a unital C^*-algebra. Then, for every rank one element r in $\mathcal{B}$, there exists a Hilbert space H and an irreducible representation $\phi : \mathcal{B} \to \mathcal{L}(H)$ such that $\phi[\mathcal{J}(r)] = \mathcal{K}(H)$ and* $\operatorname{Ker}\phi \cap \mathcal{J}(r) = \{0\}$.

So the restriction $\phi|_{\mathcal{J}(r)}$ of the representation ϕ to the ideal $\mathcal{J}(r)$ is a C^*-isomorphism from $\mathcal{J}(r)$ onto $\mathcal{K}(H)$.

Theorem 2.4 (*[32, Corollary 6.43]*). *Let r and s be rank one elements in the unital C^*-algebra $\mathcal{B}$. Then either $\mathcal{J}(r) = \mathcal{J}(s)$ or $\mathcal{J}(r) \cap \mathcal{J}(s) = \{0\}$.*

We now introduce an equivalent relation in the set of all rank one elements in a unital C^*-algebra $\mathcal{B}$ by calling two rank one elements r and s in $\mathcal{B}$ equivalent if $\mathcal{J}(r) = \mathcal{J}(s)$. Let T stand for the set of all corresponding equivalence classes. Further, given $t \in T$, choose a representative

r_t in the equivalence class t. Write $\mathcal{J}_t$ for the ideal $\mathcal{J}(r_t)$, and, in line with Theorem 2.3, select a (non-trivial) Hilbert space H_t and an irreducible representation $\pi_t : \mathcal{B} \to \mathcal{L}(H_t)$ such that $\pi_t[\mathcal{J}_t] = \mathcal{K}(H_t)$ and $\operatorname{Ker} \pi_t \cap \mathcal{J}_t = \{0\}$. From Lemma 3.6, to be presented below, it will become clear that, modulo the obvious inessential similarity transformations, the Hilbert space H_t and the irreducible representation π_t are uniquely determined by t.

Theorem 2.5 (*[32, Proposition 6.45]*)**.** *Suppose t_1 and t_2 are different equivalence classes in T. Then $\pi_{t_1}[\mathcal{J}_{t_2}] = \{0\}$.*

Corollary 2.6. *Suppose t_1 and t_2 are different equivalence classes in T. If $g_1 \in J_{t_1}$ and $g_2 \in \mathcal{J}_{t_2}$, then $g_1 g_2 = g_2 g_1 = 0$.*

Proof. Note that both $g_1 g_2$ and $g_2 g_1$ belong to $\mathcal{J}(r_{t_1}) \cap \mathcal{J}(r_{t_2})$, which is equal to $\{0\}$ by Theorem 2.4. □

Next we characterize invertibility in $\mathcal{B}$ in terms of the homomorphisms κ and π_t.

Theorem 2.7 (*[32, Theorem 6.44]*)**.** *An element $a \in \mathcal{B}$ is invertible in $\mathcal{B}$ if and only if the following conditions are satisfied:*

(i) *a is Fredholm, i.e, $\kappa(a)$ is invertible in the Calkin algebra $\mathcal{B}/\mathcal{C}(\mathcal{B})$,*
(ii) *$\pi_t(a)$ is invertible in $\mathcal{L}(H_t)$ for every $t \in T$.*

This theorem allows for a reformulation in the language of non-commutative Gelfand theory (see [35], Section 7.1 in [38,41,12,13]): the collection of homomorphisms $\{\kappa : \mathcal{B} \to \mathcal{B}/\mathcal{C}(\mathcal{B})\} \cup \{\pi_t : \mathcal{B} \to \mathcal{L}(H_t)\}_{t \in T}$ is a sufficient family for $\mathcal{B}$. Combining this with two results from [13], namely Corollary 2.3 and Theorem 2.4, we obtain

$$\mathcal{C}(\mathcal{B}) \cap \bigcap_{t \in T} \operatorname{Ker} \pi_t = \{0\}. \tag{2}$$

Here is another way to express this identity.

Theorem 2.8. *The family $\{\pi_t : \mathcal{B} \to \mathcal{L}(H_t)\}_{t \in T}$ separates the points of $\mathcal{C}(\mathcal{B})$.*

For the convenience of the reader, we give the direct proof of the theorem as it can be extracted from [13].

Proof. We need to establish (2). Take h in the left hand side of (2). For $a, b \in \mathcal{B}$ and $t \in T$, we then have $\pi_t(e_{\mathcal{B}} + ahb) = \pi_t(e_{\mathcal{B}}) + \pi_t(a)\pi_t(h)\pi_t(b) = \pi_t(e_{\mathcal{B}}) = I_t$, where I_t is the identity operator on H_t. Thus $\pi_t(e_{\mathcal{B}} + ahb)$ is invertible in $\mathcal{L}(H_t)$ for every $t \in T$. Together with h, the element ahb belongs to $\mathcal{C}(\mathcal{B})$. Hence $\kappa(e_{\mathcal{B}} + ahb)$ is invertible in the quotient algebra $\mathcal{B}/\mathcal{C}(\mathcal{B})$. Theorem 2.7 now gives that $e_{\mathcal{B}} + ahb$ is invertible in $\mathcal{B}$. As $a, b \in \mathcal{B}$ were taken arbitrarily, we may conclude that h belongs to the radical of $\mathcal{B}$. The desired result is now immediate from the well-known fact that C^*-algebras are semi-simple. □

We continue by considering the finite rank elements in $\mathcal{B}$ in more detail. First we present a somewhat strengthened version of Proposition 6.48 in [32].

Theorem 2.9. *Let $t \in T$. Then π_t maps the finite rank elements in $\mathcal{J}_t$ in a one-to-one way onto the operators in $\mathcal{L}(H_t)$ having finite range dimension. Also, if $g \in \mathcal{J}_t$ is of finite rank, then* $\operatorname{rank} g = \dim \operatorname{Im} \pi_t(g)$.

Thus the restriction of π_t to $\mathcal{J}_t \cap \mathcal{C}_0(\mathcal{B})$ is an injective rank preserving mapping onto the set of operators in $\mathcal{L}(H_t)$ having finite range dimension.

Proof. First we mention what has been obtained in Proposition 6.48 from [32]: if $r \in \mathcal{J}_t$ is of finite rank, then $\pi_t(r) \in \mathcal{L}(H_t)$ has finite range dimension and $\operatorname{rank} r = \dim \operatorname{Im} \pi_t(r)$. Next we recall that π_t maps different elements of $\mathcal{J}_t$ into different operators in $\mathcal{L}(H_t)$. Thus what remains to be proved is this: given an operator $R \in \mathcal{L}(H_t)$ having finite dimensional range, there exists a finite rank element $r \in \mathcal{J}_t$ such that $\pi_t(r) = R$. Here is the argument.

Write R as a finite sum $R = R_1 + \cdots + R_n$ of operators in $\mathcal{L}(H_t)$ having range dimension one. These operators obviously belong to $\mathcal{K}(H_t)$ which is the image under π_t of $\mathcal{J}_t$. For $k = 1, \ldots, n$ choose $g_k \in \mathcal{J}_t$ with $\pi_t(g_k) = R_k$. Clearly g_k is a non-zero element of $\mathcal{B}$. Also, if $b \in \mathcal{B}$, there exists a scalar $\mu(b)$ such that $R_k \pi_t(b) R_k = \mu(b) R_k$, and this can be rewritten as $\pi_t(g_k b g_k) = \pi_t\big(\mu(b) g_k\big)$. The latter identity is trivially true when t is replaced by $s \in T$, $s \neq t$ because in that case both sides vanish by Theorem 2.5. From Theorem 2.8 it now follows that $g_k b g_k = \mu(b) g_k$. Thus g_k is a rank one element in $\mathcal{B}$. Put $r = g_1 + \cdots + g_n$. Then r is a finite rank element in $\mathcal{J}_t$ and $\pi_t(r) = R$, as desired. $\square$

Theorem 2.10 (*[32, Proposition 6.47]*)**.** *Let $r \in \mathcal{B}$ be of finite rank. Then there exist finite rank elements $g_t \in \mathcal{J}_t,\ t \in T$, such that*

(i) *there are only finitely many $t \in T$ for which g_t is non-zero,*
(ii) *$r = \sum_{t \in T} g_t$.*

The elements g_t are uniquely determined and $\operatorname{rank} r = \sum_{t \in T} \operatorname{rank} g_t$.

It is now possible to characterize the finite rank elements in $\mathcal{B}$ among those that are compact.

Theorem 2.11. *Let g be a compact element in $\mathcal{B}$. Then g is of finite rank if and only if the following conditions are satisfied:*

(i) *for every $t \in T$, the operator $\pi_t(g) \in \mathcal{L}(H_t)$ has finite range dimension,*
(ii) *there are only finitely many $t \in T$ for which $\pi_t(g) \in \mathcal{L}(H_t)$ is non-zero.*

In that case $\operatorname{rank} g = \sum_{t \in T} \dim \operatorname{Im} \pi_t(g)$.

Proof. First assume g is of finite rank. Then (i) and (ii), as well as the expression for the rank of g, can be directly obtained by combining Theorems 2.5, 2.9 and 2.10. It remains to prove that g is of finite rank whenever (i) and (ii) are fulfilled. This is the reasoning. For $t \in T$, the representation π_t maps $\mathcal{J}_t$ in a one-to-one manner onto $\mathcal{K}(H_t)$. Hence there is a unique $r_t \in \mathcal{J}_t$ of finite rank such that $\pi_t(r_t) = \pi_t(g)$. Clearly $r_t = 0$ in case $\pi_t(g) = 0$. Therefore there are only finitely many $t \in T$ for which r_t is non-zero. Define r as being equal to the finite sum $\sum_{t \in T} r_t$. Then r is of finite rank. As is easily verified $\pi_t(r) = \pi_t(g)$ for every $t \in T$. Since both g and r are compact, Theorem 2.8 gives $g = r$. Hence g is of finite rank. $\square$

We close this section with a result on Fredholm elements taken again from [32].

Theorem 2.12 (*[32, Theorem 6.46]*)**.** *Let a be a Fredholm element in $\mathcal{B}$. Then*

(i) *for every $t \in T$, the operator $\pi_t(a) \in \mathcal{L}(H_t)$ is a Fredholm operator,*
(ii) *there are only finitely many $t \in T$ for which $\pi_t(a) \in \mathcal{L}(H_t)$ is not invertible.*

Part (i) was already noted in connection with Theorem 2.2.

3. Finite rank idempotents and traces

Later on, we will draw considerably on material developed in [11]. This means that we need to pay attention to finite rank idempotents. Also we need to introduce traces for finite rank elements. This section serves to lay the groundwork for these points. We begin with a refinement of (the first part of) Theorem 2.9. Notations are as in the previous section. Following standard practice, idempotent bounded linear operators on Hilbert or Banach spaces are called projections.

Theorem 3.1. *Let $t \in T$. Then π_t maps the finite rank idempotents in $\mathcal{J}_t$ in a one-to-one way onto the projections in $\mathcal{L}(H_t)$ having finite range dimension.*

Proof. If q is an idempotent in $\mathcal{B}$, then $\pi_t(q)$ is one in $\mathcal{L}(H_t)$. From Theorem 2.9 it is now clear that π_t maps the finite rank idempotents in $\mathcal{J}_t$ in a one-to-one way into the idempotent operators in $\mathcal{L}(H_t)$ having finite range dimension. Let P be such an operator. Again by Theorem 2.9 there exists a finite rank element $p \in \mathcal{J}_t$ with $\pi_t(p) = P$. Both p and p^2 belong to $\mathcal{J}_t$ and $\pi_t(p^2) = \pi_t(p)^2 = P^2 = P = \pi_t(p)$. As π_t is injective on $\mathcal{J}_t$ it follows that $p^2 = p$, and with this the desired result is obtained. □

Theorem 3.2. *Let p be a compact element in $\mathcal{B}$. Then p is a finite rank idempotent if and only if the following conditions are satisfied:*

(i) *for every $t \in T$, the operator $\pi_t(p) \in \mathcal{L}(H_t)$ is an idempotent having finite range dimension,*
(ii) *there are only finitely many $t \in T$ for which $\pi_t(p) \in \mathcal{L}(H_t)$ is non-zero.*

Proof. For $t \in T$, we have that $\pi_t(p^2) = \pi_t(p)^2 = \pi_t(p)$. The 'only if part' of the theorem is now covered by that of Theorem 2.11. For the 'if part' we argue as follows. If (i) and (ii) are satisfied, then according to the 'if part' of Theorem 2.11, the element p is of finite rank. But then p^2 is of finite rank too. Also $\pi_t(p^2 - p) = 0$ for all $t \in T$. Applying Theorem 2.8, we get $p^2 = p$. □

In view of Theorem 3.2 it is natural to ask whether there can exist compact idempotents that fail to be of finite rank. The answer is negative.

Proposition 3.3. *If p is a compact idempotent in $\mathcal{B}$, then p is of finite rank.*

Proof. Take $t \in T$. Then $\pi_t(p)$ is an idempotent operator in $\mathcal{L}(H_t)$. Also $\pi_t(p)$ is a compact operator by Theorem 2.2. Hence $\pi_t(p)$ has finite range dimension. Choose $r \in \mathcal{C}_0(\mathcal{B})$ such that $\|p - r\| < 1$. As π_t, being a *-homomorphism, is a contraction, we have $\|\pi_t(p) - \pi_t(r)\| < 1$. From the 'only if part' of Theorem 2.11 we know that there are only finitely many $t \in T$ for which $\pi_t(r)$ is non-zero. In case $\pi_t(r)$ is the zero operator we have $\|\pi_t(p)\| < 1$ and, $\pi_t(p)$ being an idempotent, this implies that $\pi_t(p) = 0$. So there are only finitely many $t \in T$ for which $\pi_t(p)$ is non-zero. The 'if part' of Theorem 2.11 now gives that p is of finite rank. □

The unit element $e_{\mathcal{B}}$ in $\mathcal{B}$ is an idempotent. If $e_{\mathcal{B}}$ is compact (or, equivalently, of finite rank), then all elements of $\mathcal{B}$ are compact (and even of finite rank). The converse is also true of course. In fact this situation occurs if and only if $\mathcal{B}$ is finite dimensional or, what is well-known from general C^*-theory to amount to the same, $\mathcal{B}$ is *-isomorphic to an algebra of block matrices with given block size. Here is the precise formulation and its proof.

Proposition 3.4. *The unit element in $e_{\mathcal{B}}$ in $\mathcal{B}$ is compact (or, equivalently, of finite rank) if and only if $\mathcal{B}$ is C^*-isomorphic to a finite direct sum of C^*-algebras of the type $\mathbb{C}^{m \times m}$.*

The proof of the 'if part' of this proposition is a routine matter and left to the reader. The argument for the 'only if part' (which will be used to establish Proposition 3.10 and Theorem 3.12) is somewhat more involved.

