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On well-posedness for nonlinear Schrödinger equations with power nonlinearity in fractional order Sobolev spaces

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ABSTRACT

We study the well-posedness for the nonlinear Schrödinger equation (NLS)

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda |u|^{p-1}u$$

in $\mathbb{R}^{1+n}$, where $p > 1$, $\lambda \in \mathbb{C}$, and prove that (NLS) is locally well-posed in H^s if $2 < s < 4$ and $s/2 < p < 1 + 4/(n-2s)_+$. To obtain a good lower bound for p, we systematically use Strichartz type estimates in fractional order Besov spaces for the time variable.

1. Introduction

In this paper we consider the Cauchy problem for the nonlinear Schrödinger equation

$$i\partial_t u + \frac{1}{2}\Delta u = f(u), \tag{1.1}$$

$$u(0) = \phi, \tag{1.2}$$

where $u: \mathbb{R}^{1+n} \to \mathbb{C}$ is the unknown function, and $f(u) = \lambda |u|^{p-1}u$ with $p > 1$, $\lambda \in \mathbb{C}$. Introducing the propagator $U(t) = \exp(it\Delta/2)$ and the retarded potential $Gg(t) = \int_0^t U(t-\tau)g(\tau)d\tau$, we can convert the problem (1.1)–(1.2) to the equivalent integral equation

$$u(t) = U(t)\phi - i\,(Gf(u))\,(t). \tag{1.3}$$

The solvability of (1.1)–(1.2) has been studied by many authors; see e.g. [1–10]. The problem (1.1)–(1.2) is said to be locally well-posed in H^s if (1.3) has a unique local (in time) solution $u \in C([0,T]; H^s)$ for any $\phi \in H^s$ and the flow mapping $\phi \mapsto u$ is a continuous mapping from H^s to $C([0,T]; H^s)$. Here T needs to be taken uniformly in some neighborhood of arbitrarily fixed $\phi \in H^s$. For $0 \le s < n/2$, the local solvability of (1.3) has been established for $p_0(s) < p < 1 + 4/(n-2s)$, where $p_0(s) = 1$ for $s \le 2$, $s - 1$ for $2 < s < 4$ and $s - 2$ for $s \ge 4$; if $s \ge n/2$, (1.3) is locally solvable for $p_0(s) < p < \infty$. In some cases, we need auxiliary spaces of Strichartz type (see [11]). The lower bound $p_0(s)$ mentioned above is due to [8]. This result was proved for $s = 1$ by Ginibre and Velo [3,4], $s = 0$ by Tsutsumi [9], and $s = 2$ by Tsutsumi [10] provided that $\lambda \in \mathbb{R}$, mainly by the use of the L^p–L^q estimate and the regularization technique. Kato [5,6] systematically used the Strichartz estimate (see [4,12,13]) and gave an alternative proof of solvability for $s = 0, 1, 2$. His proof is also applicable for the case $\lambda \notin \mathbb{R}$. Cazenave and Weissler [2] proved the result above for $s \notin \mathbb{Z}$ under the additional assumption $p > [s] + 1$, and this can be lowered to $p > s$ by the method of Ginibre, Ozawa and Velo [14]. Pecher [8] used fractional regularity spaces of Besov type for the time variable and proved the result for $p > p_0(s)$. Strictly speaking, his proof shows the existence of

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solutions in $C([0,T];H^{s-\epsilon})$, but the ϵ-loss of regularity can be recovered if we use Proposition 2.3 in [15] or Lemma 2.3(ii) in the present paper instead of Proposition 2.7 in [8]. On the other hand, the continuity of the flow mapping in full strength was proved for $s = 1$ in [3], for $s = 0$ in [9], and for $s = 0, 1, 2$ in [5,6]. Recently, continuity of the flow mapping for $s \notin \mathbb{Z}$ was proved for $0 < s < 1$ in [16] and for $1 < s < 2$ in [15].

In the preceding results referred to above, the natural upper bound $p < 4/(n-2s)$ comes from the scale invariance of (1.1), whereas the lower bound $p > p_0(s)$ comes from the finite (at most p times) differentiability of the nonlinear term $f(u)$. Indeed, Pecher [8] principally estimate the equation in $H^1_q(B^{s-2-\epsilon}_{r,2})$ when $2 < s < 4$, and in $H^2_q(B^{s-4-\epsilon}_{r,2})$ when $s \geq 4$, for which we would need $p > p_0(s)$. However, this condition does not seem to be natural since $p_0(4-0) > p_0(4+0)$. Taking account of the property that for the Schrödinger equation, one time derivative corresponds to two space derivatives, the optimal lower bound for $2 < s < 4$ should be $p > s/2$, which linearly connects $p_0(2)$ and $p_0(4)$. Actually, by the systematical use of fractional order Besov spaces for the time variable, we can obtain the following:

