A semi-analytical elastic solution for stress field of lined non-circular tunnels at great depth using complex variable method

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ABSTRACT

In this paper, a semi-analytical elastic plane strain solution was provided for stress field around a lined non-circular tunnel subjected to uniform ground load. Concrete lining and the surrounding rock mass were assumed as linearly elastic materials. Due to complexity of the problem for non-circular geometric configurations, complex variable method introduced by Muskhelishvili and conformal mapping functions were used to determine stress components within concrete lining and the surrounding rock mass. Finally, the solution was validated by ABAQUS finite element software through an example. Very good agreement was demonstrated between semi-analytical and numerical solution although some discrepancies were found at tunnel corners where large curvature existed. It was demonstrated that the solution predicted stress components more accurately around the tunnels, especially the corners with large stress concentration. Practical significance of the solution was placed in the fact that it could be used as a quick-solver with high accuracy.

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1. Introduction

Tunnels are main infrastructures in underground mining which are broadly used for transportation, water passage and other purposes such as underground cables. Liner is the primary support adopted to endure ground pressure. It has been of high interest for determining stress distribution within liner and rock medium with high level of accuracy. This subject has been studied by numerous researchers (Bobet, 2001, 2003; Einstein and Schwartz, 1979; Lee and Nam, 2001; Penzien and Wu, 1998; Savin, 1961; Timoshenko and Goodier, 2011).

Many studies have been carried out to determine stress and deformation field for tunnels by applying numerical methods. Exact solutions have been generally used to validate results of numerical methods. They are also largely employed in preliminary stages of design in order to find stress and deformation state. Furthermore, analytical solutions give an understanding of how final solutions are influenced by different parameters.

One of the useful analytical approaches implemented in two-dimensional elastic theory is Muskhelishvili’s complex variable method (Muskheilishvili and Radok, 1953). The method investigated based on complex potential functions and conformal mapping method can determine stress components and deformations within the materials. Based on this method, Exadaktylos et al. proposed a closed-form solution for stress and displacement of semicircular tunnels as well as a semi-analytical solution for notched circular openings in homogenous elastic media (Exadaktylos and Stavropoulou, 2002; Exadaktylos et al., 2003). A similar method was proposed for stress and displacement around a circular tunnel in an elastic half-plane (Strack and Verruijt, 2002; Verruijt, 1997). However, the interaction between support and geomaterial has been rarely considered in the above-mentioned literature.

In case the problem is considered without reinforcement layer, the solution would come up much easier. When liner is included, two different regions should be assumed which increases complexity of the problem (England, 2009). For openings with simple geometry like ellipse or regular polygonal holes, a closed form solution could be extracted. However, when the opening and consequently reinforced layer has a complex geometry, a semi-analytical method is resulted, in which complex potential and conformal mapping functions are defined as power series.

An attempt was made in this study to find stress field of a lined non-circular tunnel at a great depth with a complex geometry configuration by applying Muskheilishvili’s complex variable method. Rock mass and lining concrete were assumed as isotropic linear elastic materials.

2. General consideration

The problem involves a lined non-circular tunnel at depth $H$ within linearly elastic geomaterial. The tunnel is located at such
a great depth compared with the tunnel dimension that the problem is considered a reinforced hole in an infinite plane, subjected to a uniform stress state at infinity. Concrete and rock mass are assumed as isotropic and homogenous materials. It is supposed that liner is installed without any delay after tunnel excavation. The infinite plate on complex plane is divided into two isotropic homogenous regions of $S_1$ and $S_2$ bounded by contours $L_1$ and $L_2$ (Fig. 1). The regions $S_1$ and $S_2$ refer to rock mass and concrete lining with Young modulus $E_1$, $E_2$ and Poisson ratio $\nu_1$, $\nu_2$, respectively. Let $w(\zeta)$ be a conformal mapping function which maps boundaries $L_1$ and $L_2$ into two concentric circles $\eta_1$ and $\eta_2$ with unit and $R_0$ radii. It is assumed that the conformal mapping function has the following expansion:

$$w(\zeta) = Re^\frac{\zeta}{2} \left( \sum_{i=1}^{n} \frac{\phi_0(z)}{z^j} \right)$$

(1)

Where $i = \sqrt{-1}$, $R$ is a real constant affecting scale of the hole and $\frac{\zeta}{2}$ is the angle by which the shape is rotated from its original position.

