On the Riesz basis of a family of analytic operators in the sense of Kato and application to the problem of radiation of a vibrating structure in a light fluid

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Abstract
In this paper, we prove that the system formed by some of generalized eigenvectors of the operator $T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \cdots$ which are analytic on $\varepsilon$, forms a Riesz basis of the separable Hilbert space $\mathcal{H}$, where $\varepsilon \in \mathbb{C}$ and $T_0, T_1, T_2, \ldots$ some linear transformations on $\mathcal{H}$ which have the same domain $D \subseteq \mathcal{H}$. After that, we give an application for a problem concerning the radiation of a vibrating structure in a light fluid.
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1. Introduction

In this paper, we deal with the operator

$$T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \cdots + \varepsilon^k T_k + \cdots, \quad (1.1)$$

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where \( \epsilon \in \mathbb{C} \) and \( T_0, T_1, T_2, \ldots \) are linear transformations acting from a Hilbert space \( X \) into a Hilbert space \( Y \) and having the same domain \( D \subseteq X \). In [11], B.Sz. Nagy has proved that if \( T_0 \) is a closed operator and \( a, b, q \) are positive numbers such that
\[
\|T_k \psi\| \leq q^{k-1} \left(a\|\psi\| + b\|T_0 \psi\|\right) \quad \text{for each } k \in \mathbb{N}^*; \tag{1.2}
\]
then:

(i) The series
\[
T_0 \psi + \epsilon T_1 \psi + \epsilon^2 T_2 \psi + \cdots + \epsilon^k T_k \psi + \cdots
\]
converges for all \( \psi \in D \) and for all \( |\epsilon| < q^{-1} \); and if we denote by \( T_\epsilon \psi \) its limit then we define a linear transformation \( T_\epsilon \) with domain \( D \).

(ii) The transformation \( T_\epsilon \) is closed for \( |\epsilon| < (q + b)^{-1} \).

Now, let \((\lambda_n)_n\) be the eigenvalues of the operator \( T_0 \) with principal multiplicity one associated to the eigenvectors \((\varphi_n)_n\). B.Sz. Nagy [11] has proved that for each eigenvalue \( \lambda_n \) of \( T_0 \) there exists a neighborhood of \( \lambda_n \) in which there is a unique eigenvalue \( \lambda_n(\epsilon) \) of the operator \( T_\epsilon \) with principal multiplicity one. Let \( \varphi_n(\epsilon) \) be the eigenvector of \( \lambda_n(\epsilon) \), then one can develop \( \lambda_n(\epsilon) \) and \( \varphi_n(\epsilon) \) as entire series of \( \epsilon \) as follows:
\[
\lambda_n(\epsilon) = \lambda_n + \epsilon \lambda_n^1 + \epsilon^2 \lambda_n^2 + \cdots, \quad \varphi_n(\epsilon) = \varphi_n + \epsilon \varphi_n^1 + \epsilon^2 \varphi_n^2 + \cdots. \tag{1.3}
\]

The main purpose of this paper is to answer the following question:

*If the system \((\varphi_n)_n\) forms a Riesz basis of the separable Hilbert space \( X \), then is it so for \((\varphi_n(\epsilon))_n\), expressed in (1.3)?*

This work is inspired by [8] where the authors prove that the system of generalized eigenvectors of the operator
\[
(I + \epsilon K)^{-1} \frac{d^4}{dx^4}
\]
forms a Riesz basis of \( L^2([-L, L]) \), \( L > 0 \) where \( K \) is the integral operator with kernel the Hankel function of the first kind and order 0. Our work is the continuity of [8]. In fact, we tried here to find the same result as in [8] for an operator of the type (1.1) and to determine the conditions necessary for it.

The purpose of this work is to pursue the analysis started in [8] and to extend it to the general case. More precisely, let \( T_0 \) be a closed linear operator with domain \( D(T_0) \), dense in \( X \) and let \( T_1, T_2, \ldots \) be some linear transformations satisfying both of the conditions
\[
D(T_i) \subseteq D(T_0) \quad \text{for all } i \in \mathbb{N}^* \text{ and Eq. (1.2).}
\]
If the eigenvectors \((\varphi_n)_n\) of \( T_0 \) form a Riesz basis of \( X \) and if the eigenvalues \((\lambda_n)_n\) of the operator \( T_0 \) are with simple multiplicity and there exists a positive constant \( r \), independent of \( n \), such that
\[
\{ z \in \mathbb{C} \text{ such that } |z - \lambda_n| < r \} \cap \sigma(T_0) = \{\lambda_n\} \quad \text{for all } n \in \mathbb{N}^*;
\]
then, there exists a sequence of complex \((\epsilon_n)_n \in \mathbb{N}^*\) and a sequence of eigenvectors of the operator (1.1) having the form
\[
\varphi_n(\epsilon) = \varphi_n + \epsilon \varphi_n^1 + \epsilon^2 \varphi_n^2 + \cdots
\]
such that the system \((\varphi_n(\varepsilon_n))_n\) forms a Riesz basis of \(X\). Also, by using the result of A. Intissar [6], we prove the same result, if we suppose that the eigenvalues \(\lambda_n\) of the operator \(T_0\) are with finite multiplicity.

We organize the paper in the following way: in Section 2 some preliminary abstract results are given. In Section 3 we prove that the system formed by some of generalized eigenvectors of the operator (1.1) which are analytic on \(\varepsilon\), forms a Riesz basis of \(X\). The main result of this section is Theorem 3.2. In Section 4 we use the results obtained in Section 3 in a problem concerning the radiation of a vibrating structure in a light fluid.

2. Some preliminary results

Let \(X\) and \(Y\) be two Banach spaces. By an operator \(A\) from \(X\) to \(Y\) we mean a linear operator with domain \(D(A) \subset X\) and range \(R(A) \subset Y\). We denote by \(C(X, Y)\) (respectively \(L(X, Y)\)) the set of all closed, densely defined linear operators (respectively the Banach algebra of all bounded linear operators) from \(X\) into \(Y\). For \(A \in C(X, Y)\), we let \(\sigma(A)\), \(\rho(A)\) and \(N(A)\) denote respectively the spectrum, the resolvent set and the null space of \(A\). The nullity, \(\alpha(A)\), of \(A\) is defined as the dimension of \(N(A)\) and the deficiency, \(\beta(A)\), of \(A\) is defined as the codimension of \(R(A)\) in \(Y\). The set of Fredholm operators from \(X\) into \(Y\) is defined by

\[
\Phi(X, Y) = \left\{ A \in C(X, Y) \text{ such that } \alpha(A) < \infty, \ R(A) \text{ is closed in } Y \text{ and } \beta(A) < \infty \right\}.
\]

**Definition 2.1.** Let \(X\) and \(Y\) be two Banach spaces and let \(F \in L(X, Y)\). \(F\) is called a Fredholm perturbation if \(U + F \in \Phi(X, Y)\) whenever \(U \in \Phi(X, Y)\).