Theorem 1.1. *Let $n \geq 5$, $2 < s < \min(4, n/2)$ and $s/2 < p < 1 + 4/(n-2s)$. Let*

$$\left(\frac{n}{2} - s\right)\frac{p-1}{p+1} < \frac{2}{q} = \delta(r) \equiv \frac{n}{2} - \frac{n}{r} < \min\left\{\frac{n}{2} - s;\ \frac{n}{2}\cdot\frac{p-1}{p+1};\ \frac{2}{p+1}\right\}.$$

Then for any $\phi \in H^s$, there exists $T = T(\|\phi\|_{H^s})$ and (1.3) has a unique solution u in

$$X = C([0,T];H^s) \cap L^q(0,T;B^s_{r,q}) \cap B^{s/2}_{q,2}(0,T;L^r).$$

Moreover, the flow mapping $\phi \mapsto u$ is a continuous mapping from H^s to X.

We remark that in the preceding we have assumed $s < n/2$, which requires $n \geq 5$ in our theorem, simply because we describe the results (and the proof of the theorem) in a unified manner. If $s > n/2$, we can obtain similar results more easily because $H^s \subset L^\infty$. In particular, we can prove the result analogous to our theorem under the assumption $n \geq 1$, $2 < s < 4$ and $s/2 < p < 1 + 4/(n-2s)_+$. If $s \geq n/2$, we should choose q, r such that

$$0 < \frac{2}{q} = \delta(r) < \min\left\{\frac{n}{2}\cdot\frac{p-1}{p+1};\ \frac{2}{p+1}\right\}.$$

We conclude this section by giving the notation used in this paper. For Banach couples $\bar{A} = (A_0, A_1)$, $\bar{A}_{\theta,\alpha}$ and $\bar{A}_{[\theta]}$ denote its real and complex interpolation spaces respectively. L^r, H^s_r and $B^s_{r,\alpha}$ denote the usual Lebesgue, Sobolev and Besov spaces on $\mathbb{R}^n$ respectively; see [17,18]. H^s is an abbreviation of H^s_2. For $1 \leq r \leq \infty$, we put $r' = r/(r-1)$ and $\delta(r) = n/2 - n/r$.

2. Preliminaries

We first introduce vector-valued Sobolev/Besov spaces. For the details, see [19–21]. Let A be a Banach space. Let $I \subset \mathbb{R}$ be an open interval. For $1 \leq q < \infty$ and $m = 0, 1, 2, \ldots$, $H^m_q(I;A)$ denotes the space of all A-valued functions defined on I whose distributional derivatives up to m belong to $L^q(I;A)$. The norm of $H^m_q(I;A)$ is defined by

$$\|u\|_{H^m_q(I;A)} = \sum_{j=0}^{m} \|\partial^j u\|_{L^q(I;A)}. \tag{2.1}$$

For $1 \leq q, \alpha < \infty$ and $\mu > 0$, we define

$$B^\mu_{q,\alpha}(I;A) = \left(L^q(I;A), H^m_q(I;A)\right)_{\mu/m,\alpha}, \tag{2.2}$$

where the right-hand side is the real interpolation space, and $m > \mu$ is an integer. If μ is not an integer, we have the following equivalent representation of the norm of $B^\mu_{q,\alpha}(I;A)$ by the modulus of continuity:

$$\|u\|_{B^\mu_{q,\alpha}(I;A)} \simeq \|u\|_{H^{[\mu]}_q(I;A)} + \left\{\int_{-\infty}^{\infty} \left(\tau^{-(\mu-[\mu])}\|\partial^{[\mu]}u(\cdot+\tau) - \partial^{[\mu]}u\|_{L^q(I_\tau;A)}\right)^\alpha \frac{d\tau}{\tau}\right\}^{1/\alpha}, \tag{2.3}$$

where $[\mu]$ is the integer part of μ and $I_\tau = \{t \in I; t + \tau \in I\}$. By definition and the fundamental properties of real interpolation, we have

$$B^\mu_{q,\alpha}(I;A) \subset B^\mu_{q,\beta}(I;A) \subset B^\nu_{q,\alpha}(I;A) \subset L^q(I;A)$$

with continuous injections if $\alpha \leq \beta$ and $0 < \nu < \mu$. If $1 \leq q < r < \infty$ and $s = 1/q - 1/r$, then we have the Sobolev type inequality

$$\|u\|_{L^r(I;A)} \leq C\|u\|_{B^s_{q,r}(I;A)}.$$

On the other hand if $1 < q, \alpha < \infty$ and $1/q < \mu < 1$, we have

$$\|u\|_{L^\infty(I;A)} \leq C\min(1, |I|^{\mu-1/q})\|u\|_{B^\mu_{q,\alpha}(I;A)}.$$

In what follows, $B^\mu_{q,\alpha}(I;A)$ is often abbreviated to $B^\mu_{q,\alpha}(A)$ unless the statements do not essentially depend on the choice of I. Similarly, we simply write $L^q(A) = L^q(I;A)$, etc.