The function $w(\zeta)$ thoroughly maps each exterior point of unit circle $\eta_1$ into region $S_1$ and each point in the region $\eta_2$ bounded by circles $\eta_1$ and $\eta_2$ into region $S_2$. According to Mushkilishvili and Kolosove's method, there are complex potential functions $\varphi_1$, $\psi_1$ and $\varphi_2$, $\psi_2$ which are analytic on regions $S_1$ and $S_2$, respectively. Stress components are determined based on these complex potential functions as follows:

$$\sigma_{\rho\rho} + \sigma_{\phi\phi} = 2 \left( \frac{\partial w(\zeta)}{\partial \zeta} \right)^2 + \frac{\partial w(\zeta)}{\partial \bar{\zeta}}$$

$$\sigma_{\phi\rho} + 2i \tau_{\phi\rho} = \frac{\partial w(\zeta)}{\partial \zeta} \left( \frac{\partial w(\zeta)}{\partial \zeta} \right)^* + \psi_j(\zeta)$$

(2)

where $\sigma_{\rho\rho}$, $\sigma_{\phi\phi}$ and $\tau_{\phi\rho}$ are radial, circumferential and tangential stress components, respectively. The complex potential functions $\varphi_1$ and $\psi_1$ defined on region $\gamma_1$ are:

$$\varphi_1(\zeta) = \Gamma w(\zeta) + \varphi_0(\zeta)$$

$$\psi_1(\zeta) = \Gamma^* w(\zeta) + \psi_0(\zeta)$$

(3)

where

$$\varphi_0(\zeta) = \sum_{j=0}^{\infty} a_j \zeta^j, \quad \psi_0(\zeta) = \sum_{j=0}^{\infty} b_j \zeta^j$$

(4)

$\varphi_0(\zeta)$ and $\psi_0(\zeta)$ are holomorphic functions with $\varphi_0(\infty) = 0$ and $\psi_0(\infty) = 0$. $\Gamma$ and $\Gamma^*$ are real and complex constants with regard to stress state at infinity, which are determined as follows:

$$\Gamma^* = \frac{1}{2}(\sigma_1 + \sigma_2) = \frac{1}{2}\kappa \gamma H$$

(5)

where $\sigma_1$ and $\sigma_2$ are principal stress components at infinity, $\kappa$ is the angle made between $\sigma_1$ direction and $x$ axis and $K$ and $\gamma$ are lateral coefficient pressure and stress gradient of rock mass, respectively.

The complex functions $\varphi_2$ and $\psi_2$ defined on region $\gamma_2$ are determined as follows:

$$\varphi_2(\zeta) = R_1(\zeta) + R_2(\zeta)$$

$$\psi_2(\zeta) = Q_1(\zeta) + Q_2(\zeta)$$

(6)

where $R_1$, $R_2$ and $Q_1$, $Q_2$ are assumed analytic functions on region $\gamma_2$ which have the following series expansions:

$$Q_1(\zeta) = \sum_{j=0}^{\infty} c_j \zeta^j, \quad R_1(\zeta) = \sum_{j=0}^{\infty} c_j \zeta^j$$

(7)

$$Q_2(\zeta) = \sum_{j=0}^{\infty} d_j \zeta^j, \quad R_2(\zeta) = \sum_{j=0}^{\infty} d_j \zeta^j$$

Stress functions $\varphi_1$, $\psi_1$ and $\varphi_2$, $\psi_2$ should satisfy boundary conditions on $\eta_1$ and $\eta_2$ circles:

$$k_1 \varphi_1(t) - \frac{1}{G_1} \frac{w(t)}{w(t)} \left( \varphi_1(t) + \psi_0(t) \right) = \frac{k_2}{G_2} \varphi_2(t) - \frac{1}{G_2} \frac{w(t)}{w(t)} \left( \varphi_2(t) + \psi_2(t) \right)$$

(8)

on $\eta_1$,

$$\varphi_1(t) + \frac{w(t)}{w(t)} \varphi_1(t) + \psi_1(t) = \varphi_2(t) + \frac{w(t)}{w(t)} \varphi_2(t) + \psi_2(t)$$

(9)

on $\eta_2$,

$$\varphi_2(t) + \frac{w(t)}{w(t)} \varphi_2(t) + \psi_2(t) = 0$$

(10)

where $G_1 = \frac{k_1}{2(1+\nu_1)}$, $G_2 = \frac{k_2}{2(1+\nu_2)}$ plane stress ($i = 1, 2$).