The set of Fredholm perturbations is denoted by \(\mathcal{F}(X, Y)\). This class of operators is introduced and investigated in [5]. In particular, it is shown, in [2], that \(\mathcal{F}(X, Y)\) is a closed subset of \(L(X, Y)\), and \(X = Y\), then \(\mathcal{F}(X) = \mathcal{F}(X, X)\) is closed two-sided ideal of \(L(X) := L(X, X)\).

Let \(A \in \mathcal{C}(X) := \mathcal{C}(X, X)\), then it follows from the closedness of \(A\) that \(D(A)\) endowed with the graph norm \(\|\|_A\) (i.e., \(\|x\|_A := \|x\| + \|Ax\|\)) is a Banach space. Let \(X_A\) denote \((D(A), \|\|_A)\), in this new space the operator \(A\) satisfies \(\|Ax\| \leq \|x\|_A\), and this proves that \(A\) is a bounded operator from \(X_A\) into \(X\) (i.e., \(A \in L(X_A, X)\)).

Let \(J \in L(X)\). If \(D(A) \subset D(J)\), then \(J\) will be called \(A\)-defined. If \(J\) is \(A\)-defined, we will denote by \(\hat{J}\) its restriction to \(D(A)\). Moreover, if \(\hat{J} \in L(X_A, X)\), we say that \(J\) is \(A\)-bounded. One checks easily that if \(J\) is closed (or closable) (cf. [9, Remark 1.5, p. 191]) then \(J\) is \(A\)-bounded.

**Definition 2.2.** An operator \(J\) is called \(A\)-closed if \(x_n \to x, Ax_n \to y\) and \(Jx_n \to z\) for \((x_n)_n \in D(A)\) implies that \(x \in D(A)\) and \(Jx = z\).

It will be called \(A\)-closable if \(x_n \to 0, Ax_n \to 0\) and \(Jx_n \to z\) implies that \(z = 0\).
Remark 2.1.

(i) If $J$ is bounded, then $J$ is $A$-bounded.
(ii) If $J$ is closed, then $J$ is $A$-closed.
(iii) If $J$ is closable, then $J$ is $A$-closable.
(iv) If $A$ is closed, then, by [12, Lemma 2.1], we have $J$ is $A$-closed if and only if $J$ is $A$-closable if and only if $J$ is $A$-bounded.

Definition 2.3. Let $X$ be a Banach space and let $A \in \mathcal{C}(X)$. We define the Schechter essential spectrum of the operator $A$ by

$$\sigma_{ess}(A) = \bigcap_{K \in \mathcal{K}(X)} \sigma(A + K)$$

where $\mathcal{K}(X)$ denotes the set of all compact operators on $X$.

Let $A \in \mathcal{C}(X)$ and let $J$ be an $A$-bounded operator on $X$. The following result gives some information about the eigenvalue of the perturbed operator $A + J$ knowing some information about the spectrum of $A$.

Theorem 2.1. Let $X$ be a Banach space, $\mathcal{I}(X)$ an arbitrary nonzero two-sided ideal of $\mathcal{L}(X)$ satisfying the condition

$$\mathcal{K}(X) \subseteq \mathcal{I}(X) \subseteq \mathcal{J}(X)$$

where $\mathcal{K}(X)$ and $\mathcal{J}(X) := \{F \in \mathcal{L}(X) \text{ such that } (I - F) \in \Phi(X) \text{ and } i(I - F) = 0\}$. Let $A \in \mathcal{C}(X)$ such that $\sigma_{ess}(A) = \emptyset$ (i.e., $\sigma(A) = \sigma P(A)$ where $\sigma P(A)$ denotes the point spectrum of $A$). Let $J \in \mathcal{C}(X)$ such that $J$ is $A$-bounded and $J(\lambda - A)^{-1} \in \mathcal{I}(X)$ for some $\lambda \in \rho(A)$. Then $\sigma(A + J) = \sigma P(A + J)$.

Proof. This theorem immediately follows from [7, Theorem 2.1, Corollary 2.2 and Remark 2.1(c)].

Remark 2.2. Theorem 2.1 may be regarded as an extension of [8, Theorem 2.1] to unbounded perturbations.

Definition 2.4. Let $K$ be a compact operator on the Hilbert space $\mathcal{H}$. $K$ is said to belong to the Carleman-class $\mathcal{C}_p$, $p > 0$, with order $p$, if the series $\sum_{n=1}^{\infty} [s_n(\sqrt{K})]^p$ converge, where $s_n(\sqrt{K})$, $n = 1, 2, \ldots$, are the eigenvalues of the operator $\sqrt{K} \ast K$.

In the particular case $p = 1$ (respectively $p = 2$) $\mathcal{C}_1$ (respectively $\mathcal{C}_2$) is exactly the nuclear or trace operators (respectively Hilbert–Schmidt class). For a systematic treatment of the operators of Carleman-class we refer to the Gohberg and Krein’s book [4].

Let us now recall the result due to M.V. Keldys:

Theorem 2.2. Let $H$ be a closed densely defined linear operator on a Hilbert space $\mathcal{H}$. We suppose that the operator $H$ is self-adjoint and belongs to Carleman-class $\mathcal{C}_p$, $p > 0$. 
Let $A = H(I + S)$ where $S$ is a compact operator in $H$. If the operator $A$ vanishes only at zero, then the system of its root vectors is complete in $H$.

Now, we must recall two results very important due to B.Sz. Nagy [11].