Proof. Assume $e_{\mathcal{B}}$ is of finite rank, and let $t \in T$. Then $\pi_t : \mathcal{B} \to \mathcal{L}(H_t)$ is an irreducible representation. As $\pi(e_{\mathcal{B}})$ is the non-zero identity operator I_t on H_t, we have that $\pi_t[\mathcal{C}(\mathcal{B})] \neq \{0\}$. Hence $\pi_t[\mathcal{C}(\mathcal{B})] = \mathcal{K}(H_t)$ by Theorem 2.2. But then $I_t = \pi_t(e_{\mathcal{B}})$ is a compact operator on H_t, hence H_t is finite dimensional. Note that π_t is surjective as $\pi_t[\mathcal{J}_t] = \mathcal{K}(H_t) = \mathcal{L}(H_t)$.

By Theorem 2.10 there exist finite rank elements $e_t \in \mathcal{J}_t,\ t \in T$, such that there are only finitely many $t \in T$ for which $e_t \neq 0$ while, moreover, $e_{\mathcal{B}} = \sum_{t\in T} e_t$. Suppose T is not finite. Then there is an s in T with $e_s = 0$. Consider the rank one element $r_s \in \mathcal{J}_s$. Trivially $r_s e_s = 0$. From Corollary 2.6 it is now clear that $r_s e_t = 0$ for all $t \in T$. Hence $r_s = r_s e_{\mathcal{B}} = \sum_{t\in t} r_s e_t = 0$. But this is impossible since r_s is a rank one element in $\mathcal{B}$, and we can conclude that T is a finite set.

Consider the function π from $\mathcal{B}$ into the direct sum of the C^*-algebras $\mathcal{L}(H_t)$:

$$\pi : \mathcal{B} \to \bigoplus_{t\in T} \mathcal{L}(H_t), \qquad \pi(b)_t = \pi_t(b), \quad t \in T, \tag{3}$$

(i.e., the t-th coordinate of $\pi(b)$ in the direct sum is $\pi_t(b)$). Then, obviously, π is a *-homomorphism. By Theorem 2.8, the family $\{\pi_t : \mathcal{B} \to \mathcal{L}(H_t)\}_{t\in T}$ separates the points of $\mathcal{C}(\mathcal{B})$. But in the case considered here $\mathcal{C}(\mathcal{B}) = \mathcal{B}$. Thus the family $\{\pi_t : \mathcal{B} \to \mathcal{L}(H_t)\}_{t\in T}$ separates the points of $\mathcal{B}$, and this amounts to the same as saying that the *-homomorphism π is injective. It is also surjective. This follows in a straightforward manner from the surjectivity of the representations π_t and Theorem 2.5. The conclusion is that $\mathcal{B}$ is *-isomorphic to the (finite) direct sum featuring in (3). Of course one can identify $\mathcal{L}(H_t)$ with the C^*-algebra $\mathbb{C}^{m_t\times m_t}$, where m_t is the dimension of H_t. □

The following somewhat technical lemma will be used in Section 4.

Lemma 3.5. *Let $a, b \in \mathcal{B}$ with a Fredholm and $ab = ba = 0$ (hence b is an element of finite rank). Then there exist finite rank idempotents $p, q \in \mathcal{B}$ such that $pa = b(e_{\mathcal{B}} - p) = 0$ and $aq = (e_{\mathcal{B}} - q)b = 0$.*

As Fredholmness for elements of $\mathcal{B}$ amounts to the same as invertibility modulo the ideal $\mathcal{C}_0(\mathcal{B})$ of finite rank elements in $\mathcal{B}$, the lemma says that the collection of finite rank idempotents in $\mathcal{B}$ is a $\mathcal{C}_0(\mathcal{B})$-annihilating family of idempotents for the commuting zero divisors in $\mathcal{B}$. This terminology comes from [11].

Proof. As the element a is Fredholm, it is invertible modulo the ideal $\mathcal{C}_0(\mathcal{B})$ of finite rank elements in $\mathcal{B}$. It follows that $b \in \mathcal{C}_0(\mathcal{B})$. The existence of q is proved in the same way as that of p. Therefore we present the argument only for p.

Let $t \in T$. By Theorem 2.12(i), the operator $\pi_t(a)$ is Fredholm. Clearly $\pi_t(a)\pi_t(b) = 0$. Standard operator theory now guarantees the existence of an idempotent P_t in $\mathcal{L}(H_t)$ having finite range dimension and satisfying

$$P_t \pi_t(a) = \pi_t(b)(I_t - P_t) = 0. \tag{4}$$

For details, see Example 3.2 in [11]. By Theorem 3.1, there exists a unique finite rank idempotent $p_t \in \mathcal{J}_t$ such that $\pi_t(p_t) = P_t$. The identity (4) can now be rewritten as

$$\pi_t(p_t)\pi_t(a) = \pi_t(b)\big(I_t - \pi_t(p_t)\big) = 0. \tag{5}$$

According to Theorem 2.12(ii), there exists a finite subset T_0 of T such that for $t \in T \setminus T_0$, the operator $\pi_t(a)$ is invertible. Combining this with (4), we see that $\pi_t(p_t) = 0$, hence $p_t = 0$, for every $t \in T \setminus T_0$. This enables us to define $p \in \mathcal{B}$ by

$$p = \sum_{t \in T} p_t = \sum_{t \in T_0} p_t.$$

Then $\pi_t(p) = \pi_t(p_t)$, $t \in T$. Theorem 3.2 now gives that p is a finite rank idempotent. Also (5) can be rewritten as $\pi_t(pa) = \pi_t\big(b(e_{\mathcal{B}} - p)\big) = 0$. Along with p, the element pa is of finite rank. As has been observed already, the element b is of finite rank too and so is $b(e_{\mathcal{B}} - p)$. But then $pa = b(e_{\mathcal{B}} - p) = 0$ by Theorem 2.8. □

Operators having finite range dimension have a trace. In order to sensibly introduce such a notion for finite rank elements in the C^*-algebra $\mathcal{B}$, we need a supplement to Theorems 2.3 and 2.4.

Lemma 3.6. *Let $\mathcal{B}$ be a unital C^*-algebra, and let r be a rank one element in $\mathcal{B}$. Suppose H_1 and H_2 are Hilbert spaces, and let $\pi_1 : \mathcal{B} \to \mathcal{L}(H_1)$ and $\pi_2 : \mathcal{B} \to \mathcal{L}(H_2)$ be irreducible representations such that*

$$\pi_j[\mathcal{J}(r)] = \mathcal{K}(H_j), \qquad \operatorname{Ker} \pi_j \cap \mathcal{J}(r) = \{0\}, \quad j = 1, 2.$$

Then there exists a unitary isometry S from H_2 onto H_1 such that

$$\pi_2(b) = S^{-1}\pi_1(b)S, \quad b \in \mathcal{C}(\mathcal{B}).$$

We shall need the above identity only for the finite rank elements in $\mathcal{B}$, so for $b \in \mathcal{C}_0(\mathcal{B})$. Actually it holds for all $b \in \mathcal{B}$. This can be seen with the help of Theorem 5.7 in [32] for which there is a reference given to [20, 2.11.2 and 3.2.1].

Proof. Write $\mathcal{J}$ for the coinciding ideals $\mathcal{J}(r_1)$ and $\mathcal{J}(r_2)$. Also denote the restrictions of π_1 and π_2 to $\mathcal{J}$ by $\pi_{1,\mathcal{J}}$ and $\pi_{2,\mathcal{J}}$, respectively. Then $\pi_{1,\mathcal{J}} : \mathcal{J} \to \mathcal{K}(H_1)$ and $\pi_{2,\mathcal{J}} : \mathcal{J} \to \mathcal{K}(H_2)$ are surjective *-isomorphisms. Hence $\varrho = \pi_{2,\mathcal{J}}\pi_{1,\mathcal{J}}^{-1}$ is a *-isomorphism from $\mathcal{K}(H_1)$ onto $\mathcal{K}(H_2)$. By Theorem 5.11 in [32], for which there is a reference given to [20, 4.1.8] (see also [21, Corollary 5.43]), there exists a unitary isometry S from H_2 onto H_1 such that $\varrho(K) = S^{-1}KS$ for all K in $\mathcal{K}(H_1)$. The desired identity now follows by taking $K = \pi_1(b) \in \mathcal{K}(H_1)$ with $b \in \mathcal{C}(\mathcal{B})$. □

The trace of a square matrix M will be denoted by tr M, and the same notation is used for an operator M on a Hilbert (or Banach space) having finite range dimension. Now let $r \in \mathcal{C}_0(\mathcal{B})$ be an element in $\mathcal{B}$ of finite rank. We define the *trace* of r, written trace r, by the expression

$$\operatorname{trace} r = \sum_{t \in T} \operatorname{tr} \pi_t(r). \tag{6}$$

That this definition makes sense, we see from Theorem 2.11; that, in spite of the non-uniqueness of the Hilbert spaces H_t and the representations π_t, it is unambiguous, from Lemma 3.6. The trace on $\mathcal{C}_0(\mathcal{B})$ thus introduced is a linear functional which has the commutativity property that justifies the use of the term trace namely. Indeed, if $r \in \mathcal{C}_0(\mathcal{B})$ and b is an arbitrary element in $\mathcal{B}$, then trace $(br) =$ trace (rb). Note that the trace need not be continuous (cf., the situation for the C^*-algebra of all bounded linear operators on the Hilbert space ℓ_2).

For a projection on a Banach space having finite range dimension, the trace and rank coincide. So if p is a finite rank idempotent in $\mathcal{B}$, we have

$$\operatorname{tr} \pi_t(p) = \dim \operatorname{Im} \pi_t(p), \quad t \in T,$$

and it ensues that trace $p =$ rank p. In particular the traces of finite rank idempotents in $\mathcal{B}$ belong to $\mathbb{Z}_+$, the set of non-negative integers.

Next we turn to sums of finite rank idempotents. For matrices and operators on Banach spaces these are considered and characterized (via a rank-trace condition) in [33,44,6,7]. To get the matter at hand in a proper perspective, note that there are unital C^*-algebras where each element can be written as a finite sum of idempotents. An example is the C^*-algebra $\mathcal{L}(\ell_2)$: in [39] it is shown that each bounded linear operator on ℓ_2 is the sum of five idempotents in $\mathcal{L}(\ell_2)$.

Proposition 3.7. *Let $r \in \mathcal{B}$. If r is a finite sum of finite rank idempotents in $\mathcal{B}$, then r is a finite rank element in $\mathcal{B}$ and* rank $r \leq$ trace $r \in \mathbb{Z}_+$.

Proof. Suppose r is a sum of the finite rank idempotents $p_1, \ldots, p_n$. Then clearly r is a finite rank element in $\mathcal{B}$. For $t \in T$, the operator $\pi_t(r)$ is the sum of the projections $\pi_t(p_1), \ldots, \pi_t(p_n)$ in $\mathcal{L}(H_t)$ all having finite range dimension. Hence (see the references given above, [7] in particular)

$$\dim \operatorname{Im} \pi_t(r) \leq \operatorname{tr} \pi_t(r) \in \mathbb{Z}_+, \quad t \in T. \tag{7}$$

Combine this with the second statement in Theorem 2.11 and (6). □

The converse of Proposition 3.7 does not hold. So for a finite rank element r it may happen that rank $r \leq$ trace $r \in \mathbb{Z}_+$ while r is not a finite sum of idempotents. A simple counterexample can be constructed by considering the C^*-subalgebra of $\mathbb{C}^{2\times 2}$ consisting of the diagonal matrices.

We can do a little better with the following approach. Fix $t \in T$. For $r \in \mathcal{C}_0(\mathcal{B})$, introduce $\operatorname{trace}_t r = \operatorname{tr} \pi_t(r)$. Thus we obtain a family $\{\operatorname{trace}_t\}_{t \in T}$ of traces on the ideal $\mathcal{C}_0(\mathcal{B})$. The relationship with the trace introduced above is simple:

$$\operatorname{trace} r = \sum_{t \in T} \operatorname{trace}_t r, \quad r \in \mathcal{C}_0(\mathcal{B}). \tag{8}$$

For r a finite rank element in $\mathcal{B}$ and $t \in T$, we denote the (finite) range dimension $\dim \operatorname{Im} \pi_t(r)$ of $\pi_t(r)$ by $\operatorname{rank}_t r$.

Theorem 3.8. *Let $r \in \mathcal{B}$. The following statements are equivalent:*

(i) *$r \in \mathcal{C}_0(\mathcal{B})$ and* $\operatorname{rank}_t r \leq \operatorname{trace}_t r \in \mathbb{Z}_+$ *for every $t \in T$;*
(ii) *r is a finite sum of finite rank idempotents in $\mathcal{B}$;*
(iii) *r is a finite sum of rank one idempotents in $\mathcal{B}$.*

If r is a finite sum of rank one idempotents, the number of terms in the sum is equal to trace r.

Proof. The second part of assertion (i) is just a reformulation of (7). Thus Proposition 3.7 and its proof give the implication (ii) ⇒ (i). Obviously (iii) ⇒ (ii). It remains to prove that (iii) is a consequence from (i).

Take a finite rank element r in $\mathcal{B}$ and assume that (i), or what amounts to the same (7), is satisfied. Then (see the references preceding Proposition 3.7), the operators $\pi_t(r)$ are sums of projections in $\mathcal{L}(H_t)$. In fact these idempotents can be taken in such a way as to have range dimension one. Let T_0 be a finite subset of T such that $\pi_t(r) = 0$ for $t \in T \setminus T_0$. For $t \in T_0$, write $\pi_t(r) = P_{t,1} + \cdots + P_{t,n_t}$ with $P_{t,1} \ldots, P_{t,n_t}$ in $\mathcal{L}(H_t)$ projections having range dimension one. Combining Theorems 3.1 and 2.9, we see that there exists a unique rank one idempotent $p_{t,k} \in J_t$ such that $\pi_t(p_{t,k}) = P_{t,k}$. Now consider the sum of the elements $p_{t,k}$:

$$r_0 = \sum_{t \in T_0} \sum_{k=1}^{n_t} p_{t,k}.$$

Then r_0 is a sum of rank one idempotents in $\mathcal{B}$. For $s \in T_0$, we have $\pi_s(r_0) = \pi_s(r)$. Here we use Theorem 2.5. For $s \in T \setminus T_0$, we have that both $\pi_s(r_0)$ and $\pi_s(r)$ vanish. The upshot of this is that $\pi_t(r_0) = \pi_t(r)$ for all $t \in T$. But then $r = r_0$ by Theorem 2.8. □

Non-trivial zero sums of idempotents play an important role in [3,4] and also in the forthcoming paper [14]. When only finite rank idempotents are involved, such sums do not exist. In fact a somewhat stronger result holds.