Eqs. (9) and (10) are concerned with continuity of deformation and stress field across the rock-concrete interface due to no-slip condition (i.e., there is a perfect adherence between liner and rock mass so that they have common deformation along the interface) and Eq. (11) is about boundary condition along the observation hole (which involves zero traction along the boundary $L_2$). It should be considered that expressions $\Gamma w(\zeta)$ and $\Gamma^* w(\zeta)$ are not incorporated into continuity Eq. (9) since they define pre-stress and pre-deformation field in rock mass when tunnels are excavated.

3. Solution

To acquire a solution for stress functions $\varphi_1$, $\psi_1$ and $\varphi_2$, $\psi_2$, at first, Eqs. (9)–(11) would be conjugated considering the fact that $t = \frac{1}{\zeta}$ for each point on $\eta_1$:

$$k_1 \varphi_1(t) - \frac{1}{G_1} \frac{w(t)}{w(t)} \left( \varphi_1(t) + \psi_0(t) \right) = k_2 \frac{1}{G_2} \varphi_2(t) - \frac{1}{G_2} \frac{w(t)}{w(t)} \left( \varphi_2(t) + \psi_2(t) \right)$$

(12)

$$\varphi_1(t) + \frac{w(t)}{w(t)} \varphi_1(t) + \psi_1(t) = \varphi_2(t) + \frac{w(t)}{w(t)} \varphi_2(t) + \psi_2(t)$$

(13)

$$\varphi_2(t) + \frac{w(t)}{w(t)} \varphi_2(t) + \psi_2(t) = 0$$

(14)

By multiplying Eqs. (12)–(14) by $\zeta^j$ and integrating it along unit circle $\eta_1$ for $|\zeta| > 1$ and $|\zeta| < 1$, a set of 6 equations is obtained which include 6 unknown complex functions. For $|\zeta| > 1$:

$$- \frac{1}{G_1} \frac{1}{2\pi i} \int_{\eta_1} \frac{w(t)}{w(t)} \left( \varphi_1(t) - \psi_0(t) \right) dt = \frac{1}{G_1} \varphi_0(\zeta) - \frac{1}{G_1} \varphi_1(\zeta)$$

(15)

$$= - \frac{k_2}{G_2} R_2 \left[ \frac{1}{\zeta} \right] + \frac{k_2}{G_2} R_2 \left[ \frac{1}{\zeta} \right] \frac{1}{2\pi i} \int_{\eta_1} \frac{w(t)}{w(t)} R_1(t) \left( \frac{1}{\zeta} \right) dt$$

$$- \frac{1}{G_2} \frac{1}{2\pi i} \int_{\eta_1} \frac{w(t)}{w(t)} R_2(t) \left( \frac{1}{\zeta} \right) dt = \frac{1}{G_2} \varphi_1(\zeta)$$
\[
\frac{1}{2\pi i} \int_{\eta} \frac{w(\zeta)}{w(t')} \varphi_0(t') \, dt' - \frac{1}{2\pi i} \int_{\eta} \frac{w(\zeta)}{w(t)} R_2(t) \, dt - \frac{1}{2\pi i} R_1(t) \quad \text{for } |\zeta| < l.
\]

\[
= -R_2 \varphi_0(t') + \frac{1}{2\pi i} \int_{\eta} \frac{w(\zeta)/w(t')}{w(R_0)} R_2(t) \, dt - \frac{1}{2\pi i} R_1(t) \quad \text{for } |\zeta| > l.
\]