**Theorem 2.3.** [11, Theorem 3] Let $X$ and $Y$ be two Banach spaces and let $T_0, T_1, T_2, \ldots$ some linear transformations from $X$ into $Y$, which all have the same domain $D \subset X$. Let $T_0$ be a closed operator. We suppose that there exists $a, b, q \geq 0$ such that

$$
\|T_k\psi\| \leq q^{k-1}(a\|\psi\| + b\|T_0\psi\|), \quad k = 1, 2, \ldots
$$

In these conditions, the series

$$
T_0\psi + \varepsilon T_1\psi + \varepsilon^2 T_2\psi + \cdots + \varepsilon^k T_k\psi + \cdots
$$

converges for all $\psi \in D$ and for $|\varepsilon| < q^{-1}$. Let its limit be $T_\varepsilon\psi$. $T_\varepsilon$ is a linear transformation with domain $D$. For $|\varepsilon| < (q + b)^{-1}$, the transformation $T_\varepsilon$ is closed.

**Theorem 2.4.** [11] Under the hypotheses of Theorem 2.3 and if $\lambda_0$ is an isolated point of $T_0$’s spectrum with multiplicity one, the perturbed transformation $T_\varepsilon$ will have, for $|\varepsilon|$ enough small, a unique point of the spectrum in the neighborhood of $\lambda_0$, and this point $\lambda(\varepsilon)$ will be also with multiplicity one. $\lambda(\varepsilon)$ and corresponding eigenvectors $(\varphi(\varepsilon)$ of $T_\varepsilon$ can be developed into an entire series of $\varepsilon$.

**Definition 2.5.** A sequence $(e_n)_n$ is a basis for a separable Hilbert space $H$ if

(i) it is complete, i.e., $\langle u, e_n \rangle = 0 \ \forall \ n \in \mathbb{N} \Rightarrow u = 0$,

(ii) the $e_n$ are independent, i.e.,

$$
\text{if } \sum_{n=0}^{\infty} c_n e_n = 0 \quad \text{for some sequence } (c_n)_n \in l^2, \quad \text{then } c_n = 0 \text{ for all } n \in \mathbb{N}.
$$

A Riesz basis has the additional property that the $\langle e_n, u \rangle$ obey

$$
A \|u\|^2 \leq \sum_{n=0}^{\infty} |\langle e_n, u \rangle|^2 \leq B \|u\|^2,
$$

for strictly positive constants $A$ and $B$ independent of $u$.

**Theorem 2.5.** [13, Lemma 1.1] Let $(e_n)_n$ be a Riesz basis of a separable Hilbert space $H$. Let $(d_n)_n$ be a collection of vectors in $H$ such that

$$
\sum_{n=0}^{\infty} \|d_n - e_n\|^2 < \frac{A^2}{B}.
$$

Then, the vectors $(d_n)_n$ form a Riesz basis of $H$. 
3. The main result

Let $X$ and $Y$ be two separable Hilbert spaces and let $T_0$ be an operator from $X$ into $Y$ satisfying the hypotheses:

(H1) $T_0$ is a closed linear operator with domain $D(T_0)$ dense in $X$.
(H2) The eigenvectors $(\varphi_n)_n$ of $T_0$ form a Riesz basis of $X$.

Let $T_1, T_2, T_3, \ldots$ some linear transformations from $X$ into $Y$ having the same domain $D$ and satisfying the hypothesis:

(H3) $D(T_0) \subset D$ and there exists $a, b, q \geq 0$ such that for all $\varphi \in D$,
\[ \|T_k \varphi\| \leq q^{k-1} (a\|\varphi\| + b\|T_0 \varphi\|) \quad \forall k \in \mathbb{N}^*. \]

Let $\varepsilon \in \mathbb{C}^*$ and consider the eigenvector problem:
\[
\begin{align*}
T_0 \varphi + \varepsilon T_1 \varphi + \varepsilon^2 T_2 \varphi + \cdots + \varepsilon^k T_k \varphi + \cdots &= \lambda \varphi, \\
\varphi &\in D(T_0).
\end{align*}
\]

Theorem 3.1. Assume that the resolvent of $T_0$ belongs to Carleman-class $C_p$, $p > 0$ and let hypotheses (H1) and (H3) be satisfied. Then

(i) For $|\varepsilon| < (q + b)^{-1}$, the series
\[ T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \cdots + \varepsilon^k T_k + \cdots \]
converges and its limit $T_\varepsilon$ is a closed operator.

(ii) For some $\lambda \in \rho(T_0)$ and for $|\varepsilon| < \frac{1}{q+M(\lambda,a,b)}$, the operator $(\lambda I - T_\varepsilon)^{-1}$ belongs to Carleman-class $C_p$, $p > 0$, where
\[ M(\lambda, a, b) = a\| (\lambda I - T_0)^{-1} \| + b[1 + |\lambda|\| (\lambda I - T_\varepsilon)^{-1} \|]. \]

Proof. (i) The assertion (i) follows from Theorem 2.3.

(ii) Let $\lambda \in \rho(T_0)$, we have
\[
\begin{align*}
\lambda I - T_\varepsilon &= \lambda I - T_0 - \varepsilon T_1 - \varepsilon^2 T_2 - \cdots \\
&= \left[ I - \varepsilon T_1 (\lambda I - T_0)^{-1} - \varepsilon^2 T_2 (\lambda I - T_0)^{-1} - \cdots \right] (\lambda I - T_0).
\end{align*}
\]

Let
\[ S = T_1 (\lambda I - T_0)^{-1} + \varepsilon T_2 (\lambda I - T_0)^{-1} + \cdots. \]

So,
\[ \lambda I - T_\varepsilon = (I - \varepsilon S)(\lambda I - T_0). \tag{3.1} \]

For every $\varphi \in R(T_0)$ such that $\varphi \neq 0$, we have the following estimate
\[ \|S \varphi\| \leq \sum_{k=1}^{+\infty} |\varepsilon|^{k-1} \|T_k (\lambda I - T_0)^{-1} \varphi\|. \tag{3.2} \]
Using the hypothesis (H3), we have
\[ \| T_k(\lambda I - T_0)^{-1} \varphi \| \leq q_k^{-1} \left[ a \| (\lambda I - T_0)^{-1} \varphi \| + b \| T_0(\lambda I - T_0)^{-1} \varphi \| \right] \leq q_k^{-1} \left[ a \| (\lambda I - T_0)^{-1} \varphi \| + b - \varphi + \lambda (\lambda I - T_0)^{-1} \varphi \| \right] \leq q_k^{-1} \left[ a \| (\lambda I - T_0)^{-1} \varphi \| + b (\| \varphi \| + |\lambda| \| (\lambda I - T_0)^{-1} \varphi \|) \right]. \]