Proposition 3.9. *Let n be a positive integer, let $p_1, \ldots, p_n$ be finite rank idempotents in $\mathcal{B}$, and assume the sum $p_1 + \cdots + p_n$ is a quasinilpotent. Then $p_1, \ldots, p_n$ are all equal to the zero element in $\mathcal{B}$ (and so, in fact, the sum $p_1 + \cdots + p_n$ vanishes).*

Proof. Put $s = p_1 + \cdots + p_n$. Then s is quasinilpotent and has finite rank. Take $t \in T$. If λ is a non-zero complex number, then $\lambda e_{\mathcal{B}} - s$ is invertible in $\mathcal{B}$, and hence $\lambda I_t - \pi_t(r) = \pi_t(\lambda e_{\mathcal{B}} - s)$ is invertible in $\mathcal{L}(H_t)$. Thus $\pi_t(s)$ is quasinilpotent. Also $\pi_t(s)$ has finite range dimension by Theorem 2.11. But then $\pi_t(s)$ is nilpotent and tr $\pi_t(s) = 0$. From the definition of the trace on $\mathcal{C}_0(\mathcal{B})$ it is now clear that trace $s = 0$. As was observed earlier, for finite rank idempotents, the rank and the trace coincide. This gives

$$\sum_{k=1}^{n} \text{rank } p_k = \sum_{k=1}^{n} \text{trace}\, p_k = \text{trace}\left(\sum_{k=1}^{n} p_k\right) = \text{trace } s = 0,$$

and it follows that $p_k = 0,\ k = 1, \ldots, n$. □

An idempotent $p \in \mathcal{B}$ is said to be of *finite co-rank* if the complementary idempotent $e_{\mathcal{B}} - p$ is of finite rank.

Proposition 3.10. *Let n be a positive integer, let $p_1, \ldots, p_n$ be idempotents of finite co-rank in $\mathcal{B}$, and assume the sum $p_1 + \cdots + p_n$ is quasinilpotent. Then $\mathcal{B}$ is *-isomorphic to an algebra of block matrices with given block size, (hence) all elements of $\mathcal{B}$ are of finite rank, and the idempotents $p_1, \ldots, p_n$ are all equal to the zero element in $\mathcal{B}$ (so, actually, the sum $p_1 + \cdots + p_n$ vanishes).*

Proof. Taking into account Propositions 3.4 and 3.9, it suffices to show that the unit element $e_{\mathcal{B}}$ of $\mathcal{B}$ is compact. Put $s = p_1 + \cdots + p_n$. Then s is quasinilpotent and, consequently, $ne_{\mathcal{B}} - s$ is invertible. On the other hand

$$ne_{\mathcal{B}} - s = \sum_{k=1}^{n} (e_{\mathcal{B}} - p_k)$$

is a finite rank element in $\mathcal{B}$. But then so is $e_{\mathcal{B}} = (ne_{\mathcal{B}} - s)^{-1}(ne_{\mathcal{B}} - s)$. □

The next theorem involving general Banach algebras is a generalization of Theorem 4.3 in [3]. The latter deals with a zero sum of four idempotents, so it corresponds to the case where the integer ν featuring below is a priori assumed to be equal to zero. For the proof of the theorem (inspired by the material in Section 3 of [23]) we refer to [14].

Theorem 3.11. *Let q_1, q_2, q_3 and q_4 be idempotents in a Banach algebra $\mathcal{A}$ with unit element $e_{\mathcal{A}}$, and let ν be a non-negative integer. If*

$$q_1 + q_2 + q_3 + q_4 + \nu e_{\mathcal{A}} = 0,$$

then $\nu = 0$ and $q_1 = q_2 = q_3 = q_4 = 0$.

In the C^*-setting considered here, Theorem 3.11 leads to the following result.

Theorem 3.12. *Let n be a positive integer, let $p_1, \ldots, p_n$ be idempotents in $\mathcal{B}$, and suppose the sum $p_1 + \cdots + p_n$ is compact. Then (precisely) one of the following statements holds:*

(a) *$p_1, \ldots, p_n$ are all of finite rank (hence so is their sum),*
(b) *$n \geq 5$ and at least five among the idempotents $p_1, \ldots, p_n$ are neither of finite rank nor of finite co-rank.*

It is worthwhile to say a few words on the situation when all the idempotents $p_1, \ldots, p_n$ are of finite co-rank. If that is the case and, in addition, $p_1 + \cdots + p_n$ is compact, then (a) holds, i.e., $p_1, \ldots, p_n$ are all of finite rank as well. It follows that $e_{\mathcal{B}} = (e_{\mathcal{B}} - p_1) + p_1$ is of finite rank, and we arrive at one of the conclusions also appearing in Proposition 3.10, namely that $\mathcal{B}$ is *-isomorphic to an algebra of block matrices with given block size (i.e., a direct sum of C^*-algebras of the type $\mathbb{C}^{m \times m}$). So a compact sum of finite co-rank idempotents can only occur in finite dimensional unital C^*-algebras.

Proof. First assume that each of the idempotents $p_1, \ldots, p_n$ is either of finite rank or of finite co-rank. Write k for the number of idempotents among $p_1, \ldots, p_n$ that are of finite co-rank. If $k = 0$, we have (a). So suppose k is at least one. Renumbering (if necessary), we can achieve the situation where $p_1, \ldots, p_k$ are of finite co-rank and $p_{k+1}, \ldots, p_n$ are of finite rank. Now

$$\sum_{j=1}^{k} p_k = \sum_{j=1}^{n} p_k - \sum_{j=k+1}^{n} p_k,$$

where the first sum in the right hand side is compact (by hypothesis) and the second of finite rank. So $p_1 + \cdots + p_k$ is compact. The idempotents $(e_{\mathcal{B}} - p_1), \ldots, (e_{\mathcal{B}} - p_k)$ are of finite rank. Further

$$e_{\mathcal{B}} = \frac{1}{k}\left(\sum_{j=1}^{k} p_k + \sum_{j=1}^{k} (e_{\mathcal{B}} - p_k)\right).$$

It follows that $e_{\mathcal{B}}$ is compact and Proposition 3.4 gives that $\mathcal{B}$ is *-isomorphic to an algebra of block matrices with given block size (cf., the remark made prior to the proof). Hence all elements of $\mathcal{B}$ are of finite rank and (a) holds in particular.

Next consider the case when among $p_1, \ldots, p_n$, there are idempotents which are neither of finite rank nor of finite co-rank. Let there be m of those. We may assume (renumbering if necessary) that $p_1, \ldots, p_m$ are of this type and (hence) $p_{m+1}, \ldots, p_n$ are not, i.e., they are of finite rank or finite co-rank. Let ν be the number of idempotents among $p_{m+1}, \ldots, p_n$ that are of finite co-rank, and suppose (without loss of generality) that $p_{m+1}, \ldots, p_{m+\nu}$ are of that kind. Then $p_{m+\nu+1}, \ldots, p_n$ are of finite rank. The same is true for $(e_{\mathcal{B}} - p_{m+1}), \ldots, (e_{\mathcal{B}} - p_{m+\nu})$, and it follows that $p_1 + \cdots + p_m + \nu e_{\mathcal{B}}$ is compact. Write κ for the canonical homomorphism from $\mathcal{B}$ onto the Calkin algebra $\mathcal{B}/\mathcal{C}(\mathcal{B})$ of $\mathcal{B}$. Then $\kappa(p_1), \ldots, \kappa(p_m)$ are idempotents in $\mathcal{B}/\mathcal{C}(\mathcal{B})$ and, with $\kappa(e_{\mathcal{B}})$ the (non-zero) unit element in $\mathcal{B}/\mathcal{C}(\mathcal{B})$,

$$\kappa(p_1) + \cdots + \kappa(p_m) + \nu\kappa(e_{\mathcal{B}}) = 0.$$

Now, if m is at most four, it follows from Theorem 3.11 that $\nu = 0$ and all of $\kappa(p_1), \ldots, \kappa(p_m)$ vanish. The latter means that all the idempotents $p_1, \ldots, p_m$ are compact. But then, by Proposition 3.3, they are of finite rank, which is impossible in view of how the number m has been introduced. So m (and a fortiori n) must be at least five, as claimed in (b). □

In the situation where the sum of idempotents in Theorem 3.12 is both compact and quasinilpotent (for instance because it vanishes), the conclusion of the theorem can be sharpened.

Theorem 3.13. *Let n be a positive integer, let $p_1, \ldots, p_n$ be idempotents in $\mathcal{B}$, and suppose the sum $p_1 + \cdots + p_n$ is compact and quasinilpotent. Then (precisely) one of the following statements holds:*

(a) $p_k = 0, \quad k = 1, \ldots, n$ *(so, in fact, the sum $p_1 + \cdots + p_n$ vanishes),*
(b) $n \geq 5$ *and at least five among the idempotents $p_1, \ldots, p_n$ are neither of finite rank nor of finite co-rank.*

Proof. Combine Theorem 3.12 and Proposition 3.9. □

Corollary 3.14. *Let p_1, p_2, p_3, p_4 and p_5 be idempotents in $\mathcal{B}$, not all equal to the zero element in $\mathcal{B}$, and assume $p_1 + p_2 + p_3 + p_4 + p_5 = 0$. Then all five idempotents p_1, p_2, p_3, p_4 and p_5 have both infinite rank and co-rank.*

The role of the number five in Theorems 3.12, 3.13 and Corollary 3.14 is directly related to that of the number four in Theorem 3.11. From [39], cited in the paragraph prior to Proposition 3.7, it is immediate that the number five in question cannot be replaced by a larger integer. Indeed, every bounded linear operator on ℓ_2 can be written as a sum of five idempotents in $\mathcal{L}(\ell_2)$ and so, in particular, there do exist zero sums of six idempotents in $\mathcal{L}(\ell_2)$ involving one idempotent of rank one. The significance of the number five in the present context is further underlined by the fact that there exist unital C^*-algebras featuring non-trivial zero sums of exactly five idempotents, all necessarily neither of finite rank nor of finite co-rank (Corollary 3.14). Until recently, essentially the only known example was the C^*-algebra $\mathcal{L}(\ell_2)$ of all bounded linear operators on ℓ_2 (see [4]). Meanwhile several other examples have been found; see [14].

We close this section with an analogue of Proposition 3.9 for selfadjoint idempotents.

Proposition 3.15. *Let $p_1, \ldots, p_n \in \mathcal{B}$ be selfadjoint idempotents, and assume that $p_1 + \cdots + p_n$ is quasinilpotent. Then $p_k = 0$ for each $k = 1, \ldots, n$.*

Proof. Put $r = p_1 + \cdots + p_n$. Then r is selfadjoint, so its spectral radius and norm coincide. As r is quasinilpotent, we may conclude that $r = 0$. For a selfadjoint idempotent p we have $p = p^2 = p^* p$, hence p is a nonnegative element in $\mathcal{B}$. It is a well-known fact that a sum of nonnegative elements in a C^*-algebra can only vanish when all terms do. □

4. Fredholm functions

When f is a function with values in a unital Banach algebra $\mathcal{A}$, the *resolvent* of f is the function f^{-1} given by the expression $f^{-1}(\lambda) = f(\lambda)^{-1}$. It is defined on the *resolvent set* of f, that is the set Res f of all λ in the domain of f for which $f(\lambda)$ is an invertible element in $\mathcal{A}$. If Res f is non-empty and f is analytic, then so is f^{-1}.

In the remainder of this section, $\mathcal{B}$ will be a unital C^*-algebra. Also notations are as in the preceding section. The following results contain analogues of material presented in Section XI.8 of [26] and Chapter 4 in [29]; see also the references given there, in particular the seminal paper [30].

Lemma 4.1. *Let f be a $\mathcal{B}$-valued function defined and analytic on an open neighborhood of $\mu \in \mathbb{C}$. Suppose there exists a finite rank element $r \in \mathcal{B}$ such that $f(\mu) + r$ is invertible. Also assume that μ is an accumulation point of* Res f. *Then f takes invertible values on a deleted neighborhood of μ.*

Proof. It is convenient to adopt the following notation: U_δ stands for open disc with center μ and radius δ.

Put $g(\lambda) = f(\lambda) + r$. Then g is analytic on an open neighborhood of μ and $g(\mu)$ is invertible. Hence there exists $\delta > 0$ such that $g(\lambda)$ is invertible for all $\lambda \in U_\delta$. Clearly, $f(\lambda) = g(\lambda) - r$ is a Fredholm element in $\mathcal{B}$ for these values of λ.

Let $t \in T$, and define the functions F_t and G_t on U_δ by $F_t(\lambda) = \pi_t\big(f(\lambda)\big)$ and $G_t(\lambda) = \pi_t\big(g(\lambda)\big)$. Then both F_t and G_t are analytic $\mathcal{L}(H_t)$-valued functions. Also the values of F_t are Fredholm operators, and those of G_t are invertible operators on H_t. Let T_0 be a finite subset of T such that $\pi_t(r) = 0$ for every $t \in T \setminus T_0$. For such t and $\lambda \in U_\delta$, one has $F_t(\lambda) = \pi_t\big(g(\lambda)\big) - \pi_t(r) = \pi_t\big(g(\lambda)\big) = G_t(\lambda)$, and hence F_t takes invertible values on U_δ. Next take $t \in T_0$. As Res $f \subset$ Res F_t, the latter set has μ as an accumulation point. But then it is known from the theory for Fredholm operator valued functions (see, e.g., [26, Section XI.8]) that F_t takes invertible values on a deleted neighborhood of the origin. In other words, there exists $\delta_t \in (0, \delta)$ such that $F_t(\lambda)$ is invertible for $\lambda \in U_{\delta_t} \setminus \{\mu\}$. Let ε be a positive real number not exceeding δ and δ_t, $t \in T_0$. Then $\pi_t\big(f(\lambda)\big) = F_t(\lambda)$ is invertible for every $t \in T$ and $\lambda \in U_\varepsilon \setminus \{\mu\}$. For these values of λ, the element $f(\lambda) \in \mathcal{B}$ is Fredholm too, and we see from Theorem 2.7 that $f(\lambda)$ is invertible. $\square$

A function will be called a *Fredholm function* (on a set D) if its values (on D) are Fredholm elements.

Theorem 4.2. *Let D be a non-empty connected open subset of the complex plane $\mathbb{C}$, let $f : D \to \mathcal{B}$ be an analytic Fredholm function, and assume* Res f *is non-empty. Then the following two statements hold:*

(i) *the set $D \setminus$ Res f of all λ in D for which $f(\lambda)$ is not invertible has no accumulation point in D (and is therefore at most countable);*
(ii) *at each point $\mu \in D \setminus$ Res f, the resolvent f^{-1} of f has a pole and the coefficients of the principal part of the Laurent expansion of f^{-1} at μ are finite rank elements in $\mathcal{B}$.*

Transferring terminology from the literature on analytic Fredholm operator valued functions (see, e.g., [26] or [29]) in a straightforward manner to the present situation, the conclusion of the theorem can be summarized by saying that the resolvent of f is finitely meromorphic on D.

Proof. Let D_0 be the set of all $\mu \in D$ such that f takes invertible values on some deleted neighborhood of μ. Then D_0 is non-empty because Res f is contained in D_0. Also D_0 is clearly an open subset of D. We shall presently prove that $D \setminus D_0$ is open too. Assuming this for the moment, the connectedness of D gives that D_0 is all of D. Thus for each μ in D, the function f takes invertible values on some deleted neighborhood of μ. This immediately gives (i).