\[
\sigma_\alpha = -\frac{1}{\pi(\pi - 2\pi)} \text{Re} \left( C_1 \sum_{j=1}^{n} j a_1^{j-1} + (C_1 + i \delta) \left( \sum_{j=1}^{n} j a_1^{j-1} + \mu_1 \sum_{j=1}^{n} j f_1^{j-1} \right) \right)
\]

\[
\sigma_\rho = -\frac{1}{\pi(\pi - 2\pi)} \text{Re} \left( C_1 \sum_{j=1}^{n} j a_1^{j-1} + (C_1 + i \delta) \left( \sum_{j=1}^{n} j a_1^{j-1} + \mu_1 \sum_{j=1}^{n} j f_1^{j-1} \right) \right)
\]

\[
\tau_\mu = -\frac{1}{\pi(\pi - 2\pi)} \text{Im} \left( C_1 \sum_{j=1}^{n} j a_1^{j-1} + (C_1 + i \delta) \left( \sum_{j=1}^{n} j a_1^{j-1} + \mu_1 \sum_{j=1}^{n} j f_1^{j-1} \right) \right)
\]

Furthermore, circumferential stress along circle \( \eta_1 \), from the side of the region \( \mathcal{R}_1 \) (\( |\zeta| = 1 \)), is calculated as follows:

\[
\sigma_\alpha = 4 \frac{C_1 a_1 + d_1 b_1}{C_1^2 + d_1^2} - \sigma_\rho
\]

where

\[
a_1' = \text{Re} \left\{ \sum_{j=1}^{n} j a_1^{j-1} \right\}, \quad b_1' = \text{Im} \left\{ \sum_{j=1}^{n} j a_1^{j-1} \right\},
\]

\[
c_1' = \text{Re} \left\{ \frac{w'(\zeta)}{R} \right\}, \quad d_1' = \text{Im} \left\{ \frac{w'(\zeta)}{R} \right\}
\]

\[
\alpha_2 = \text{Re} \left\{ \sum_{j=1}^{n} j \delta_1^{j-1} - \left( \sum_{j=1}^{n} j \delta_1^{j-1} + t \delta_1 \sum_{j=1}^{n} j \delta_1^{j-1} \right) \right\}
\]

\[
\beta_1 = r_1 (\cos(a_1^{j-1}) + i \sin(a_1^{j-1})), \quad \beta_2 = r_2 (\cos(a_2^{j-1}) + i \sin(a_2^{j-1}))
\]

Functions \( \chi_{ik} \) and \( \mu_i \) are defined as follows:

\[
\chi_{ik} = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}, \quad \mu_i = \begin{cases} 1 & 0 \leq i < n \\ 0 & i \geq n \end{cases}
\]

By applying series solution to solve the equation set (15)-(20) as well as equaling multipliers of the same order variables, the following equation set is concluded:
\[
T_k = \begin{cases} 
\sum_{j=1}^{n-(k-1)} j a_j q_{j+k-1} & k > 0 \\
\mu_2 \sum_{j=1}^{n-1} j a_j q_{j+1} & k = 0 \\
\sum_{j=0}^{n} (j-k-1) a_{j-k-1} q_j & k < 0 
\end{cases} 
\]

Note that, in this section, compressive stress is assumed a positive quantity for convenience. ABAQUS grid for tunnel cross-section is presented in Fig. 3. Fig. 4 shows magnitude of circumferential stress along internal lining periphery predicted by semi-analytical solution and ABAQUS finite element software. It could be observed that semi-analytical solution and numerical model had reasonable agreement apart from \( \theta = 70^\circ, 130^\circ \) denoting lower and upper corners of tunnel. The boundary values for normal and shear stress components along the inner lining periphery are also presented in Fig. 5, demonstrating good agreement between the semi-analytical solution and finite element code, except at corners. Note that, ABAQUS software predicted finite radial and shear stress at outline corners while the solution predicted almost zero traction and shear stress, which proved its high accuracy.

In general, most discrepancies between the semi-analytical solution and ABAQUS finite element code are found at tunnel corners where large curvature exists, which may be due to the fact that grid size in ABAQUS modeling (Fig. 3) is not small enough considering corner curvature to produce accurate results. Furthermore, there is always some sources of error in finite element numerical analysis such as approximation functions and solving equation systems methods.