So,
\[ \| T_k(\lambda I - T_0)^{-1} \varphi \| \leq q_k^{-1} \left[ a \| (\lambda I - T_0)^{-1} \| + b \| 1 + |\lambda| \| (\lambda I - T_0)^{-1} \| \right] \| \varphi \|. \tag{3.3} \]

Substituting the inequality (3.3) in (3.2), we get
\[ \| S \varphi \| \leq \sum_{k=1}^{+\infty} (|\varepsilon| q)^{k-1} \left[ a \| (\lambda I - T_0)^{-1} \| + b (1 + |\lambda| \| (\lambda I - T_0)^{-1} \|) \right] \| \varphi \|. \]

Therefore
\[ \frac{\| S \varphi \|}{\| \varphi \|} \leq \sum_{k=1}^{+\infty} (|\varepsilon| q)^{k-1} \left[ a \| (\lambda I - T_0)^{-1} \| + b (1 + |\lambda| \| (\lambda I - T_0)^{-1} \|) \right]. \]

So,
\[ \| S \| \leq M(\lambda, a, b) \sum_{k=1}^{+\infty} (|\varepsilon| q)^{k-1}, \]

where \( M(\lambda, a, b) = a \| (\lambda I - T_0)^{-1} \| + b (1 + |\lambda| \| (\lambda I - T_0)^{-1} \|) \). Since \( |\varepsilon| q < 1 \), then
\[ \| S \| \leq \frac{M(\lambda, a, b)}{1 - |\varepsilon| q}, \]

and therefore for \( |\varepsilon| < \frac{1}{q + M(\lambda, a, b)} \), we have \( |\varepsilon| \| S \| < 1 \) and \( I - \varepsilon S \) is invertible. Using Eq. (3.1), we infer that \( \lambda I - T_\varepsilon \) is invertible and
\[ (\lambda I - T_\varepsilon)^{-1} = (\lambda I - T_0)^{-1} (I - \varepsilon S)^{-1}. \tag{3.4} \]

Since the resolvent of \( T_0 \) belongs to Carleman-class \( C_p, p > 0 \), the rest of the proof of (ii) follows from Eq. (3.4) and the boundedness of the linear operator \( (I - \varepsilon S)^{-1} \). \( \Box \)

**Proposition 3.1.** Let \( T_0 \) a self-adjoint operator which resolvent’s belongs to Carleman-class \( C_p, p > 0 \). Suppose that \( T_0 \) satisfies the hypotheses (H1) and (H3) and that, for all \( \delta > 0 \), there exists a constant \( c_\delta > 0 \) such that
\[ \| T_i \varphi \| \leq \delta \| T_0 \varphi \| + c_\delta \| \varphi \| \ \forall \varphi \in D(T_0) \text{ and } \forall i \in \mathbb{N}^*. \]

Then
(i) \( T_i \) is \( T_0 \)-compact for all \( i \in \mathbb{N}^* \);
(ii) for \( |\varepsilon| < \frac{1}{q + M(\lambda, a, b)} \), the system of generalized eigenvectors of the operator \( T_\varepsilon \) is complete in \( X \).
Proof. (i) This assertion is an immediate consequence of [1].

(ii) If $\lambda \in \rho(T_0) \cap \mathbb{R}$ and if $|\varepsilon| < \frac{1}{q + M(\lambda, a, b)}$, then $\lambda \in \rho(T_{\varepsilon})$ and using Eq. (3.1) we have,

$$(\lambda I - T_{\varepsilon})^{-1} = (\lambda I - T_0)^{-1} (I - \varepsilon S)^{-1} = (\lambda I - T_0)^{-1} (I + \varepsilon S + \varepsilon^2 S^2 + \cdots).$$

Next, using the assertion (i), we infer that the operator $\varepsilon S + \varepsilon^2 S^2 + \cdots$ is a compact operator on $X$. The result of (ii) follows from Eq. (3.5) and the Keldys theorem (see Theorem 2.2). □

Theorem 3.2. Assume that (H1)–(H3) hold. If the eigenvalues $(\lambda_n)_n$ of the operator $T_0$ are with simple multiplicity and there exists a positive constant $r$, independent of $n$, such that

$$\{ z \in \mathbb{C} \text{ such that } |z - \lambda_n| < r \} \cap \sigma(T_0) = \{ \lambda_n \} \quad \text{for all } n \in \mathbb{N}^*,$$

then there exists a sequence of complex $(\epsilon_n)_n \in \mathbb{N}^*$ and a sequence of eigenvectors of $T_{\varepsilon}$ having the form

$$\varphi_n(\varepsilon) = \varphi_n + \varepsilon \varphi_n^1 + \varepsilon^2 \varphi_n^2 + \cdots$$

such that the system $\{ \varphi_n(\varepsilon) \}_n \in \mathbb{N}^*$ forms a Riesz basis of $X$.

Proof. Let $n \in \mathbb{N}^*$, we note $\sigma_0 = \{ \lambda_n \}$ and $\sigma_1 = \sigma(T_0) \setminus \sigma_0$ where $\lambda_n$ is the eigenvalue number $n$, with principal multiplicity one, of the operator $T_0$. Let $C_n = C(\lambda_n, r)$ the closed circle with center $\lambda_n$ and with radii $r$, lying in the resolvent set of $T_0$, and having $\sigma_0$ inside and $\sigma_1$ outside. Since $(\lambda I - T_0)^{-1}$ is a regular analytic function of $\lambda \in \rho(T_0)$, $\| (\lambda I - T_0)^{-1} \|$ is a continuous function. So, we denote by

$$\mathcal{M} = \sup_{\lambda \in C_n} \| (\lambda I - T_0)^{-1} \|,$n

$$\mathcal{N} = \sup_{\lambda \in C_n} \| T_0 (\lambda I - T_0)^{-1} \| = \sup_{\lambda \in C_n} \| -I + \lambda (\lambda I - T_0)^{-1} \|,$n

$$\alpha = a \mathcal{M} + b \mathcal{N},$$

and

$$\omega = \| \varphi_n \|,$n

where $\varphi_n$ is an eigenvector of $T_0$ associated to the eigenvalue $\lambda_n$. Due to [11], for

$$|\varepsilon| < \frac{1}{(q + \alpha + \omega^2 r \mathcal{M} \alpha)},$$n

the perturbed transformation $T_{\varepsilon}$ will have a unique point of the spectrum inside of the circle $C_n = C(\lambda_n, r)$ and this point $\lambda_n(\varepsilon)$ will be also with principal multiplicity 1 (see Theorem 2.4). The eigenvalue $\lambda_n(\varepsilon)$ and the eigenvector $\varphi_n(\varepsilon)$ of the perturbed operator $T_{\varepsilon}$ can be developed as entire series of $\varepsilon$ and we have

$$\lambda_n(\varepsilon) = \lambda_n + \varepsilon \varphi_n^1 + \varepsilon^2 \varphi_n^2 + \cdots,$n

$$\varphi_n(\varepsilon) = \varphi_n + \varepsilon \varphi_n^1 + \varepsilon^2 \varphi_n^2 + \cdots,$n