Take $\lambda_0 \in D \setminus D_0$. We wish to see that there is an open neighborhood of λ_0 which is contained in $D \setminus D_0$. First we shall prove that there exists a finite rank element $r \in \mathcal{B}$ such that $f(\lambda_0) + r$ is invertible. For $t \in T$, let F_t be the function defined on D by $F_t(\lambda) = \pi_t\big(f(\lambda)\big)$. Then F_t is Fredholm operator valued by Theorem 2.12. As Res f is non-empty, so is Res F_t. This, together with the connectedness of D, gives that the values of F_t are Fredholm operators with index zero. In particular $F_t(\lambda_0)$ is Fredholm with index zero. But then there exists an operator R_t on H_t having finite range dimension such that $F_t(\lambda_0) + R_t$ is invertible. In view of Theorem 2.12(ii) we may assume that $R_t = 0$ for all but a finite number of $t \in T$. Now let r_t be the unique unique finite rank element in $\mathcal{J}_t$ such that $\pi_t(r_t) = R_t$. Then $r_t = 0$ for all but a finite number of elements in T. Thus it makes sense to put $r = \sum_{t \in T} r_t$. Clearly r is a finite rank element in $\mathcal{B}$ and $\pi_t(f(\lambda_0) + r) = F_t(\lambda_0) + R_t$ is invertible for every $t \in T$. Also $f(\lambda_0) + r$ is a Fredholm element in $\mathcal{B}$. It follows from Theorem 2.7 that $f(\lambda_0) + r$ is invertible.

As λ_0 is not in D_0, each deleted neighborhood of λ_0 contains points where f takes non-invertible values. Applying Lemma 4.1, we get that λ_0 is not an accumulation point of Res f. In other words, there is an open neighborhood V of λ_0 such that f takes non-invertible values on $V \setminus \{\lambda_0\}$. But then $f(\lambda_0)$ is non-invertible too. Clearly V is contained in $D \setminus D_0$.

We have now proved (i). So we turn to (ii). Take $\mu \in D \setminus \text{Res } f$. Then there is a deleted neighborhood of μ on which f takes invertible values. We only need to prove that f^{-1} has a pole at μ. The statement about the coefficients in the principal part of the Laurent expansion of f^{-1} at μ is then immediate from Lemma 2.5 in [11] since the Fredholm element $f(\mu) \in \mathcal{B}$ is invertible modulo the ideal $\mathcal{C}_0(\mathcal{B})$ of finite rank elements in $\mathcal{B}$.

Consider the Laurent expansion

$$f^{-1}(\lambda) = \sum_{k=-\infty}^{\infty} (\lambda - \mu)^k f_k,$$

of f^{-1} at μ, and write F for $f \circ \kappa$. Here, as before, κ is the canonical homomorphism from $\mathcal{B}$ onto the Calkin algebra $\mathcal{B}/\mathcal{C}(\mathcal{B})$. As f is Fredholm valued, the function F has invertible values. Also $F^{-1}(\lambda) = \kappa\big(f^{-1}(\lambda)\big)$ for $\lambda \in \text{Res } f$. Hence, for the Laurent expansion of F^{-1} at μ, we have

$$F^{-1}(\lambda) = \sum_{k=-\infty}^{\infty} (\lambda - \mu)^k \kappa(f_k) = \sum_{k=0}^{\infty} (\lambda - \mu)^k \kappa(f_k).$$

Thus $\kappa(f_k) = 0$, in other words $f_k \in \mathcal{C}(\mathcal{B})$, for all negative integers k.

Take $t \in T$. Then F_t has invertible values on a deleted neighborhood of μ, and the Laurent expansion of F_t^{-1} at μ has the form

$$F_t(\lambda)^{-1} = \sum_{k=-\infty}^{\infty} (\lambda - \mu)^k \pi_t(f_k).$$

Now $F_t(\mu)$ is invertible for all but a finite number of $t \in T$, and for these values of t, we have that $\pi_t(f_k) = 0$ for all negative integers k. For the other values of t, only a finite number, the situation is as follows. By the theory for Fredholm operator valued functions as presented in [26, Section XI.8], the point μ is not an essential singularity for F_t^{-1}. So there is a non-negative integer n_t such that $\pi_t(f_k) = 0$ for all integers k not exceeding $-n_t$. The upshot of all this is that for some non-negative integer n, one has $\pi_t(f_k) = 0$ for all $t \in T$ and for all integers k with

$k \le -n$. For these values of k, Theorem 2.8 now gives $f_k = 0$. To see that μ is a genuine pole of f^{-1} (i.e., that it has positive order), note that the non-invertibility of $f(\mu)$ implies that not all coefficients of the principal part of the Laurent expansion of f^{-1} at μ can vanish. □

Theorem 4.3. *Let D be a non-empty open subset of the complex plane $\mathbb{C}$, and let $f : D \to \mathcal{B}$ be an analytic Fredholm function. Suppose $D \setminus \operatorname{Res} f$ is finite, i.e., f takes invertible values on D except for a finite number of points where the poles of f^{-1} are located. Let $\alpha_1, \dots, \alpha_n$ be the poles of f^{-1}, in any order, but with pole orders taken into account. Then there exist analytic functions $g, h : D \to \mathcal{B}$ taking invertible values on all of D, and finite rank idempotents $p_1, \dots, p_n, q_1, \dots, q_n \in \mathcal{B}$, such that*

$$f(\lambda) = g(\lambda)\big(e_{\mathcal{B}} - p_1 + (\lambda - \alpha_1)p_1\big)\cdots\big(e_{\mathcal{B}} - p_n + (\lambda - \alpha_n)p_n\big), \quad \lambda \in D,$$
$$f(\lambda) = \big(e_{\mathcal{B}} - q_1 + (\lambda - \alpha_1)q_1\big)\cdots\big(e_{\mathcal{B}} - q_n + (\lambda - \alpha_n)q_n\big)h(\lambda), \quad \lambda \in D.$$

By the expression 'pole orders taken into account' we mean the following. If α is a pole of f^{-1} of order k, then the value α occurs precisely k times among $\alpha_1, \dots, \alpha_n$. Clearly, n is the sum of the orders of the poles of f^{-1}. In the scalar case $\mathcal{B} = \mathbb{C}$, the expressions involving the (non-zero) idempotents $p_1, \dots, p_n, q_1, \dots, q_n$ correspond to linear factors of the type $\lambda - \alpha$.

Proof. As already indicated earlier, in terms of [11] the content of Lemma 3.5 is that the collection of finite rank idempotents in $\mathcal{B}$ is a $\mathcal{C}_0(\mathcal{B})$-annihilating family of idempotents for the commuting zero divisors in $\mathcal{B}$. The assumption that f is a Fredholm function can be reformulated by saying that the values of f are invertible modulo $\mathcal{C}_0(\mathcal{B})$. The theorem is now a special case of Theorem 2.6 in [11]. □

Theorem 4.3 says that under the assumptions holding there, the function f is analytically equivalent on D to a *finite rank elementary polynomial*, i.e., one that is a product of factors of the form $e_{\mathcal{B}} - p + (\lambda - \alpha)p$ with p a finite rank idempotent in $\mathcal{B}$. Here analytic equivalence is taken in the sense of [28]; cf., [26, Chapter III]. In fact, in the theorem we even have one-sided equivalence, in the first expression for f with the equivalence function g only at the right, in the second expression with the equivalence function h only at the left. Allowing for equivalence functions both in the left and the right position, we enter the situation where f is analytically equivalent to a finite rank elementary polynomial in the middle. It is a remarkable fact that in the Fredholm operator case the middle term can be chosen to be of diagonal type involving mutually disjoint (commuting) projections, as indicated in [26, Chapter XI] and [31]. For the matrix case, things come down to what is called the Smith canonical form (see Chapter VI in [24], Chapter 7 in [36] or Section 4.3 in [29]). This canonical form is essentially unique and carries with it certain invariants. We consider it likely that an analogue can be obtained in the present situation but we will not pursue this issue here.

5. Logarithmic residues and spectral regularity

In this section we consider logarithmic residues of analytic functions and spectral regularity in a C^*-setting. These notions were mentioned in the introduction in a somewhat loose manner. Here we shall give the formal definitions and develop an adequate terminology.

A *spectral configuration* is a triple $(\mathcal{B}, \Delta, f)$ where $\mathcal{B}$ is a unital complex Banach algebra, Δ is a bounded Cauchy domain in $\mathbb{C}$ (see [42] or [26]) and f is a $\mathcal{B}$-valued analytic function on an open neighborhood of the closure of Δ which has invertible values on all of the boundary $\partial\Delta$

of Δ. With such a spectral configuration, taking $\partial\Delta$ to be positively oriented, one can associate the contour integral

$$LR(f;\Delta) = \frac{1}{2\pi i}\int_{\partial\Delta} f'(\lambda)f(\lambda)^{-1}d\lambda.$$

We call it the *logarithmic residue associated with* $(\mathcal{B}, \Delta, f)$; sometimes the term *logarithmic residue of* f *with respect to* Δ is used as well.

In the remainder of this section, as in the previous ones, $\mathcal{B}$ will be a unital C^*-algebra. Notations will be as before.

Theorem 5.1. *Let* $(\mathcal{B}, \Delta, f)$ *be a spectral configuration, and assume* f *is a Fredholm function on* Δ. *Then the logarithmic residue*

$$LR(f;\Delta) = \frac{1}{2\pi i}\int_{\partial\Delta} f'(\lambda)f(\lambda)^{-1}d\lambda$$

of f *with respect to* Δ *is a finite sum of finite rank idempotents in* $\mathcal{B}$. *In particular* $LR(f;\Delta)$ *is a finite rank element in* $\mathcal{B}$ *and the rank of* $LR(f;\Delta)$ *does not exceed the trace of* $LR(f;\Delta)$ *which is a nonnegative integer.*

In the conclusion of the theorem, finite rank idempotents in $\mathcal{B}$ may be replaced by rank one idempotents (see Theorem 3.8).

Proof. A routine argument based on the results obtained in the previous section gives that $LR(f;\Delta)$ is a finite rank element in $\mathcal{B}$. Here are the main ingredients of the argument. First note that $\Delta \setminus \operatorname{Res} f$ is a finite subset of Δ consisting of poles of f^{-1}. Next recall that at each such pole, the principal part of the Laurent expansion of f^{-1} has finite rank coefficients. Finally observe that the same is true when f^{-1} is replaced by the logarithmic derivative $f'f^{-1}$.

To finish the proof, it is sufficient to show that

$$\operatorname{rank}_t LR(f;\Delta) \le \operatorname{trace}_t LR(f;\Delta) \in \mathbb{Z}_+, \quad t \in T. \tag{9}$$

Indeed, once this has been established we can simply apply Theorem 3.8 to get that $LR(f;\Delta)$ is a finite sum of finite rank idempotents in $\mathcal{B}$, and Proposition 3.7 to obtain $\operatorname{rank} LR(f;\Delta) \le \operatorname{trace} LR(f;\Delta) \in \mathbb{Z}_+$.

For $t \in T$, put $F_t = \pi_t \circ f$. Then $(\mathcal{L}(H_t), \Delta, F_t)$ is a spectral configuration and $LR(F_t;\Delta) = \pi_t\big(LR(f;\Delta)\big)$. By Theorem 2.12, the values of F_t on Δ are Fredholm operators. From Theorem 3.4 in [7] we now see that the operator $LR(F_t;\Delta)$ has finite range dimension while $\dim \operatorname{Im} LR(F_t;\Delta) \le \operatorname{tr} LR(F_t;\Delta) \in \mathbb{Z}_+$. Rewriting this as $\dim \operatorname{Im} \pi_t\big(LR(f;\Delta)\big) \le \operatorname{tr} \pi_t\big(LR(f;\Delta)\big) \in \mathbb{Z}_+$ we arrive at (9). $\square$

We supplement Theorem 5.1 with the following comment. A finite sum of finite rank idempotents in $\mathcal{B}$ is always a logarithmic residue of some (entire) $\mathcal{B}$-valued Fredholm function. The proof is based on [22] and similar to the argument used in [7] to establish that statement (i) in [7, Theorem 3.4] implies statement (iv).

In the operator case, the trace of a logarithmic residue of a Fredholm operator function has an interpretation in terms of the algebraic multiplicity as defined in [26, Section XI.9] (cf. also Chapter 4 in [29] where the term index is used). As we shall see now, there is something of the same flavor in the present context (see the remark made just before the proof of Theorem 4.3).

Theorem 5.2. *Let* $(\mathcal{B}, \Delta, f)$ *be a spectral configuration, and suppose that on* Δ *the function* f *is represented in the form*

$$f(\lambda) = g(\lambda)\big(e_{\mathcal{B}} - p_1 + (\lambda - \alpha_1)p_1\big) \cdots \big(e_{\mathcal{B}} - p_n + (\lambda - \alpha_n)p_n\big)h(\lambda), \tag{10}$$

with $\alpha_1, \ldots, \alpha_n \in \Delta$ *(not necessarily distinct),* $p_1, \ldots, p_n$ *finite rank idempotents in* $\mathcal{B}$*, and* $g, h : \Delta \to \mathcal{B}$ *analytic functions taking invertible values on* Δ*. Then*

$$\operatorname{trace} LR(f; \Delta) = \sum_{k=1}^{n} \operatorname{rank} p_k. \tag{11}$$

As was already observed in the first part of the proof of Theorem 5.1, the set $\Delta \backslash \operatorname{Res} f$ is finite. Thus Theorem 4.3 guarantees that a representation of the type (10) does exist. For completeness we mention that, for each $t \in T$, the identity (11) also holds with trace and rank replaced by trace_t and rank_t, respectively.

Proof. A direct application of Theorem 5.1 in [11], where the ideal featuring there is taken to be $\mathcal{C}_0(\mathcal{B})$, gives $\operatorname{trace} LR(f; \Delta) = \sum_{k=1}^{n} \operatorname{trace} p_k$. Recall now from Section 3 that for finite rank idempotents in $\mathcal{B}$, the rank and the trace coincide. $\square$

The sum of the ranks appearing in (11) is an invariant for f (with respect to the Cauchy domain Δ) in the sense that it is obviously independent of the choice of the (non-unique) representation (10) of f. In the following analogue of Rouché's Theorem we prove that it is stable under small perturbations of f.

Theorem 5.3. *Let* $(\mathcal{B}, \Delta, f)$ *be a spectral configuration, and assume* f *is a Fredholm function on* Δ*. Further let* g *be a* $\mathcal{B}$*-valued function, defined and analytic on an open neighborhood of the closure of* Δ*, and suppose*

$$\max_{\lambda \in \partial\Delta} \|\big(g(\lambda) - f(\lambda)\big)f^{-1}(\lambda)\| < 1.$$

Then $(\mathcal{B}, \Delta, g)$ *is a spectral configuration,* g *is a Fredholm function on* Δ*, and* $\operatorname{trace} LR(g; \Delta) = \operatorname{trace} LR(f; \Delta)$.