4. Discussion

In this section, the derived solution is applied to an example and a comparison is provided with ABAQUS finite element code in order to verify the solution; thereafter, the number of leading terms of the series expansions with a substantial contribution to the convergence of the solution and rate of convergence are investigated.

4.1. Comparison of the analytical solution for non-circular tunnel configuration with ABAQUS finite element code

Here, the solution is compared with ABAQUS finite element software through an examples. Input data are represented in Table 1.

Geometry of tunnel is presented in Fig. 2.

Coefficients of conformal mapping function are given in Table 2. It should be mentioned that tunnel cross-section after mapping was used for analytical and numerical analyses.

Table 1

<table>
<thead>
<tr>
<th>Rock type</th>
<th>Elastic properties of rock</th>
<th>Elastic properties of concrete</th>
<th>Lateral pressure coefficient (K)</th>
<th>Stress gradient (kPa/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Limestone</td>
<td>( E_1 ) (GPa) ( v_1 )</td>
<td>( E_2 ) (GPa) ( v_2 )</td>
<td>( C_1 )</td>
<td>0.5</td>
</tr>
<tr>
<td></td>
<td>45</td>
<td>0.3</td>
<td>30</td>
<td>0.25</td>
</tr>
<tr>
<td></td>
<td>30</td>
<td>0.25</td>
<td>0.5</td>
<td>0.027</td>
</tr>
</tbody>
</table>

![Fig. 2. Tunnel cross-section 1, (a) before mapping and (b) after mapping.](image-url)
4.2. Convergence of the semi-analytical solution

The average of absolute relative approximation error of circumferential stress, predicted by semi-analytical solution, with respect to the number of terms for series expansions $u_0(n)$ and $R_2(n)$ is illustrated in Tables 3 and 4. It is observed that, after almost 10 and 40 terms for $n_1$ and $n_2$, error was reduced to less than 1%. In other words, the first 10 and 40 terms for $u_0(n)$ and $R_2(n)$, respectively, had more than 99% contribution to the convergence of circumferential stress. This trend was similar for other stress components.

Considering convergence rate of the solution, it highly depended on the material properties and shape of tunnel cross-sections. Effect of material properties on the rate of convergence of circumferential stress along inner lining periphery is presented in Fig. 7. It shows that the solution converged more rapidly with respect to $n_1$ when $G_2/G_1 \approx 1$ whereas rate of convergence increased versus $n_2$ with decreasing rate of $G_2$. The trend was similar for circumferential stress along rock-concrete interface. On the other hand, with respect to the shape of mapped cross-section, rate of convergence decreases for the shapes with corners of high curvature due to high stress concentration.

5. Conclusion

An elastic plane strain solution based on Muskhelishvili’s complex variable method was provided in this study in order to calculate stress components around non-circular tunnels and within their lining supports. The materials were supposed to be isotropic undergoing uniform remote stress field. The stress components were predicted by employing complex potential functions which
were consequently determined by Couchy integration and series solution method.

Even though it was shown in the example that results of the new solution prediction and numerical finite element method had good agreement, there were some prominences for semi-analytical solution which could be described as follows: (a) the new solution predicted stress components more accurately at corners with great curvature and high stress concentration than numerical finite element method, (b) the solution was independent from numerical method constraint, like grid size, and could predict the results much faster with high accuracy. Attempts should be made in upcoming works to improve accuracy of conformal mapping functions in order to preserve shape after mapping.

Although numerical methods are capable of solving elaborate problems in elasticity, complex variable method introduced by Muskhelishvili comes up to be a straightforward way for solving complicated problem in theory of elasticity in a more precise and time-saving process.