(3.7) (3.8)
where \( \lambda_n^i \) and \( \varphi_n^i \) satisfy the following estimations:
\[
\begin{align*}
|\lambda_n^i| &\leq \omega r^2 M_\alpha (q + \alpha + \omega^2 r M_\alpha)^{i-1}, \\
\|\varphi_n^i\| &\leq \omega r M (q + \alpha + \omega^2 r M_\alpha)^i.
\end{align*}
\]
(3.9)

For the details we refer to [11]. On the other hand, by hypothesis (H2) the eigenvectors \((\varphi_n)_n\) of \( T_0 \) form a Riesz basis of \( X \). So, there exists \( A > 0 \) and \( B > 0 \) such that
\[
A \|u\|^2 \leq \sum_{n=1}^{+\infty} |\langle \varphi_n, u \rangle|^2 \leq B \|u\|^2 \quad \text{for all } u \in X.
\]
Let \( n \in \mathbb{N}^* \). For each eigenvalue \( \lambda_n \) of \( T_0 \), we fix an \( \varepsilon_n \in \mathbb{C} \) such that
\[
|\varepsilon_n| \in \left[ 0, \frac{A}{(\beta \omega r M\sqrt{B} + A)(q + \alpha + \omega^2 r M_\alpha)} \right].
\]
where
\[
\beta^2 = \sum_{k=1}^{\infty} \frac{1}{k^2}.
\]
Since
\[
\left[ 0, \frac{A}{(\beta \omega r M\sqrt{B} + A)(q + \alpha + \omega^2 r M_\alpha)} \right] \subset \left[ 0, \frac{1}{(q + \alpha + \omega^2 r M_\alpha)} \right],
\]
using Eqs. (3.7) and (3.8), the eigenvalue \( \lambda_n(\varepsilon_n) \) and the eigenvector \( \varphi_n(\varepsilon_n) \) can be developed as entire series of \( \varepsilon_n \) and we have
\[
\begin{align*}
\lambda_n(\varepsilon_n) &= \lambda_n + \varepsilon_n \lambda_n^1 + \varepsilon_n^2 \lambda_n^2 + \cdots, \\
\varphi_n(\varepsilon_n) &= \varphi_n + \varepsilon_n \varphi_n^1 + \varepsilon_n^2 \varphi_n^2 + \cdots.
\end{align*}
\]
Using the last equation and the estimate (3.9), we obtain the following estimation
\[
\|\varphi_n(\varepsilon_n) - \varphi_n\| \leq \sum_{k=1}^{+\infty} |\varepsilon_n|^k \omega r M (q + \alpha + \omega^2 r M_\alpha)^k \\
\leq \omega r M \sum_{k=1}^{+\infty} |\varepsilon_n|(q + \alpha + \omega^2 r M_\alpha)^k \\
\leq \frac{\omega r M |\varepsilon_n|(q + \alpha + \omega^2 r M_\alpha)}{1 - |\varepsilon_n|(q + \alpha + \omega^2 r M_\alpha)} \\
\leq \frac{A}{\beta n \sqrt{B}}.
\]

We deduce that
\[
\sum_{n=1}^{+\infty} \|\varphi_n(\varepsilon_n) - \varphi_n\|^2 \leq \sum_{n=1}^{+\infty} \frac{A^2}{\beta^2 n^2 B} = \frac{A^2}{B}.
\]
Using Theorem 2.5, we deduce that the system \((\varphi_n(\varepsilon_n))_{n \in \mathbb{N}^*}\) forms a Riesz basis of the space \( X \) and this achieves the proof. \( \square \)
Remark 3.1. If the eigenvalues $(\lambda_n)_n$ of the operator $T_0$ are with simple multiplicity ordered as an ascending sequence in modulus and satisfying the condition
\[
\lim_{n \to +\infty} |\lambda_{n+1}| - |\lambda_n| = +\infty,
\]
then the condition (3.6) is verified.

Arguing as the proof of the last theorem and using the result of A. Intissar [6, Theorem 3], we have a more general result than Theorem 3.2.

Theorem 3.3. Assume that (H1)–(H3) hold. If the operator $P_j T_1|_{E_j}$, has at least one eigenvalue $\sigma_j^1$ with simple multiplicity, where $E_j$ is the eigenspace with $m_j$ dimension associated to the eigenvalue $\lambda_j$ of $T_0$ and $P_j$ is the spectral projector on $E_j$. Then

(i) for a sufficiently small $|\varepsilon|$, the eigenvalues $\{\lambda_n\}_{n \in \mathbb{N}^*}$ and the eigenvectors $\{\varphi_n\}_{n \in \mathbb{N}^*}$ of $T_\varepsilon$ can be developed as entire series of $\varepsilon$
\[
\lambda_n(\varepsilon) = \lambda_n + \varepsilon \sigma_n^1 + \varepsilon^2 \lambda_n^2 + \cdots,
\]
\[
\varphi_n(\varepsilon) = \varphi_n + \varepsilon \varphi_n^1 + \varepsilon^2 \varphi_n^2 + \cdots;
\]
(ii) there exists a sequence of complex $(\varepsilon_n)_{n \in \mathbb{N}^*}$ and a sequence of eigenvectors of $T_\varepsilon$ having the form
\[
\varphi_n(\varepsilon_n) = \varphi_n + \varepsilon_n \varphi_n^1 + \varepsilon_n^2 \varphi_n^2 + \cdots \tag{3.10}
\]
such that the system $\{\varphi_n(\varepsilon_n)\}_{n \in \mathbb{N}^*}$ forms a Riesz basis of $X$.