In fact, the latter identity will follow from the fact that

$$\operatorname{trace}_t LR(g; \Delta) = \operatorname{trace}_t LR(f; \Delta), \quad t \in T. \tag{12}$$

Theorem 5.3 is of course inspired by the corresponding theorem for Fredholm operator valued functions in [30]; see also Theorem 9.2 in [26, Section XI.9] or Theorem 4.4.3 in [29].

Proof. Clearly $g(\lambda)$ is invertible for every $\lambda \in \partial\Delta$. Hence $(\mathcal{B}, \Delta, g)$ is a spectral configuration. Write $F = \kappa \circ f$ and $G = \kappa \circ g$ where, as before, κ is the canonical mapping from $\mathcal{B}$ onto the Calkin algebra $\mathcal{B}/\mathcal{C}(\mathcal{B})$. Then F takes invertible values on all of the closure of Δ and

$$\begin{aligned}\max_{\lambda \in \partial\Delta} \|\big(G(\lambda) - F(\lambda)\big)F^{-1}(\lambda)\| &= \max_{\lambda \in \partial\Delta} \|\kappa\Big(\big(g(\lambda) - f(\lambda)\big)f^{-1}(\lambda)\Big)\| \\ &\le \max_{\lambda \in \partial\Delta} \|\big(g(\lambda) - f(\lambda)\big)f^{-1}(\lambda)\| < 1.\end{aligned}$$

Applying the maximum principle (see, e.g., [29, Theorem 1.2.1]), it follows that

$$\max_{\lambda \in \Delta} \|\big(G(\lambda) - F(\lambda)\big)F^{-1}(\lambda)\| < 1,$$

and we may conclude that $\kappa\big(g(\lambda)\big) = G(\lambda)$ is invertible in $\mathcal{B}/\mathcal{C}(\mathcal{B})$ for every $\lambda \in \Delta$. But then g is a Fredholm function on Δ.

The logarithmic residues $LR(g; \Delta)$ and $LR(f; \Delta)$ are finite rank elements in $\mathcal{B}$. It remains to prove (12).

Take $t \in T$, and introduce $F_t = \pi_t \circ f$ and $G_t = \pi_t \circ g$. Then $(\mathcal{L}(H_t), \Delta, F_t)$ and $(\mathcal{L}(H_t), \Delta, G_t)$ are spectral configurations. Also the values of F_t and G_t on Δ are Fredholm operators. Recall that *-homomorphisms are always contractive. Hence

$$\max_{\lambda \in \partial\Delta} \|\big(G_t(\lambda) - F_t(\lambda)\big)F_t^{-1}(\lambda)\| \leq \max_{\lambda \in \partial\Delta} \|\big(g(\lambda) - f(\lambda)\big)f^{-1}(\lambda)\| < 1.$$

From Rouché's Theorem for the operator case referred to above, we now get that $LR(G_t; \Delta)$ and $LR(F_t; \Delta)$ are operators with finite range dimension while, moreover, $\operatorname{tr} LR(G_t; \Delta) = \operatorname{tr} LR(F_t; \Delta)$. Clearly

$$LR(G_t; \Delta) = \pi_t\big(LR(g; \Delta)\big), \qquad LR(F_t; \Delta) = \pi_t\big(LR(f; \Delta)\big),$$

and it follows that $\operatorname{tr} \pi_t\big(LR(g; \Delta)\big) = \operatorname{tr} \pi_t\big(LR(f; \Delta)\big)$. In view of our definition of trace_t, this is just the identity in (12). □

The spectral configuration $(\mathcal{B}, \Delta, f)$ is called *winding free* when the logarithmic residue $LR(f; \Delta) = 0$, *spectrally winding free* if $LR(f; \Delta)$ is quasinilpotent, and *spectrally trivial* in case f takes invertible values on all of Δ. This terminology is taken from [12].

Theorem 5.4. *Let $(\mathcal{B}, \Delta, f)$ be a spectral configuration, and assume f is a Fredholm function on Δ. The following statements are equivalent:*

(1) *$(\mathcal{B}, \Delta, f)$ is spectrally trivial;*
(2) *$(\mathcal{B}, \Delta, f)$ is winding free;*
(3) *$(\mathcal{B}, \Delta, f)$ is spectrally winding free.*

Proof. By Cauchy's Theorem 1 implies (2). Also (2) trivially gives (3). So we need to prove that (1) follows from (3). Recall that $\Delta \setminus \operatorname{Res} f$ is a finite subset of Δ consisting of the poles of f^{-1}. Hence, according to Theorem 4.3, the function f admits a representation on Δ of the form (10). (Even with one of the equivalence functions g and h being identically equal to the unit element in $\mathcal{B}$.) This gives the identity (11). Assume now that $LR(f; \Delta)$ is quasinilpotent. Combining Theorem 5.1 and Proposition 3.15, we see that $LR(f; \Delta) = 0$. But then $\sum_{k=1}^{n} \operatorname{rank} p_k = 0$, and we conclude that $p_k = 0$, $k = 1, \ldots, n$. Thus f is simply the product of the functions g and h. In particular f takes invertible values on all of Δ. □

We call a unital Banach algebra $\mathcal{A}$ *spectrally regular* if a spectral configuration having $\mathcal{A}$ as the underlying Banach algebra is spectrally trivial whenever it is spectrally winding free. The following result is a modification of Theorem 3.1 in [12]. The proof of the latter requires only slight adaptations to serve as an argument in the present context.

Theorem 5.5. *Let $\mathcal{A}$ be a unital Banach algebra. For ω in an index set Ω, let $\mathcal{B}_\omega$ be a spectrally regular Banach algebra, and let $\phi_\omega : \mathcal{B} \to \mathcal{B}_\omega$ be a continuous homomorphism (possibly non-unital). Further, for γ in an index set Γ, let $\mathcal{B}_\gamma$ be a C^*-algebra with unit element e_γ, and let $\psi_\gamma : \mathcal{B} \to \mathcal{B}_\gamma$ be a continuous homomorphism (possibly non-unital). Write $\mathcal{F}(\mathcal{B}_\gamma)$ for the set of*

Fredholm elements in $\mathcal{B}_\gamma$, and assume the following two inclusions hold:

(a) $\bigcap_{\omega\in\Omega} \operatorname{Ker}\phi_\omega \subset \bigcap_{\gamma\in\Gamma} \psi_\gamma^{-1}[\mathcal{F}(\mathcal{B}_\gamma) - \{e_\gamma\}]$,
(b) $\bigcap_{\gamma\in\Gamma} \operatorname{Ker}\psi_\gamma \subset \mathcal{R}(\mathcal{B})$,

where $\mathcal{R}(\mathcal{B})$ stand for the radical of $\mathcal{B}$. Then $\mathcal{B}$ is spectrally regular.

The following corollary relates to Theorem 5.5 in the same way as [12, Corollary 3.2] relates to [12, Theorem 3.1].

Corollary 5.6. *Let $\mathcal{A}$ be a closed subalgebra of $\mathcal{B}$, where (as before) $\mathcal{B}$ stands for a unital C^*-algebra with unit element $e_\mathcal{B}$. For ω in an index set Ω, let $\mathcal{B}_\omega$ be a spectrally regular Banach algebra, and let $\phi_\omega : \mathcal{A} \to \mathcal{B}_\omega$ be a continuous homomorphism. Write $\mathcal{F}(\mathcal{B})$ for the set of Fredholm elements in $\mathcal{B}$, and suppose*

$$\bigcap_{\omega\in\Omega} \operatorname{Ker}\phi_\omega \subset \mathcal{F}(\mathcal{B}) - \{e_\mathcal{B}\}. \tag{13}$$

Then $\mathcal{A}$ is spectrally regular.

The next theorem can be obtained as a simple consequence of Corollary 5.6. We prefer, however, to give a direct proof based on Theorem 5.4.

Theorem 5.7. *If $\mathcal{J}$ is a closed two-sided ideal contained in $\mathcal{C}(\mathcal{B})$ and the quotient algebra $\mathcal{B}/\mathcal{J}$ is spectrally trivial, then so is $\mathcal{B}$.*

It is not a priori required that $\mathcal{J}$ is closed under the *-operation in $\mathcal{B}$. However, by Proposition 1.8.2 in [20] it is, and hence the quotient algebra $\mathcal{B}/\mathcal{J}$ (endowed with the natural involutive structure and the quotient norm) is a C^*-algebra. In case $\mathcal{J} = \mathcal{B}$, i.e., the quotient algebra $\mathcal{B}/\mathcal{J}$ is trivial, we have $\mathcal{C}(\mathcal{B}) = \mathcal{B}$. Proposition 3.4 then gives that $\mathcal{B}$ is *-isomorphic to an algebra of block matrices with given block size, hence $\mathcal{B}$ is spectrally regular, as stated in the conclusion of the above theorem.

Proof. Let the spectral configuration $(\mathcal{B}, \Delta, f)$ be spectrally winding free. Writing ϱ for the canonical mapping from $\mathcal{B}$ onto $\mathcal{B}/\mathcal{J}$, we introduce the function $F = \varrho \circ f$. Then $(\mathcal{B}/\mathcal{J}, \Delta, F)$ is a spectral configuration. Along with $(\mathcal{B}, \Delta, f)$, the spectral configuration $(\mathcal{B}/\mathcal{J}, \Delta, F)$ is spectrally winding free (cf., the proof of Proposition 3.15). Thus we may conclude that F has invertible values on Δ. In other words, for each $\lambda \in \Delta$, the element $f(\lambda) \in \mathcal{B}$ is invertible modulo $\mathcal{J}$. But then $f(\lambda)$ is invertible modulo $\mathcal{C}(\mathcal{B})$ too. So f is a Fredholm function on Δ. Theorem 5.4 now gives that $(\mathcal{B}, \Delta, f)$ is spectrally trivial. □

The following example is placed in a broader context in Section 8.

Example 5.8. By way of illustration, consider the unital C^*-algebras generated by block Toeplitz operators appearing in [27, Sections XXXII.2 and XXXII.4]. Depending on the continuity requirements imposed on the so called defining (or generating) function, the algebras in question are denoted there by $\mathcal{T}_m(C)$ and $\mathcal{T}_m(PC)$. In fact, $\mathcal{T}_m(C)$ and $\mathcal{T}_m(PC)$ are the smallest closed subalgebra of $\mathcal{B}(\ell_2^m)$ containing all block Toeplitz operators for which the defining function is a continuous, respectively, a piecewise continuous, $\mathbb{C}^{m\times m}$-valued function. Theorem 5.7 can now be used to recover Theorem 4.14 in [12], which states that the C^*-algebras $\mathcal{T}_m(C)$ and $\mathcal{T}_m(PC)$ are all spectrally regular. The ingredients for a proof based on Theorem 5.7 can be found in [12]. There is no need to give the detailed argument here. □

Specializing in Theorem 5.7 to the case $\mathcal{J} = \mathcal{C}(\mathcal{B})$, one gets that $\mathcal{B}$ is spectrally regular whenever the Calkin algebra $\mathcal{B}/\mathcal{C}(\mathcal{B})$ has this property (as can be seen from Proposition 3.4 also true when $\mathcal{B}/\mathcal{C}(\mathcal{B})$ is trivial, i.e., $\mathcal{C}(\mathcal{B}) = \mathcal{B}$). We conjecture that it may happen that $\mathcal{B}$ is spectrally regular while the Calkin algebra $\mathcal{B}/\mathcal{C}(\mathcal{B})$ is not. An example showing this might be difficult to find. A complication is that the known supply of Banach algebras for which it is known that they fail to be spectrally regular is restricted. As a matter of fact, until now $\mathcal{L}(\ell_2)$ has been essentially the only example that appeared in the literature (cf., [3,4]). In the forthcoming paper [14], other Banach algebras failing to be spectrally regular are identified (see also Theorem 6.2 below). Nevertheless, the conjecture formulated above is still unconfirmed.

6. Simple C^*-algebras

In this short section, we consider the case when the unital C^*-algebra $\mathcal{B}$ is *simple*. The latter means that the only closed two-sided ideals of $\mathcal{B}$ are $\{0\}$ and $\mathcal{B}$.

Theorem 6.1. *Suppose the unital C^*-algebra $\mathcal{B}$ is simple. Then either $\mathcal{C}(\mathcal{B}) = \{0\}$ or $\mathcal{B}$ is *-isomorphic to the C^*-algebra $\mathbb{C}^{m\times m}$ for some positive integer m.*

Proof. Suppose $\mathcal{C}(\mathcal{B}) \neq \{0\}$. Then $\mathcal{C}(\mathcal{B}) = \mathcal{B}$, and we get from Proposition 3.4 that $\mathcal{B}$ is C^*-isomorphic to a finite direct sum of C^*-algebras of the type $\mathbb{C}^{m\times m}$. As $\mathcal{B}$ is simple, the number of terms in this direct sum cannot exceed one. □

Elaborating on Theorem 6.1, so assuming that $\mathcal{B}$ is simple, we note the following. In the situation where $\mathcal{B}$ is *-isomorphic to the C^*-algebra $\mathbb{C}^{m\times m}$, each element in $\mathcal{B}$ is Fredholm and $\mathcal{B}$ is spectrally regular. In the case when $\mathcal{C}(\mathcal{B}) = \{0\}$, Fredholmness in $\mathcal{B}$ amounts to nothing else than invertibility in $\mathcal{B}$, and so the main results of Sections 4 and 5 collapse into trivialities.

It can happen that in spite of being simple, a unital C^*-algebra fails to be spectrally regular. In fact this is the case for the so-called Cuntz algebras.

Theorem 6.2. *Cuntz algebras are not spectrally regular.*

The *Cuntz algebra* $\mathcal{O}_n$ is the universal unital C^*-algebra generated by n isometries $v_1, \ldots, v_n \in \mathcal{O}_n$ satisfying the identities

$$v_k^* v_l = \delta_{k,l} e_{\mathcal{O}_n}, \quad k, l = 1, \ldots, n, \qquad \sum_{j=1}^{n} v_j v_j^* = e_{\mathcal{O}_n},$$

where $e_{\mathcal{O}_n}$ is the unit element in $\mathcal{O}_n$. Here n is an integer, $n \geq 2$. The first to consider this algebra was Cuntz [18]. The Cuntz algebras are universal in the sense that for fixed n, any two concrete realization generated by isometries $v_1, \ldots, v_n$ and $\tilde{v}_1, \ldots, \tilde{v}_n$, respectively, are *-isomorphic to each other and that the isomorphism sends v_k into $\tilde{v}_k$ (cf., [18,19]). Cuntz algebras are infinite dimensional and simple.

Theorem 6.2 is true even for the weaker version of spectral regularity where only winding free spectral configurations are taken into account (see Section 5 above and Section 2 in [14]). For the proof we refer to [14]. As has always been the case up to now, the failure to be spectrally regular is brought to light via the construction of non-trivial (finite) zero sums of idempotents (cf., the paragraph prior to Proposition 3.9). No examples are known of Banach algebras lacking the property of being spectrally regular and having only trivial zero sums of idempotents.