Appendix A

Coefficients of equation set (26) are presented as follows:

\[
A_{lmk} = kR_0^{k-1} s \left( \mu_{k-m-n} \chi_{m-k-1} + \mu_{k-m-1} \mu_{m-k-1} (q_{k-m-1} - R_0^{(k-m-1)} q_{k-m+1}) \right)
\]  

(A1)
are determined as follows:

\[
\begin{align*}
A_{2m,k} &= k R_0^{-m} \left( t \mu_{x} H_{m-k} \sum_{j=1}^{n-1} \int_{C_{0}}^{C_{1}} \tilde{q}_{j-1,m-1} - q_{j-1,m-1} \right) \tilde{q}_{j,k-1} \\
D_{1m,k} &= -k R_0^{-m-1} \sum_{j=1}^{n} \beta_{j} \left( R_0^{-1} l_j - l_j \right) \\
D_{2m,k} &= \chi_{m,k} \left( R_0^{-m} + \frac{g}{R_0} - \frac{g}{R_0} \right) \\
A_{1m,k} &= -k R_0^{-m} (-\tilde{q}_{j,m+k} H_{k-m+1} + s R_0^{(k-m+1)} \mu_{k,m} q_{k+1}) + t R_0^{-m} \mu_{k,m} q_{k+1} \\
A_{2m,k} &= \chi_{m,k} \left( s R_0^{-m} + t R_0^{-m} \right) + \tilde{t} \mu_{m+2} R_0^{-1} \sum_{j=1}^{n-1} \mu_{j} \left( R_0^{-1} l_j - l_j \right) q_{j+1,k-1} \\
D_{1m,k} &= k \left( \mu_{k,m+1} R_0^{-1} \tilde{q}_{m-k-1} - R_0^{-1} \sum_{j=1}^{n-1} \beta_{j} \left( R_0^{-1} l_j - l_j \right) \right) \\
D_{2m,k} &= 0 \\
B_{2m} &= \gamma \mu_{m+1} \left( \frac{k_2 R_0^{m} + R_0^{-m}}{1 + k_2} \left( -\frac{1}{2} \frac{K}{\Omega_m + 1 - \frac{K}{2} \Omega_{m-1}} \right) \\
&\quad - \frac{\mu_{m+2}}{1 + k_2} \left( C_{0}^{m} - C_{0}^{m} \right) \right) \\
\end{align*}
\]

Coefficients \(q_m^r, C_m, C_m^e, C_m^m, C_m^g\) and \(C_m^c\) are determined as follows:

\[
\begin{align*}
q_k^m &= \left\{ \begin{array}{ll}
\sum_{j=1}^{n-1} \mu_{j} \left( R_0^{-1} l_j - l_j \right) q_{j+k-1} & k > 2n + 1 \\
\sum_{j=1}^{n-1} \mu_{j} \left( R_0^{-1} l_j - l_j \right) q_{j+k-1} & k = 2n + 1 \\
\sum_{j=1}^{n-1} \mu_{j} \left( R_0^{-1} l_j - l_j \right) q_{j+k-1} & n + 1 < k \leq 2n \\
0 & k < n \\
\end{array} \right. \\
C_k &= \left\{ \begin{array}{ll}
\gamma \mu_{k+1} \left( q_{k+2} \sum_{j=1}^{n} \mu_{j} q_{j+k-1} \right) & k \geq 0 \\
\gamma \mu_{k+1} \left( q_{k+2} \sum_{j=1}^{n} \mu_{j} q_{j+k-1} \right) & k = -1 \\
\gamma \mu_{k+1} \left( q_{k+2} \sum_{j=1}^{n} \mu_{j} q_{j+k-1} \right) & k < -1 \\
\end{array} \right.
\]

Fig. 7. Average of absolute relative approximation error of circumferential stress versus number of terms for series expansions (a) \(q_\phi(\xi)\) and (b) \(R_\ell(\xi)\) along internal lining periphery \((r_1 = r_2 = 0.3 \text{ & } K = 0.5)\).
\[
C_i' = \begin{cases} 
\gamma HR \left( q_0 R_0^2 \frac{1}{k^2} \sum_{j=1}^{n} (j-k-1)q_{j+k} \right) & k > 0 \\
\gamma HR \left( q_0 R_0^2 \frac{1}{k^2} \sum_{j=1}^{n} q_{j+k} \right) & k = -1 \\
\gamma HR \left( q_0 R_0^2 \frac{1}{k^2} \sum_{j=1}^{n} j q_{j+k} \right) & k < -1 
\end{cases}
\]

(A13)

Reference