4. Application

We consider an elastic membrane lying the domain $-a < x < a$ of the plane $y = 0$. It is embedded along the two straights $x = -a$ and $x = a$, in the two half-planes perfectly rigid $(x < -a, \ y = 0)$ and $(x > a, \ y = 0)$. The two half-spaces $y < 0$ and $y > 0$ are filled with gas. Finally, the membrane is excited by a harmonic force $\exp(-i\omega t)$ with an amplitude $f(x)$ which is independent of the third space variable. Thus, the equation of the motion of the membrane is reduced to the equation of the vibrant cord. The fluid motion is described by a Holmholtz equation in $\mathbb{R}^2$. The physical characteristics of the system are:

- $\rho_0$: fluid density.
- $c_0$: fluid sound speed.
- $\rho_1$: surface density of the membrane.
- $T_1$: the membrane tightness.
- $c_1 := (\rho_1/T_1)^{-1/2}$: the flexion wave speed in the membrane.

Let $u$ denote the displacement of the membrane and $p$ the acoustic pressure in the fluid. The motion equations are given by:
\[ (\Delta + \frac{\omega^2}{c_0^2}) p(M) = 0 \quad \text{in } y < 0 \text{ and } y > 0; \tag{4.1} \]

\[ \left( \frac{d^2}{dx^2} + \frac{\omega^2 \rho^2}{T_1^2} \right) u(x) = -\frac{1}{T_1^2} (f(x) - p^+(x) + p^-(x)); \tag{4.2} \]

\[ p^+(x) = \lim_{y \to 0^+} p[M(x, y)], \quad p^-(x) = \lim_{y \to 0^-} p[M(x, y)], \tag{4.3} \]

\[ \frac{\partial p}{\partial y} [M(x, 0)] = \omega^2 \rho_0 u(x), \quad \text{for } -a < x < a; \tag{4.4} \]

\[ \frac{\partial p}{\partial y} [M(x, 0)] = 0, \quad \text{for } x < -a \text{ and } x > a; \tag{4.5} \]

\[ u(-a) = u(a) = 0; \tag{4.6} \]

Sommerfeld condition for \( p \). \tag{4.7} \]

Using Eqs. (4.1), (4.4), (4.5) and (4.7), the pressure \( p(M) \) is given by the following integral:

\[ p(M) = -i \frac{\omega^2 \rho_0}{2} \int_{-a}^{a} H_0(k_0 \sqrt{(x - x')^2 + y^2}) u(x') \, dx', \quad \text{for } y > 0, \tag{4.8} \]

\[ p(M) = +i \frac{\omega^2 \rho_0}{2} \int_{-a}^{a} H_0(k_0 \sqrt{(x - x')^2 + y^2}) u(x') \, dx', \quad \text{for } y < 0, \tag{4.9} \]

where \( H_0(z) = J_0(z) + i Y_0(z) \) is the Hankel function of the first kind and order 0, and \( k_0 = \omega / c_0 \) is the wave number of the fluid. Using (4.2), (4.3), (4.6), (4.8) and (4.9) lead to the boundary value problem

\[ \frac{d^2 u}{dx^2}(x) + \omega^2 \left\{ \rho_1 \frac{u(x)}{T_1} + \frac{i \rho_0}{T_1} \int_{-a}^{a} H_0(k_0|x - x'|) u(x') \, dx' \right\} = \frac{f(x)}{T_1}, \quad \forall -a < x < a, \tag{4.10} \]

\[ u(-a) = u(a) = 0. \tag{4.11} \]

In the sequel we shall need the following operators:

\[
\begin{aligned}
T_0: & \mathcal{D}(T_0) \subset L^2([-a, a]) \to L^2([-a, a]), \\
\psi & \mapsto T_0 \psi(x) = -\frac{d^2 \psi}{dx^2}(x), \\
\mathcal{D}(T_0) & = H_0^1([-a, a]) \cap H^2([-a, a]),
\end{aligned}
\]

and

\[
\begin{aligned}
K: & L^2([-a, a]) \to L^2([-a, a]), \\
\psi & \mapsto K \psi(x) = \frac{i}{\pi} \int_{-a}^{a} H_0(k_0|x - x'|) \psi(x') \, dx'.
\end{aligned}
\]

From the problem (4.10)–(4.11) P.J.T. Filippi in [3] has considered the eigenvalue problem:
Find the values \( \lambda \in \mathbb{C} \) for which there is a solution \( u \in H^1_0(]−a, a[) \cap H^2(]−a, a[) \), \( u \neq 0 \) for the equation

\[
T_0u = \lambda (I + \varepsilon K)u, 
\]

where \( \lambda = \omega^2 \rho_1/T_1 \) and \( \varepsilon = 2\rho_0/\rho_1 \).

According to the definition given in [10, Chapter 9, Section 4], \( \lambda \) is the eigenvalue and \( u \) is the eigenmode. Note that \( \lambda \) and \( u \) each depend on the value of \( \varepsilon \). So, we denote \( \lambda := \lambda(\varepsilon) \) and \( u := u(\varepsilon) \).

**Lemma 4.1.** We have the following assertions:

(i) \( T_0 \) is a closed operator.
(ii) The injection from \( \mathcal{D}(T_0) \) into \( L^2(]−a, a[) \) is compact.
(iii) The resolvent set of \( T_0 \) is not empty. In fact \( 0 \in \rho(T_0) \).
(iv) \( T_0 \) is a self-adjoint operator with compact resolvent.
(v) The continuous spectrum and the residual spectrum of \( T_0 \) are empty.
(vi) The spectrum of \( T_0 \) is constituted only of eigenvalues which are positive, denumerable and of which the multiplicity is one and which have no finite limit points and satisfies

\[
0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \to +\infty \quad \text{as} \quad n \to +\infty.
\]

Further,

\[
\lambda_n = \left( \frac{n\pi}{2a} \right)^2.
\]