Taking the opportunity to elaborate somewhat on the issue of non-trivial zero sums of idempotents, we make a few observations.

Proposition 6.3. *Let $\mathcal{B}$ be a unital C^*-algebra. If $\mathcal{B}$ allows for a non-trivial zero sum of idempotents, then so does the Calkin algebra $\mathcal{B}/\mathcal{C}(\mathcal{B})$.*

So in that situation $\mathcal{B}/\mathcal{C}(\mathcal{B})$ is not spectrally regular (along with $\mathcal{B}$). Also $\mathcal{C}(\mathcal{B})$ cannot be equal to $\mathcal{B}$ (hence the Calkin algebra $\mathcal{B}/\mathcal{C}(\mathcal{B})$ is non-trivial). Indeed, if $\mathcal{C}(\mathcal{B}) = \mathcal{B}$, the algebra $\mathcal{B}$ does not allow for non-trivial zero sums of idempotents. This is clear from the proof below; see also Proposition 3.4, which says that under these circumstances $\mathcal{B}$ is C^*-isomorphic to a finite direct sum of C^*-algebras of the type $\mathbb{C}^{m\times m}$.

Proof. Let $p_1, \ldots, p_m$ be idempotents in $\mathcal{B}$, not all zero, which add up to the zero element in $\mathcal{B}$ (m necessarily at least five), and write κ for the canonical mapping of $\mathcal{B}$ onto $\mathcal{B}/\mathcal{C}(\mathcal{B})$. Then $\kappa(p_1), \ldots, \kappa(p_m)$ are idempotents in $\mathcal{B}/\mathcal{C}(\mathcal{B})$ adding up to the zero element in $\mathcal{B}/\mathcal{C}(\mathcal{B})$. Suppose $\mathcal{B}/\mathcal{C}(\mathcal{B})$ does not allow for a non-trivial zero sum of idempotents. Then $\kappa(p_k) = 0$, $k = 1, \ldots, m$. Thus $p_1, \ldots, p_m$ are compact idempotents in $\mathcal{B}$. But then, by Proposition 3.3, the idempotents $p_1, \ldots, p_m$ are finite rank elements in $\mathcal{B}$. So $p_1, \ldots, p_m$ are the terms in a zero sum of finite rank idempotents in $\mathcal{B}$. Proposition 3.9 now gives that $p_1, \ldots, p_m$ are all zero, contrary to our assumption. □

Specializing to the case $\mathcal{B} = \mathcal{L}(\ell_2)$, in which we have $\mathcal{C}(\mathcal{B}) = \mathcal{K}(\ell_2)$ we obtain the following corollary.

Corollary 6.4. *The Calkin algebra $\mathcal{L}(\ell_2)/\mathcal{K}(\ell_2)$ allows for non-trivial zero sums of idempotents; hence it is not spectrally regular.*

As is well-known, $\mathcal{L}(\ell_2)/\mathcal{K}(\ell_2)$ is simple. So besides the Cuntz algebra featuring above, we have another instance here of a simple unital C^*-algebra which fails to be spectrally regular.

7. Examples

In order to give an idea in what situations the results obtained in Sections 4 and 5 apply, we are now going to consider a couple of concrete C^*-algebras and analyze what the Fredholm theory means in these specific examples. Besides the issue of Fredholmness, it is of interest to identify the set of equivalence classes T along with the corresponding representations $\pi_t : \mathcal{B} \to \mathcal{L}(H_t)$ and the ideas $\mathcal{J}_t$, $t \in T$. Moreover, we will describe the set of finite rank and compact elements. With this information and the results in Section 3, it is then possible to characterize the rank one (or finite rank) idempotents and their finite sums. Furthermore, our main results from Sections 4 and 5 can then be specialized to the concrete situation in question. We refrain from giving all the pertinent details. It is a straightforward matter to fill them in.

Our first example makes the connection with the archetypical C^*-situation.

Example 7.1. Let $\mathcal{B} = \mathcal{L}(H)$ with H a Hilbert space. Then all relevant notions (rank one element, finite rank element, compact element, Calkin algebra, Fredholm element, rank and trace) coincide with the corresponding concepts from operator theory. Also T is a singleton, i.e., $T = \{t\}$ where t stands for the set of all bounded linear operators on H having range dimension one. The Hilbert space H_t and the irreducible representation π_t associated with t can be chosen to be H and the identity map on $\mathcal{L}(H)$, respectively. □

It is possible that a unital C^*-algebra $\mathcal{B}$ has no non-zero finite rank elements, i.e., $\mathcal{C}_0(\mathcal{B}) = \{0\} = \mathcal{C}(\mathcal{B})$. In that situation Fredholmness amounts to invertibility, T is the empty set, and our main results become trivialities. Here is an example (see also Theorem 6.1).

Example 7.2. Let $\mathcal{B} = C(\mathbb{T})$ be the C^*-algebra of all continuous complex-valued functions on the unit circle $\mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}$ provided with the sup-norm. In this case there are no rank one elements. □

Let us now consider unital C^*-algebras which may have rank one elements. The following ℓ_∞-type C^*-algebras were already considered in [12, Section 4.2] and [13] in connection with the issue of spectral regularity,.

Example 7.3. Let Ω be a non-empty set, and let $\mathbf{B} = \{\mathcal{B}_\omega\}_{\omega\in\Omega}$ be a family of unital C^*-algebras. By $\|\cdot\|_\omega$ we denote the norm on $\mathcal{B}_\omega$, and we write $\ell_\infty^{\mathbf{B}}$ for the ℓ_∞-direct product of the family $\mathbf{B}$ (cf., [38, Subsection 1.3.1]). Thus $\ell_\infty^{\mathbf{B}}$ consists of all $\boldsymbol{f}$ in the Cartesian product $\prod_{\omega\in\Omega} \mathcal{B}_\omega$ such that

$$|\!|\!|\boldsymbol{f}|\!|\!| = \sup_{\omega\in\Omega} \|\boldsymbol{f}(\omega)\|_\omega < \infty.$$

With the operations of addition, scalar multiplication and multiplication defined pointwise, and with $|\!|\!|\cdot|\!|\!|$ as norm, $\ell_\infty^{\mathbf{B}}$ is a unital C^*-algebra.

An element $\boldsymbol{f} \in \ell_\infty^{\mathbf{B}}$ is of rank one if and only if there exists a (unique) $\omega_0 \in \Omega$ such that $\boldsymbol{f}(\omega_0)$ is a rank one element in $\mathcal{B}_{\omega_0}$ and, in addition, $\boldsymbol{f}(\omega) = 0$ whenever $\omega \neq \omega_0$. The finite rank elements in $\ell_\infty^{\mathbf{B}}$ are those $\boldsymbol{f} \in \ell_\infty^{\mathbf{B}}$ for which $\boldsymbol{f}(\omega) \in \mathcal{C}_0(\mathcal{B}_\omega)$ for all $\omega \in \Omega$ and, in addition, $\boldsymbol{f}(\omega) = 0$ for all but finitely many $\omega \in \Omega$. In that case rank $\boldsymbol{f}$, respectively trace $\boldsymbol{f}$, is the (finite) sum of the ranks, respectively traces, which the elements $\boldsymbol{f}(\omega) \neq 0$ have as finite rank elements in the corresponding C^*-algebras $\mathcal{B}_\omega$. An element $\boldsymbol{f} \in \ell_\infty^{\mathbf{B}}$ is compact if and only if $\boldsymbol{f}(\omega) \in \mathcal{C}(\mathcal{B}_\omega)$ for each $\omega \in \Omega$ and, in addition, for every $\varepsilon > 0$ there exists a a finite subset F of Ω (depending on ε) with $\|\boldsymbol{f}(\omega)\|_\omega < \varepsilon$ for each $\omega \in \Omega \setminus F$.

An element $\boldsymbol{f} \in \ell_\infty^{\mathbf{B}}$ is Fredholm if and only if $\boldsymbol{f}(\omega) \in \mathcal{F}(\mathcal{B}_\omega)$ for each $\omega \in \Omega$ and, in addition, there exists a finite set $F \subset \Omega$ such that $\boldsymbol{f}(\omega)$ is invertible for all $\omega \in \Omega \setminus F$ and

$$\sup_{\omega\in\Omega\setminus F} \|\boldsymbol{f}(\omega)^{-1}\|_\omega < \infty. \tag{14}$$

Employing a notation which in the present context is self-evident, the set T can be identified with the set of all pairs (ω, t) with $\omega \in \Omega$ such that $\mathcal{C}(\mathcal{B}_\omega) \neq \{0\}$ and $t \in T_\omega$. The corresponding irreducible representations are given by the expression $\pi_{(\omega,t)}(\boldsymbol{f}) = \pi_t\big(\boldsymbol{f}(\omega)\big)$ and $\mathcal{J}_{(\omega,t)}$ consists of all $\boldsymbol{f} \in \ell_\infty^{\mathbf{B}}$ having $\boldsymbol{f}(\omega)$ as its sole possibly non-zero value which is an element of $\mathcal{J}_t \subset \mathcal{B}_\omega$. □

The previous rather general example can be specialized to more concrete situations which, besides occurring in [12,13], feature prominently in the numerical analysis [16,40] of bounded linear operators of ℓ_2.

Example 7.4. Consider the case when $\mathcal{B}_\omega = \mathbb{C}^{m_\omega\times m_\omega}$, where the m_ω are positive integers. Then $\mathcal{C}_0(\mathcal{B}_\omega) = \mathcal{C}(\mathcal{B}_\omega) = \mathcal{F}(\mathcal{B}_\omega) = \mathcal{B}_\omega = \mathbb{C}^{m_\omega\times m_\omega}$. So in this situation the finite rank elements in $\ell_\infty^{\mathbf{B}}$ are those $\boldsymbol{f} \in \ell_\infty^{\mathbf{B}}$ for which $\boldsymbol{f}(\omega) = 0$ for all but finitely many $\omega \in \Omega$, and rank $\boldsymbol{f}$, respectively trace $\boldsymbol{f}$, is the (finite) sum of the ranks, respectively traces, which the elements $\boldsymbol{f}(\omega) \neq 0$ have as matrices of the appropriate size. An element $\boldsymbol{f} \in \ell_\infty^{\mathbf{B}}$ is compact if and only if for every $\varepsilon > 0$ there exists a a finite subset F of Ω (depending on ε) with $\|\boldsymbol{f}(\omega)\|_\omega < \varepsilon$ for each $\omega \in \Omega \setminus F$. Mimicking a standard notation, this can be rephrased by saying that $\boldsymbol{f}$ belongs to $c_0^{\mathbf{B}}$. An element $\boldsymbol{f} \in \ell_\infty^{\mathbf{B}}$ is Fredholm if and only if $\boldsymbol{f}(\omega)$ is invertible for all but finitely many $\omega \in \Omega$ and the condition (14) is satisfied. The set T can be identified with the index set Ω. Given $\omega \in \Omega$, one can take $\mathbb{C}^{m_\omega}$ for the Hilbert space H_ω, and the coordinate mapping

$$\ell_\infty^{\mathbf{B}} \ni \boldsymbol{f} \mapsto \boldsymbol{f}(\omega) \in \mathbb{C}^{m_\omega \times m_\omega} = \mathcal{B}_\omega$$

for the irreducible representation $\pi_\omega : \ell_\infty^{\mathbf{B}} \to \mathcal{L}(H_\omega)$. Here, of course, $\mathcal{L}(H_\omega) = \mathcal{L}(\mathbb{C}^{m_\omega})$ is identified with $\mathcal{B}_\omega = \mathbb{C}^{m_\omega \times m_\omega}$. Finally, $\mathcal{J}_\omega$ consists of those $\boldsymbol{f} \in \ell_\infty^{\mathbf{B}}$ having the singleton set $\{\omega\}$ as support.

We can specialize this example further by taking $\Omega = \mathbb{N}$ and $m_k = k$. Thus $\mathcal{B}_k = \mathbb{C}^{k \times k}$ for $k \in \mathbb{N}$. Then $\boldsymbol{f}$ is Fredholm in $\ell_\infty^{\mathbf{B}}$ if and only if $\boldsymbol{f}$ considered as a sequence $(\boldsymbol{f}(k))_{k \in \mathbb{N}}$ is stable. The latter means that there exists an $n_0 \in \mathbb{N}$ such that $\boldsymbol{f}(k)$ is invertible for all $k \geq n_0$ and in addition $\sup_{k \geq n_0} \|\boldsymbol{f}(k)^{-1}\| < \infty$. □

8. C^*-algebras generated by one non-unitary isometry

Let $\mathcal{B}$ be a C^*-algebra with unit element $e_\mathcal{B}$. We say that $\mathcal{B}$ is *generated by a non-unitary isometry* v if $v \in \mathcal{B}$ is a *non-unitary isometry*, which by definition means that $v^*v = e_\mathcal{B} \neq vv^*$, and $\mathcal{B}$ coincides with the smallest C^*-subalgebra of $\mathcal{B}$ containing the unit elements v, v^* and $e_\mathcal{B}$.

Given such a C^*-algebra generated by the non-unitary isometry v, let us introduce the element $p_1 = e_\mathcal{B} - vv^*$, which is a non-zero, selfadjoint idempotent. Because $v^*p_1 = p_1v^* = 0$ it is easy to see that the set of all elements of the form

$$\alpha_0 e_\mathcal{B} + \sum_{k=1}^{N} \left(\alpha_k v^k + \alpha_{-k} (v^*)^k \right) + \sum_{j,k=0}^{N} \beta_{jk} v^j p_1 (v^*)^k \tag{15}$$

with $\alpha_k, \beta_{jk} \in \mathbb{C}$ and $N \in \mathbb{N}$ forms an algebra. Since $\mathcal{B}$ must contain all elements of the form (15) and since $\mathcal{B}$ is generated by v, v^* and $e_\mathcal{B}$, it is clear that $\mathcal{B}$ is the closure of the set of all elements (15). Moreover, the set of all elements which take the form of the last term (double sum) in the above expression form a *-ideal. Thus, the closure of the set

$$\left\{ \sum_{j,k=0}^{N} \beta_{jk} v^j p (v^*)^k \mid \beta_{jk} \in \mathbb{C}, \; N \in \mathbb{N} \right\}$$

is a *-ideal in $\mathcal{B}$. In fact, it is the smallest closed ideal in $\mathcal{B}$ containing p_1. We shall denote it by $\mathcal{J}_1$. Introducing another notation, we define a map T_v which sends a trigonometric polynomial $\sum_{k=-N}^{N} \alpha_k \tau^k$, $\tau \in \mathbb{T}$, defined on the unit circle $\mathbb{T}$, into an element of $\mathcal{B}$:

$$T_v : \sum_{k=-N}^{N} \alpha_k \tau^k \mapsto \alpha_0 e_\mathcal{B} + \sum_{k=1}^{N} \left(\alpha_k v^k + \alpha_{-k} (v^*)^k \right).$$

These notations are needed for the further analysis given below. But before we turn to that, we mention that an example of the type of C^*-algebra considered here is the Toeplitz algebra $\mathcal{T}_1(C) \subset \mathcal{L}(\ell_2)$; see Example 5.8. In this case, the non-unitary isometry v is given by the simple forward shift,

$$V : \{x_n\}_{n=0}^{\infty} \in \ell_2 \mapsto \{0, x_0, x_1, x_2, \ldots\} \in \ell_2.$$

The following results are known; see [17,25], or [40, Sections 4.23–4.25], or [27, Section XXXII.1].