Lemma 2.1. *Let $\bar{A} = (A_0, A_1)$ be a compatible Banach couple and let $A = \bar{A}_{\theta,\alpha}$ with $0 < \theta < 1$, $1 \le \alpha \le \infty$. Let $\mu > 0$, $1 < q_0, q_1, \beta < \infty$ and $1/q = (1-\theta)/q_0 + \theta/q_1$. Then for any $u \in B^{\mu}_{q_0,\beta}(A_0) \cap L^{q_1}(A_1)$, we have*

$$\|u\|_{B^{(1-\theta)\mu}_{q,\beta/(1-\theta)}(A)} \le C\|u\|^{1-\theta}_{B^{\mu}_{q_0,\beta}(A_0)} \|u\|^{\theta}_{L^{q_1}(A_1)}.$$

Proof. Similar to the proof of [8, Lemma 4.1]. □

Lemma 2.2. *Let $0 < \mu < 1$, and let $1 < q_j, r_j, \alpha < \infty$ with $1/q_0 = 1/q_1 + 1/q_2 = 1/q_3 + 1/q_4$, $1/r_0 = 1/r_1 + 1/r_2 = 1/r_3 + 1/r_4$. Then for any $u \in B^{\mu}_{q_1,\alpha}(L^{r_1}) \cap L^{q_3}(L^{r_3})$ and $v \in L^{q_2}(L^{r_2}) \cap B^{\mu}_{q_4,\alpha}(L^{r_4})$, the following inequality holds:*

$$\|uv\|_{B^{\mu}_{q_0,\alpha}(L^{r_0})} \le C\|u\|_{B^{\mu}_{q_1,\alpha}(L^{r_1})}\|v\|_{L^{q_2}(L^{r_2})} + C\|u\|_{L^{q_3}(L^{r_3})}\|v\|_{B^{\mu}_{q_4,\alpha}(L^{r_4})}.$$

Proof. We can easily prove the inequality by using (2.3) and the Hölder inequality. □

We next introduce Strichartz type estimates used in the proof of the theorem.

Lemma 2.3. *Let $s > 0$, $0 < \theta_- < \theta < \theta_+ < 1$ and let $0 < 2/q = \delta(r) < 1$. Then we have the following:*

(i) *if $\phi \in H^s$, then $U(\cdot)\phi \in C(H^s) \cap L^q(B^s_{r,2}) \cap B^{s/2}_{q,2}(L^r)$ with the estimate*

$$\|U(\cdot)\phi\|_{L^\infty(H^s)\cap L^q(B^s_{r,2})\cap B^{s/2}_{q,2}(L^r)} \le C\|\phi\|_{H^s};$$

(ii) *if $f \in B^{\theta}_{q',2}(L^{r'}) \cap \bigcap_{\pm} L^{q_*(\theta\pm)}(L^{r_*(\theta\pm)})$, then $Gf \in C(H^{2\theta})$ with the estimate*

$$\|Gf\|_{L^\infty(H^{2\theta})} \le C\|f\|_{B^{\theta}_{q',2}(L^{r'})} + C\max_{\pm}\|f\|_{L^{q_*(\theta\pm)}(L^{r_*(\theta\pm)})},$$

where $1/q_*(\theta) = (1-\theta)/q'$ and $1/r_*(\theta) = (1-\theta)/r' + \theta/2$;

(iii) *if $f \in B^{\theta}_{q',2}(L^{r'}) \cap \bigcap_{\pm} L^{\bar{q}(\theta\pm)}(L^{r_*(\theta\pm)})$, then $Gf \in L^q(B^{2\theta}_{r,q}) \cap B^{\theta}_{q,2}(L^r)$ with the estimate*

$$\|Gf\|_{L^q(B^{2\theta}_{r,q})\cap B^{\theta}_{q,2}(L^r)} \le C\|f\|_{B^{\theta}_{q',2}(L^{r'})} + C\max_{\pm}\|f\|_{L^{\bar{q}(\theta\pm)}(L^{\bar{r}(\theta\pm)})},$$

where $1/\bar{q}(\theta) = (1-\theta)/q' + \theta/q$ and $1/\bar{r}(\theta) = (1-\theta)/r' + \theta/r$.