**Proof.** (i) It suffices to proof that the graph \( \Gamma(T_0) \) is closed. In fact, let \( \lambda \in \rho(T_0) \), and \( \langle \psi_n, T_0 \psi_n \rangle \in \Gamma(T_0) \) which converges to \( \langle \psi, \varphi \rangle \). We must show that \( \psi \in H^1_0(]−a, a[) \cap H^2(]−a, a[) \) and \( \varphi = T_0 \psi \). The sequence \( \psi_n \) converges to \( \psi \) in \( H^1(]−a, a[) \) and \( \psi_n \in H^1_0(]−a, a[) \) so \( \psi \in H^1_0(]−a, a[) \). The fact that \( \psi_n \) converges to \( \psi \) in \( H^1(]−a, a[) \) implies \( \psi_n \) converges to \( \psi \) in \( L^2(]−a, a[) \). Hence, we infer that \( \lambda \psi_n - T_0 \psi_n \) converges to \( \lambda \psi - \varphi \) in \( L^2(]−a, a[) \). Since \( \lambda \in \rho(T_0) \) then \( \psi_n \) converges to \( (\lambda I - T_0)^{-1}(\lambda \psi - \varphi) \) in \( L^2(]−a, a[) \). Since \( \psi_n \) converges to \( \psi \) in \( L^2(]−a, a[) \) we have \( \psi = (\lambda I - T_0)^{-1}(\lambda \psi - \varphi) \) and \( (\lambda I - T_0)\psi = \lambda \psi - \varphi \) which implies that \( T_0 \psi = \varphi \). On the other hand \( \varphi \in L^2(]−a, a[) \) so we get \( T_0 \psi \in L^2(]−a, a[) \) and as \( \psi \in H^1(]−a, a[) \) then \( \psi \in H^2(]−a, a[) \). This completes the proof of (i).

The proofs of the items (ii), (iv) and (v) are evident.

(iii) Let \( \varphi \in N(T_0) \). Since \( T_0 \) is \( V \)-elliptic then \( \varphi = 0 \) since \( \varphi \) is continuous. This implies that \( N(A) = \{0\} \). Let \( \psi \in L^2(]−a, a[) \), we look for \( \varphi \in H^1_0(]−a, a[) \cap H^2(]−a, a[) \) such that \( T_0 \varphi = \psi \) with the form

\[
\varphi(x) = ax + \beta + \int_{]−a, a[}^{x} \psi(s) \, ds \, d\alpha, \\
\varphi(-a) = 0 \quad \text{and} \quad \varphi(a) = 0.
\]
We let
\[ \eta = \int_{-a}^{a} \int_{-a}^{a} \psi(s) \, ds \, d\alpha. \]

We find ourselves with a linear system of two equations with two unknowns,
\[ \begin{pmatrix} -a & 1 \\ a & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ \eta \end{pmatrix}. \]

The last system has a unique solution since the matrix’s determinant is nonzero. As
\[ |\eta| \leq (2a)^2 \|\psi\| \]
and
\[ \left| \int_{-a}^{a} \int_{-a}^{a} \psi(s) \, ds \, d\alpha \right| \leq (2a)^2 \|\psi\|, \]
then
\[ |\varphi(x)| \leq a|\alpha| + |\beta| + \left| \int_{-a}^{a} \int_{-a}^{a} \psi(s) \, ds \, d\alpha \right|. \]

Hence
\[ |\varphi(x)| \leq c \|\psi\| \]
and
\[ \|(0 - T_0)^{-1}\| = \sup_{\|\psi\| \leq 1} \|(0 - T_0)^{-1}\psi\| \leq c. \]

This achieves the proof of (ii).

(vi) We consider the eigenvalue problem
\[ \begin{cases} T_0 \varphi = \lambda \varphi, \\ \varphi \in \mathcal{D}(T_0). \end{cases} \tag{4.13} \]

The solution of the problem (4.13) is formally given by
\[ \varphi(x) = \alpha e^{i \sqrt{\lambda_n} x} + \beta e^{-i \sqrt{\lambda_n} x} \]
and satisfies the following boundary conditions:
\[ \varphi(-a) = \varphi(a) = 0. \]

So, we find ourselves with a linear system of two equations with two unknowns and which are vanish. To avoid trivial solution it suffices that the determinant of the following system be zero:
\[ \det \begin{pmatrix} e^{i \sqrt{\lambda_n} a} & e^{-i \sqrt{\lambda_n} a} \\ e^{-i \sqrt{\lambda_n} a} & e^{i \sqrt{\lambda_n} a} \end{pmatrix} = 0. \]
So

\[ \sin \left( 2a \sqrt{\lambda_n} \right) = 0, \]

i.e.,

\[ \lambda_n = \frac{n^2 \pi^2}{4a^2}. \]

This completes the proof. \( \square \)

**Theorem 4.1.** The resolvent of the operator \( T_0 \) belongs to Carleman-class \( C_p \) for any \( p > 1/2 \).

**Proof.** This follows from Lemma 4.1(v) and the fact that \( T_0 \) is a self-adjoint operator. \( \square \)

**Proposition 4.1.** (i) Let \( (\lambda_n)_n \) the eigenvalues of \( T_0 \). Then for each \( \lambda \neq \lambda_n, n \in \mathbb{N}^* \), we have

\[ (\lambda I - T_0)^{-1} \varphi = \sum_{n=1}^{\infty} \frac{1}{\lambda - \lambda_n} \langle \varphi, \varphi_n \rangle \varphi_n, \]

where \( (\varphi_n)_n \) are the corresponding eigenfunctions of \( (\lambda_n)_n \). Moreover, if \( \text{Im} \lambda \neq 0 \) and if \( \lambda \) belongs to a ray with origin zero and of angle \( \theta \) with \( \theta \neq 0 \) and \( \theta \neq \pi \), we have

\[ \| (\lambda I - T_0)^{-1} \| \leq \frac{1}{|\text{Im} \lambda|} = \frac{c(\theta)}{|\lambda|}, \]

where \( c(\theta) \) is a positive constant that depends on \( \theta \).

(ii) There exists a sequence of circles \( C(O, r_n) \) \( n \in \mathbb{N}^* \) with radii \( r_n \) going to infinity such that

\[ \| (\lambda I - T_0)^{-1} \| \leq \frac{a \sqrt{10}}{\pi |\lambda|^{1/2}} \quad \text{and} \quad |\lambda| = r_n. \]

**Proof.** (i) This follows immediately from Lemma 4.1(iv).

(ii) Let \( n \in \mathbb{N}^* \), we have

\[ \lambda_{n+1} - \lambda_n \geq \frac{\pi^2}{2a^2 n}. \]

We set

\[ r_n = \frac{\lambda_{n+1} + \lambda_n}{2} = \lambda_n + \frac{\lambda_{n+1} - \lambda_n}{2}. \]