Theorem 8.1. *Let $\mathcal{B}$ be a unital C^*-algebra generated by a non-unitary isometry v.*

(a) *The map T_v extends by continuity to an isometry $T_v : C(\mathbb{T}) \to \mathcal{B}$, which is multiplicative modulo $\mathcal{J}_1$, i.e., $T_v(ab) - T_v(a)T_v(b) \in \mathcal{J}_1$ for all $a, b \in C(\mathbb{T})$.*

(b) *There exists a unique *-homomorphism from $\mathcal{B}$ into $C(\mathbb{T})$, denoted by smb_v, whose kernel is $\mathcal{J}_1$ and for which $\mathrm{smb}_v \circ T_v$ is the identity map on $C(\mathbb{T})$.*

(c) *There exists a unique *-isomorphism $\pi : \mathcal{B} \to \mathcal{T}_1(C)$ satisfying $\pi(v) = V$. This *-isomorphism maps $\mathcal{J}_1$ onto the ideal $\mathcal{K}(\ell_2)$ of compact operators on ℓ_2 which is contained in $\mathcal{T}_1(C)$ (universality).*

Item (c) implies that two unital C^*-algebra generated by a non-unitary isometry are always *-isomorphic (universality property).

The expression smb appearing in (b) is the abbreviation of the word symbol, commonly featuring in material concerning Toeplitz operators. In view of the above characterizations it is clear that smb_v maps elements of the form (15) into the trigonometric polynomial $\sum_{k=-N}^{N} \alpha_k \tau^k$, $\tau \in \mathbb{T}$. Also, the statements (a) and (b) can be summarized by saying that the diagram

$$0 \longrightarrow \mathcal{J} \xrightarrow{\text{id}} \mathcal{B} \underset{T_v}{\overset{\mathrm{smb}_v}{\rightleftarrows}} C(\mathbb{T}) \longrightarrow 0$$

is a short exact sequence with a continuous cross-section T_v. It follows that one has the decomposition $\mathcal{B} = T_v(C(\mathbb{T})) \dotplus \mathcal{J}_1$. Furthermore, we obtain that $\mathcal{B}/\mathcal{J}$ is *-isomorphic to $C(\mathbb{T})$ and that the isomorphism and its inverse can be defined with the help of smb_v and T_v.

Let us now discuss the Fredholm theory of the unital C^*-algebras generated by a non-unitary isomorphy. First of all p_1 is a rank one element. One way to see is this is to multiply the elements of the form (15) with p_1 both from the left and the right, and to carry out the appropriate computations. Another approach uses the isomorphism π with the Toeplitz algebra. Then $\pi(p_1) = \pi(e_{\mathcal{B}} - vv^*) = I - VV^*$ is a projection in $\mathcal{L}(\ell_2)$ with range dimension one. Hence it is a rank one element in the subalgebra $\mathcal{T}_1(C)$, and this implies that p_1 is a rank one element in $\mathcal{B}$. As noted before, p_1 generates $\mathcal{J}_1$. A possible representation corresponding to $\mathcal{J}_1$ is the map π introduced in (c) of the above theorem. Indeed, this is an isomorphism between the ideals $\mathcal{J}_1$ and $\mathcal{K}(\ell_2)$.

Next we argue that up to equivalence p_1 is the only rank one element in $\mathcal{B}$, i.e., the set T is a singleton. One way to see this is again by passing to the (isometric) Toeplitz algebra. It is known from [17] that $\mathcal{K}(\ell_2)$ is the minimal ideal there. The latter means that if there were another ideal $\mathcal{J}_2$ and $\mathcal{J}_2 \neq \{0\}$, then $\mathcal{J}_2 \supset \mathcal{K}(\ell_2)$. Since we know that $\mathcal{J}_1$ and $\mathcal{J}_2$ corresponding to different equivalence classes in T have only the zero element in common, it follows that T is indeed a singleton (see Theorems 2.4 and 2.5).

Another (perhaps easier) argument is as follows. Since π is a representation corresponding to $\mathcal{J}$ (see Theorem 2.3 and Lemma 3.6), it follows that another ideal $\mathcal{J}_2 \neq \mathcal{J}_1$ generated by a non-equivalent rank one element p_2 must annihilate the corresponding ideal and in particular the rank one element p_2. That is $\pi(p_2) = 0$. But π is an isomorphism on all of $\mathcal{B}$ and we obtain $p_2 = 0$, which is a contradiction.

As we have seen T is a singleton set. The unique corresponding ideal is $\mathcal{J}_1$ and the corresponding representation is π. Using this, we can now characterize Fredholmness in $\mathcal{B}$ as follows: an element $a \in \mathcal{B}$ is Fredholm if and only if $\mathrm{smb}_v(a) \in C(\mathbb{T})$ is invertible in $C(\mathbb{T})$. Notice that the set of all compact elements is $\mathcal{C}(\mathcal{B}) = \mathcal{J}_1$.

The observations made above combined with Theorem 5.7 and the fact that commutative Banach algebras are spectrally regular (see see [2,4] or [5]), immediately give the following result.

Theorem 8.2. *If $\mathcal{B}$ is a unital C^*-algebra generated by a non-unitary isometry, then $\mathcal{B}$ is spectrally regular.*

We will now use the results on C^*-algebras generated by a non-unitary isometry in order to construct more elaborate examples.

Example 8.3. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two C^*-algebras with unit elements e_1 and e_2 generated by the non-unitary isometries $v_1 \in \mathcal{B}_1$ and $v_2 \in \mathcal{B}_2$, respectively. Consider the C^*-algebra $\mathcal{B}_1 \times \mathcal{B}_2$ with component-wise algebraic operations and the maximum norm. The unit element in $\mathcal{B}_1 \times \mathcal{B}_2$ is of course $e = (e_1, e_2)$. Now let $\mathcal{B}$ be the smallest unital C^*-subalgebra of $\mathcal{B}_1 \times \mathcal{B}_2$ containing the element $w = (v_1, v_2^*)$ and (hence) the element $w^* = (v_1^*, v_2)$. Below we will give two examples of concrete realizations of this C^*-algebra.

Write $p_1 = e_1 - v_1 v_1^*$ and $p_2 = e_2 - v_2 v_2^*$. Then p_1 and p_2 are rank one idempotents in $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. The *-ideals in $\mathcal{B}_1$ and $\mathcal{B}_2$ generated by p_1 and p_2, respectively, will be denoted by $\mathcal{K}_1$ and $\mathcal{K}_2$. Notice that q_1 and q_2, given by

$$q_1 = (p_1, 0) = e - ww^*, \qquad q_2 = (0, p_2) = e - w^*w,$$

are rank one idempotents in $\mathcal{B}_1 \times \mathcal{B}_2$ and hence in $\mathcal{B}$. Let us define

$$\mathcal{J}_1 = \mathcal{K}_1 \times \{0\}, \qquad \mathcal{J}_2 = \{0\} \times \mathcal{K}_2,$$

which are ideals in $\mathcal{B}_1 \times \mathcal{B}_2$, and as we will see also in $\mathcal{B}$.

Before we proceed, let us state a characterization of $\mathcal{B}$, which is proved below. For a function $\phi \in C(\mathbb{T})$, denote by $\phi^\sim \in C(\mathbb{T})$ the function $\phi^\sim(t) = \phi(t^{-1})$, $t \in \mathbb{T}$. The map $\phi \mapsto \phi^\sim$ is a *-automorphism of $C(\mathbb{T})$. We will also make use of the *-homomorphism (symbol) introduced in item (b) of Theorem 8.1 and discussed in the second paragraph after that result. Given the presence of two non-unitary isometries v_1 and v_2, there are two of these symbols here: $\mathrm{smb}_{v_1} : \mathcal{B}_1 \to C(\mathbb{T})$ and $\mathrm{smb}_{v_2} : \mathcal{B}_2 \to C(\mathbb{T})$. Instead of smb_{v_j} we will write smb_j.

Theorem 8.4. *The C^*-algebra $\mathcal{B}$ consists of those elements $(x_1, x_2) \in \mathcal{B}_1 \times \mathcal{B}_2$ for which $\mathrm{smb}_1(x_1) = \mathrm{smb}_2(x_2)^\sim$. Moreover,*

$$\mathcal{B} = \mathcal{J}_1 \dotplus \mathcal{J}_2 \dotplus \left\{(T_{v_1}(a), T_{v_2}(a^\sim)) \mid a \in C(\mathbb{T}) \right\}, \quad j = 1, 2. \tag{16}$$

The proof of this result will be given below. It is illustrative to compare (16) with the representations

$$\mathcal{B}_j = \mathcal{K}_j \dotplus \{T_{v_j}(a) : a \in C(\mathbb{T})\}.$$

As a consequence of Theorem 8.4 we obtain that $\mathcal{B}$ is a proper *-subalgebra of $\mathcal{B}_1 \times \mathcal{B}_2$, and that $\mathcal{J}_1$ and $\mathcal{J}_2$ are contained in $\mathcal{B}$ and therefore *-ideals in $\mathcal{B}$. Furthermore, $\mathcal{J}_1$ and $\mathcal{J}_2$ are the smallest closed ideals of $\mathcal{B}$ containing q_1 and q_2, respectively. Indeed, this can be seen using the fact that for each $x_1 \in \mathcal{B}_1$ there exists $x_2 \in \mathcal{B}_2$ such that $(x_1, x_2) \in \mathcal{B}$, and that, similarly, for each $x_2 \in \mathcal{B}_2$ there exists $x_1 \in \mathcal{B}_1$ such that $(x_1, x_2) \in \mathcal{B}$.

It is easy to describe the irreducible representations of $\mathcal{B}$ corresponding to $\mathcal{J}_1$ and $\mathcal{J}_2$, provided that we are given the irreducible representations for $\mathcal{B}_j$ corresponding to $\mathcal{K}_j$, written

$\pi_j : \mathcal{B}_j \to \mathcal{L}(\ell_2)$. Indeed, the representations in question are then given by $\mathcal{B} \ni (x_1, x_2) \mapsto \hat{\pi}_j(x_1, x_2) = \pi_j(x_j) \in \mathcal{L}(\ell_2)$. Recall that π_j is an injective homomorphism and thus $\hat{\pi}_j$ maps $\mathcal{J}_j$ isometrically onto $\mathcal{K}(\ell_2)$, while it annihilates the other ideal.

Up to this point we have shown that T consists of at least two elements. We are now going to argue that there can be no more. Suppose that $\mathcal{J}_3$ is another ideal of $\mathcal{B}$ generated by a rank one element, and suppose $\mathcal{J}_3$ is different from $\mathcal{J}_1$ and $\mathcal{J}_2$. Then $\mathcal{J}_3$ must be annihilated by $\hat{\pi}_1$ and $\hat{\pi}_2$. Take $(x_1, x_2) \in \mathcal{J}_3$. Then it follows that $\pi_1(x_1) = 0$ and $\pi_2(x_2) = 0$. Since π_1 and π_2 are injective on $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively, it follows that $\mathcal{J}_3 = \{0\}$. But this is a contradiction.

Thus T can be identified with the set $\{1, 2\}$ and it follows that $\mathcal{C}(\mathcal{B}) = \mathcal{J}_1 \dotplus \mathcal{J}_2$. In particular, $\mathcal{B}/\mathcal{C}(\mathcal{B})$ is isomorphic to $C(\mathbb{T})$, and the isomorphism in question can be given by

$$C(\mathbb{T}) \ni a \mapsto \big(T_{v_1}(a), T_{v_2}(a^{\sim})\big) \bmod \mathcal{C}(\mathcal{B}).$$

We can conclude that $\mathcal{B}$ is spectrally regular (as is already $\mathcal{B}_1 \times \mathcal{B}_2$). $\square$

Proof of Theorem 8.4. It is clear that an arbitrary element in $\mathcal{B}_1 \times \mathcal{B}_2$ is of the form

$$x = \big(K_1 + T_{v_1}(a), K_2 + T_{v_2}(b)\big). \tag{17}$$

Using the *-homomorphisms $\mathrm{smb}_j : \mathcal{B}_j \to C(\mathbb{T})$ defined previously, we can define a *-homomorphism $\mathrm{smb} : \mathcal{B}_1 \times \mathcal{B}_2 \to C(\mathbb{T}) \times C(\mathbb{T})$, by stipulating

$$\mathrm{smb}(x_1, x_2) = (\mathrm{smb}_1(x_1), \mathrm{smb}_2(x_2)^{\sim}), \quad x_1 \in \mathcal{B}_1,\ x_2 \in \mathcal{B}_2.$$

Notice that $\phi \mapsto \phi^{\sim}$ is a *-isomorphism on $C(\mathbb{T})$. Applying smb to the element x given by (17), we obtain $\mathrm{smb}(x) = (a, b^{\sim})$. On the other hand, if we consider the generating element of $\mathcal{B}$, that is $w = (v_1, v_2^*)$, then we get $\mathrm{smb}(w) = (\chi_1, \chi_{-1}^{\sim}) = (\chi_1, \chi_1)$, where $\chi_k(t) = t^k$. Thus, the first and the second component of $\mathrm{smb}(w)$ coincide. The same holds when applying smb to w^* or the unit element. Since $\mathcal{B}$ is the closure (in $\mathcal{B}_1 \times \mathcal{B}_2$) of all linear combinations of (non-commutative) products of w, w^*, and e, it is easily seen that for each $x = (x_1, x_2) \in \mathcal{B}$, the first and the second component in $\mathrm{smb}(x)$ coincide. But this means $\mathrm{smb}_1(x_1) = \mathrm{smb}_2(x_2)^{\sim}$ or, equivalently, $a = b^{\sim}$. This proves the first assertion and the statement that $\mathcal{B}$ is contained in the right hand side of (16). In passing, note that it is also clear that we have direct sums in the right hand side of (16).

Now we need to establish that the right hand side of (16) is contained in $\mathcal{B}$. For this, it suffices to show that a dense subset of it is contained in $\mathcal{B}$. It is easy to see that for each trigonometric polynomial a, the element $\big(T_{v_1}(a), T_{v_2}(a^{\sim})\big)$ is in $\mathcal{B}$. Indeed, given a trigonometric polynomial a, we have (by a straightforward computation)

$$\big(T_{v_1}(a), T_{v_2}(a^{\sim})\big) = a_0 e + \sum_{k=1}^{n} \big(a_k w^k + a_{-k}(w^*)^k\big).$$

It is now enough to prove that elements of the form

$$x = \big(v_1^j p_1 (v_1^*)^k, 0\big), \qquad y = \big(0, (v_2)^j p_2 (v_2^*)^k\big)$$

with j and k non-negative integers, belong to $\mathcal{B}$. Recalling that $q_1 = (p_1, 0) = e - ww^*$ and $q_2 = (0, p_2) = e - w^*w$, we have

$$x = w^j (e - ww^*)(w^*)^k, \qquad y = (w^*)^j (e - w^*w) w^k,$$

which proves the claim. (In the above one can assume without loss of generality that v_1 and v_2 coincide with S and from this one sees that we are dealing with specific finite rank operators whose linear space is dense in the set of all compacts.) $\square$

The C^*-algebra described in the previous example is also universal. For a first concrete instance of it, we draw on [43]. The paper [34] is relevant in this context too.