Using Lemma 4.1(iv), we have

\[ \| (\lambda I - T_0)^{-1} \| = \frac{1}{d(\lambda, \sigma(T_0))}, \]

where \( \lambda \) belongs to \( \rho(T_0) \) and \( d(\lambda, \sigma(T_0)) \) denotes the distance between the point \( \lambda \) and the spectrum of the operator \( T_0 \). From the Eq. (4.14) we obtain

\[ \| (\lambda I - T_0)^{-1} \| \leq \frac{2}{\lambda_{n+1} - \lambda_n}, \]
where \( r_n = |\lambda| \). On the other hand,
\[
r_n = \frac{\lambda_{n+1} + \lambda_n}{2} = \left( (n + 1)^2 + n^2 \right) \frac{\pi^2}{8a^2} \leq \frac{5\pi^2}{8a^2} r_n^{-1}.
\]
So,
\[
n^{-2} \leq \frac{5\pi^2}{8a^2} r_n^{-1}.
\]
Since
\[
(\lambda_{n+1} - \lambda_n)^{-1} \leq \frac{2a^2}{\pi^2 n} = \frac{2a^2}{\pi^2} (n^{-2})^{1/2} \leq \frac{a\sqrt{5}}{\pi\sqrt{2}} (r_n^{-1})^{1/2} \leq \frac{a\sqrt{5}}{\pi\sqrt{2}|\lambda|^{1/2}},
\]
we have the desired result. \( \square \)

**Theorem 4.2.** The system of eigenvectors of \( T_0 \) forms a Riesz basis in \( L^2([-a, a]) \).

**Proof.** Using [4, Chapter 6] the system of eigenvectors of the operator \( T_0 \) forms a basis in \( L^2([-a, a]) \). Now, the rest of the proof follows from Parseval’s formula and Definition 2.5. \( \square \)

For \( |\varepsilon| < 1/\|K\| \), the operator \( (I + \varepsilon K)^{-1} \) is invertible. So, the problem (4.12) becomes:

Find the values \( \lambda(\varepsilon) \in \mathbb{C} \) for which there is a solution \( \varphi \in H^1_0([-a, a]) \cap H^2([-a, a]), \varphi \neq 0 \) for the equation

\[
(I + \varepsilon K)^{-1} T_0 \varphi = \lambda(\varepsilon) \varphi. \tag{4.15}
\]

The last problem (4.15) is equivalent to:

Find the values \( \lambda(\varepsilon) \in \mathbb{C} \) for which there is a solution \( \varphi \in H^1_0([-a, a]) \cap H^2([-a, a]), \varphi \neq 0 \) for the equation

\[
(T_0 - \varepsilon K T_0 + \varepsilon^2 K^2 T_0 - \cdots + (-1)^k \varepsilon^k K^k T_0 + \cdots) \varphi = \lambda(\varepsilon) \varphi. \tag{4.16}
\]

We recall the following result obtained by A. Jeribi and A. Intissar [8], which is a preparation for the proof of the results in the rest of this section.

**Theorem 4.3.** [8, Theorem 4.2] The Hankel operator \( K \) is compact on \( L^2([-a, a]) \).

**Theorem 4.4.** (i) For \( |\varepsilon| < 1/\|K\| \), the series

\[
T_\varepsilon \psi := T_0 \psi - \varepsilon K T_0 \psi + \varepsilon^2 K^2 T_0 \psi - \cdots + (-1)^k \varepsilon^k K^k T_0 \psi + \cdots
\]

converges for all \( \psi \in H^1_0([-a, a]) \cap H^2([-a, a]). \)

(ii) For \( |\varepsilon| < 1/(2\|K\|) \), the transformation \( T_\varepsilon \) is closed.

(iii) For \( |\varepsilon| < 1/(2\|K\|) \), \( T_\varepsilon \) is an operator with a compact resolvent and its spectrum consists only of eigenvalues with nonzero imaginary parts and can be ordered as a sequence going to \( +\infty \) in modulus and the correspondent eigenspaces are finite dimensional.

(iv) For \( |\varepsilon| \neq 1/\|K\| \), the system of generalized eigenvectors of the problem (4.12) is dense in \( L^2([-a, a]) \).
(v) There exists a sequence of complex \((\epsilon_n)_n\) and a sequence of eigenmodes \((\varphi_n(\epsilon_n))_n\) of the operator \(T_\epsilon\) having the form (3.10) such that the system \((\varphi_n(\epsilon_n))_n\) forms a Riesz basis in \(L^2([-a, a])\).

**Proof.** (i) Let \(k \in \mathbb{N}\). We have,
\[
\|(-1)^k K^k T_0 \psi\| \leq \|K\|^k \|T_0 \psi\|.
\]
When taking \(b = \|K\|, q = \|K\|\) and \(a = 0\), then we are surely in the hypotheses of Theorem 2.3. So, the series
\[
T_\epsilon \psi := T_0 \psi - \epsilon K T_0 \psi + \epsilon^2 K^2 T_0 \psi - \cdots + (-1)^k \epsilon^k K^k T_0 \psi + \cdots
\]
converges for all \(\psi \in H^1([−a, a]) \cap H^2([−a, a])\) and for all \(|\epsilon| < 1/\|K\|\).

(ii) This assertion follows immediately from Theorem 2.3.

(iii) We claim that \(0 \in \rho(T_\epsilon)\).

In fact, let \(\psi \in N(T_\epsilon)\), i.e., \(T_\epsilon \psi = 0\) (since \(|\epsilon| < 1/(2\|K\|)\)). So, \(\psi = 0\) because \(N(T_0) = \{0\}\). This implies that \(N(T_\epsilon) = \{0\}\).

Let \(\psi \in N(T_\epsilon^*), \) i.e., \(T_\epsilon^* \psi = 0\). Since \(T_0 (I + \epsilon K^*)^{-1} \psi = 0\) then \((I + \epsilon K^*)^{-1} \psi = 0\) because \(N(T_\epsilon^*) = \{0\}\). This proves that \(0 \in \rho(T_\epsilon^*)\) and concludes the proof of the claim.

On the other hand, the canonical injection \(i\) from \(D(T_\epsilon^*)\) into \(L^2([-a, a])\) is compact. So, \(T_\epsilon\) has a compact resolvent. Since \(T_0\) is an accretive operator and \(K\) has nonvanishing imaginary part of the eigenvalues of the problem (4.16) are nonvanishing. This achieves the proof of (iii).

(iv) The result follows from Theorem 4.3 and Proposition 3.1(ii).

(v) The result follows from Lemma 4.1, Remark 3.1 and Theorem 3.2. \(\Box\)

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**References**