Example 8.5. Let $\Pi \subset \mathbb{C}$ be the upper half-plane, and consider on $L^2(\Pi)$ the two-dimensional singular integral operator S_Π, along with its adjoint S_Π^*,

$$(S_\Pi f)(z) = -\frac{1}{\pi}\int_\Pi \frac{f(w)}{(z-w)^2}\,dA(w), \qquad (S_\Pi^* f)(z) = -\frac{1}{\pi}\int_\Pi \frac{f(w)}{(\bar{z}-\bar{w})^2}\,dA(w),$$

with $dA = dx\,dy$ (where $w = x + iy$) stands for the Lebesgue area measure. It is known that $L^2(\Pi)$ is the orthogonal sum of two subspaces H and $\widetilde{H}$ such that both S_Π and S_Π^* have these two spaces as invariant subspaces, S_Π is a non-unitary isometry on H (i.e., $S_\Pi^* S_\Pi$ is the identity operator H), and S_Π^* is a non-unitary isometry on $\widetilde{H}$ (i.e., $S_\Pi S_\Pi^*$ is the identity operator on $\widetilde{H}$). These facts can be deduced from the material presented in [43]. For the convenience of the reader, we give some details.

The orthogonal decomposition of $L^2(\Pi)$ meant above appears in [43, Theorem 2.1]. It is constructed with the help of the so-called poly-Bergman spaces $\mathcal{A}_n^2(\Pi)$ and $\widetilde{\mathcal{A}}_n^2(\Pi)$. The space $\mathcal{A}_n^2(\Pi)$, n a positive integer, consists of all n-analytic functions on the half-plane Π which are characterized as being solutions of the differential equation

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)^n \phi = 0.$$

Similarly, $\widetilde{\mathcal{A}}_n^2(\Pi)$ consists of all n-anti-analytic functions on Π which are determined by being solutions of

$$\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)^n \phi = 0.$$

As 1-analyticity is the same as analyticity, $\mathcal{A}_1^2(\Pi)$ coincides with the space $\mathcal{A}^2(\Pi)$ of functions analytic on Π. Similarly, 1-anti-analyticity just amounts to anti-analyticity, so $\widetilde{\mathcal{A}}_1^2(\Pi)$ is the space $\widetilde{\mathcal{A}}^2(\Pi)$ of functions which are anti-analytic on Π. In view of the inclusions

$$\mathcal{A}_n^2(\Pi) \subset \mathcal{A}_{n+1}^2(\Pi), \qquad \widetilde{\mathcal{A}}_n^2(\Pi) \subset \widetilde{\mathcal{A}}_{n+1}^2(\Pi), \quad n = 1, 2, 3, \ldots,$$

it makes sense to put $\mathcal{A}_{(n)}^2(\Pi) = \mathcal{A}_{n+1}^2(\Pi) \ominus \mathcal{A}_n^2(\Pi)$ and, analogously, $\widetilde{\mathcal{A}}_{(n)}^2(\Pi) = \widetilde{\mathcal{A}}_{n+1}^2(\Pi) \ominus \widetilde{\mathcal{A}}_n^2(\Pi)$. Besides this, we let $\mathcal{A}_{(1)}^2(\Pi)$ coincide with $\mathcal{A}_1^2(\Pi) = \mathcal{A}^2(\Pi)$, and $\widetilde{\mathcal{A}}_{(1)}^2(\Pi)$ with $\widetilde{\mathcal{A}}_1^2(\Pi) = \widetilde{\mathcal{A}}^2(\Pi)$. In [43], the elements of $\mathcal{A}_{(n)}^2(\Pi)$ and $\widetilde{\mathcal{A}}_{(n)}^2(\Pi)$ are called true-n-analytic and true-n-anti-analytic functions, respectively. Obviously

$$\mathcal{A}_n^2(\Pi) = \bigoplus_{k=1}^{n} \mathcal{A}_{(k)}^2(\Pi), \qquad \widetilde{\mathcal{A}}_n^2(\Pi) = \bigoplus_{k=1}^{n} \widetilde{\mathcal{A}}_{(k)}^2(\Pi).$$

Theorem 2.1 in [43] (third part) now contains the observation that

$$L^2(\Pi) = \bigoplus_{k=1}^{\infty} \mathcal{A}_{(k)}^2(\Pi) \oplus \bigoplus_{k=1}^{\infty} \widetilde{\mathcal{A}}_{(k)}^2(\Pi).$$

Now write $H = \bigoplus_{k=1}^{\infty} \mathcal{A}^2_{(k)}(\Pi)$ and $\widetilde{H} = \bigoplus_{k=1}^{\infty} \widetilde{\mathcal{A}}^2_{(k)}(\Pi)$. Then we have the orthogonal decomposition $L^2(\Pi) = H \oplus \widetilde{H}$. By Theorem 3.5 in [43]

$$S_\Pi\big[\mathcal{A}^2_{(k)}\big] \subset \mathcal{A}^2_{(k+1)}, \quad k = 1, 2, 3, \ldots,$$
$$S^*_\Pi\big[\mathcal{A}^2_{(1)}\big] = \{0\},$$
$$S^*_\Pi\big[\mathcal{A}^2_{(k)}\big] \subset \mathcal{A}^2_{(k-1)}, \quad k = 2, 3, 4, \ldots,$$

hence $S_\Pi[H] \subset H$ and $S^*_\Pi[H] \subset H$. Again by [43, Theorem 3.5]

$$S_\Pi\big[\widetilde{\mathcal{A}}^2_{(k)}\big] \subset \widetilde{\mathcal{A}}^2_{(k-1)}, \quad k = 2, 3, 4, \ldots,$$
$$S_\Pi\big[\widetilde{\mathcal{A}}^2_{(1)}\big] = \{0\},$$
$$S^*_\Pi\big[\widetilde{\mathcal{A}}^2_{(k)}\big] \subset \mathcal{A}^2_{(k+1)}, \quad k = 1, 2, 3, \ldots,$$

and therefore $S_\Pi[\widetilde{H}] \subset \widetilde{H}$ and $S^*_\Pi[\widetilde{H}] \subset \widetilde{H}$. So both S_Π and S^*_Π have H and $\widetilde{H}$ as invariant subspaces, as desired. Next we apply Theorem 3.7 of [43]. This gives that $S^*_\Pi S_\Pi$ and $S_\Pi S^*_\Pi$ are the orthogonal projections of $L^2(\Pi)$ along $\widetilde{\mathcal{A}}^2_{(1)}$ and $\mathcal{A}^2_{(1)}$, respectively. Thus $S^*_\Pi S_\Pi$ acts as the identity operator on the orthogonal complement of $\widetilde{\mathcal{A}}^2_{(1)}$ in $L^2(\Pi)$, and $S_\Pi S^*_\Pi$ acts as the identity operator on the orthogonal complement of $\mathcal{A}^2_{(1)}$ in $L^2(\Pi)$. But these orthogonal complements contain the spaces H and $\widetilde{H}$, respectively. It follows that $S^*_\Pi S_\Pi$ is the identity operator H, and that $S_\Pi S^*_\Pi$ is the identity operator on $\widetilde{H}$, again as desired.

So far for the details concerning the decomposition meant in the first paragraph of this example. Returning to the main line of the argument, consider the smallest closed subalgebra $\mathcal{B}$ of $\mathcal{L}\big(L^2(\Pi)\big)$ containing S_Π, S^*_Π and the identity operator I on $L^2(\Pi)$. Then $\mathcal{B}$ is a unital C^*-algebra, which can be identified with a subalgebra of $\mathcal{L}(H \dotplus \widetilde{H})$. Under this identification, since H and $\widetilde{H}$ are invariant subspaces of S_Π and S^*_Π,

$$S_\Pi = \begin{pmatrix} S_\Pi|_H & 0 \\ 0 & S_\Pi|_{\widetilde{H}} \end{pmatrix}, \qquad S^*_\Pi = \begin{pmatrix} S^*_\Pi|_H & 0 \\ 0 & S^*_\Pi|_{\widetilde{H}} \end{pmatrix}.$$

In other words, $S_\Pi = (v_1, v_2^*)$ and $S^*_{C_+} = (v_1^*, v_2)$, where $v_1 = S_\Pi|_H \in \mathcal{L}(H)$ and $v_2 = S^*_\Pi|_{\widetilde{H}} \in \mathcal{L}(\widetilde{H})$ are both non-unitary isometries. Thus the C^*-algebra $\mathcal{B}$ is a concrete instance of the (universal) C^*-algebra described in Example 8.3. □

Next we give another concrete realization of the C^*-algebra described in Example 8.3 (see also [16, Section 3.2] and [32, Section 5.1.4, Example 6]). It occurs in the numerical analysis of Toeplitz operators.

Example 8.6. Consider the setting of Example 7.3 with $\mathbf{B} = \{\mathbb{C}^{n\times n}\}_{n=1}^{\infty}$. Denote by $\mathcal{S} = \ell^{\mathbf{B}}_\infty$, i.e., the C^*-algebra consisting of all bounded sequences $A = \{A_n\}_{n=1}^{\infty}$. Furthermore, let $\mathcal{N} = c_0^{\mathbf{B}}$ stand for the *-ideal of $\mathcal{S}$ consisting of all sequences converging in the norm to zero. Finally, consider the quotient algebra $\mathcal{S}^\pi = \mathcal{S}/\mathcal{N}$.

We are going to define a C^*-subalgebra of $\mathcal{S}^\pi$ and we show that it is isomorphic to the abstract C^*-algebra considered in Example 8.3. First introduce the following operators:

$$P_n^{\uparrow} : (x_1, \ldots, x_n)^T \in \mathbb{C}^n \mapsto (x_1, \ldots, x_n, 0, 0, \ldots)^T \in \ell_2, \tag{18}$$
$$P_n^{\downarrow} : (x_1, \ldots, x_n, x_{n+1}, \ldots)^T \in \ell_2 \mapsto (x_1, \ldots, x_n)^T \in \mathbb{C}^n, \tag{19}$$
$$P_n : (x_1, \ldots, x_n, x_{n+1}, \ldots)^T \in \ell_2 \mapsto (x_1, \ldots, x_n, 0, 0, \ldots)^T \in \ell_2. \tag{20}$$

Notice that $P_n^{\downarrow} P_n^{\uparrow} = I_n$, the identity matrix in $\mathbb{C}^{n \times n}$ and that $P_n^{\uparrow} P_n^{\downarrow} = P_n$, which is a projection on ℓ_2. Furthermore, we need the following flip matrix:

$$W_n : (x_1, \ldots, x_n)^T \in \mathbb{C}^n \mapsto (x_n, \ldots, x_1)^T \in \mathbb{C}^n.$$

Finally, for $a \in C(\mathbb{T})$ with Fourier coefficients a_k, $k \in \mathbb{Z}$, we define for each $n \geq 1$, the $n \times n$ Toeplitz matrix,

$$T_n(a) = (a_{j-k})_{j,k=1}^n.$$

For the specific symbols $a(t) = t^{\pm 1}$ notice that $T_n(t^{\pm 1})$ are the finite forward and backward shift matrices. Now define the element W of $\mathcal{S}$ and its adjoint W^* as

$$W = \{T_n(t)\}_{n=1}^\infty, \qquad W^* = \{T_n(t^{-1})\}_{n=1}^\infty.$$

The smallest closed subalgebra of $\mathcal{S}$ which contains W, W^* and the identity element, will be denoted by $\mathcal{S}(C)$. The notation refers to the fact that, as one can show, this is the algebra of all sequences generated by the sequences of $\{T_n(a)\}_{n=1}^\infty$ with $a \in C(\mathbb{T})$. In fact, one prove show that each element $A \in \mathcal{S}(C)$ has the following unique representation,

$$A = \left\{ T_n(a) + P_n^{\downarrow} K_1 P_n^{\uparrow} + W_n P_n^{\downarrow} K_2 P_n^{\uparrow} W_n + N_n \right\}_{n=1}^\infty$$

where $a \in C(\mathbb{T})$, $K_1, K_2 \in \mathcal{K}(\ell_2)$ and $\{N_n\} \in \mathcal{N}$. In particular, $\mathcal{S}(C)$ contains $\mathcal{N}$ as a *-ideal and hence we can define the quotient algebra $\mathcal{S}^\pi(C) = \mathcal{S}(C)/\mathcal{N}$. This is the algebra we want to consider. Alternatively, it can be defined as the smallest closed subalgebra of $\mathcal{S}^\pi$ which contains the elements $W + \mathcal{N}$, $W^* + \mathcal{N}$, and the identity element.

In order to see that $\mathcal{S}^\pi(C)$ is isomorphic to the abstract algebra considered in Example 8.3, one needs to make use of two particular representations which this C^*-algebra possesses. These are *-homomorphisms and can be introduced via strong limits on $\mathcal{S}(C)$. One can show that they act on the elements $A + \mathcal{N}$ of $\mathcal{S}^\pi(\mathbb{C})$ as follows (where A is given in the above form):

$$\Phi_1 : A + \mathcal{N} \mapsto T(a) + K_1 \in \mathcal{L}(\ell_2), \qquad \Phi_2 : A + \mathcal{N} \mapsto T(\tilde{a}) + K_2 \in \mathcal{L}(\ell_2).$$

Here $T(a)$ denotes the usual Toeplitz operator with symbol a and, as before, $\tilde{a}(\tau) = a(\tau^{-1})$, $\tau \in \mathbb{T}$. Next we form the direct sum of these two representations:

$$\Phi : A + \mathcal{N} \in \mathcal{S}^\pi(C) \mapsto \big(\Phi_1(A + \mathcal{N}), \Phi_2(A + \mathcal{N})\big) \in \mathcal{L}(\ell_2) \times \mathcal{L}(\ell_2),$$

and we observe that Φ has a trivial kernel. Hence the image of Φ is *-isomorphic to $\mathcal{S}^\pi(C)$. On the other hand, this image is obviously generated by the unit element and the elements $w = \Phi(W + \mathcal{N}) = (V, V^*)$ and $w^* = \Phi(W + \mathcal{N}) = (V^*, V)$. To make the connection with Example 8.3, we mention that in this context $v_1 = v_2 = V$, the simple forward shift on ℓ_2. This algebra $\Phi[\mathcal{S}^\pi(C)]$ is precisely the universal algebra described in Example 8.3. $\square$

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