Proof. (i) See [2, Theorem 2.2] and [8, Proposition 2.5].

(ii) This is a refinement of [8, Proposition 2.7]. For the proof, see [15, Proposition 2.3].

(iii) This is a refinement of [8, Proposition 2.6]. By the usual Strichartz estimate, G maps $L^{q'}(L^{r'})$ into $L^q(L^r)$. On the other hand, G maps $H^1_{q'}(L^{r'}) \cap L^q(L^r)$ into $H^1_q(L^r)$ and also into $L^q(H^2_r)$ by virtue of [8, Proposition 2.3]. Therefore, by real interpolation, G maps

$$\begin{aligned}\left(L^{q'}(L^{r'}), H^1_{q'}(L^{r'}) \cap L^q(L^r)\right)_{\theta,2} &= B^{\theta}_{q',2}(L^{r'}) \cap \left(L^{q'}(L^{r'}), L^q(L^r)\right)_{\theta,2} \\ &\supset B^{\theta}_{q',2}(L^{r'}) \cap \bigcap_{\pm}\left(L^{q'}(L^{r'}), L^q(L^r)\right)_{[\theta\pm]} = B^{\theta}_{q',2}(L^{r'}) \cap \bigcap_{\pm} L^{\bar{q}(\theta\pm)}(L^{\bar{r}(\theta\pm)})\end{aligned}$$

into $\left(L^q(L^r), H^1_q(L^r)\right)_{\theta,2} = B^{\theta}_{q,2}(L^r)$. (See [8, Lemma 2.1].) Similarly, G maps $\left(L^{q'}(L^{r'}), H^1_{q'}(L^{r'}) \cap L^q(L^r)\right)_{\theta,2}$ into $\left(L^q(L^r), L^q(H^2_r)\right)_{\theta,q} = L^q(B^{2\theta}_{r,q})$ by virtue of [18, Theorem 1.18.4]. □

To estimate the nonlinear term, we need the following lemma:

Lemma 2.4. *Let $0 < s < p$, $1 < r_0, r_1, \rho, \alpha < \infty$ and $f(u) = |u|^{p-1}u$. Let $1/r_0 = 1/r_1 + (p-1)/\rho$. Then for any $u \in L^\rho \cap B^s_{r_1,\alpha}$ we have*

$$\|f(u)\|_{B^s_{r_0,\alpha}} \le C\|u\|^{p-1}_{L^\rho}\|u\|_{B^s_{r_1,\alpha}}.$$

Proof. See [14, Lemma 3.4]. □

3. Proof of Theorem 1.1

In this section we arbitrarily fix the exponents q, r satisfying the assumption of Theorem 1.1. Let $X_0 = L^\infty(0, T; H^s) \cap L^q(0, T; B^s_{r,q}) \cap B^{s/2}_{q,2}(0, T; L^r)$ and

$$\mathscr{B} = \{u \in X_0;\ \|u\|_{X_0} \le R, u(0) = \phi\}.$$

$\mathscr{B}$ is closed in X_0 and complete, with metric $d(u, v) = \|u - v\|_{L^q(L^r)}$. We show that for suitable choices of T and R,

$$\Phi(u) = U(\cdot)\phi - iGf(u)$$

is a contraction mapping on $\mathscr{B}$. Since $(i\partial_t + \Delta/2)\Phi(u) = f(u)$, it suffices to estimate $\Phi(u)$ in $L^\infty(L^2) \cap L^q(L^r)$,

$$i\partial_t\Phi(u) = U(\cdot)\left(-(\Delta/2)\phi + f(\phi)\right) - iG\left(\partial_t f(u)\right)$$

in $Y = L^\infty(H^{s-2}) \cap L^q(B^{s-2}_{r,q}) \cap B^{s/2-1}_{q,2}(L^r)$ and $f(u)$ in $L^\infty(H^{s-2}) \cap L^q(B^{s-2}_{r,q})$.

Step 1. The usual Strichartz estimate and the Hölder inequality show that

$$\begin{aligned}\|\Phi(u)\|_{L^\infty(L^2)\cap L^q(L^r)} &\le C\|\phi\|_{L^2} + C\|f(u)\|_{L^{q'}(L^{r'})}\\ &\le C\|\phi\|_{L^2} + CT^\kappa\|u\|^{p-1}_{L^q(L^{\rho_0})}\|u\|_{L^q(L^r)}.\end{aligned}$$

Here $\kappa = 1 - (p+1)/q > 0$ and ρ_0 is determined by $1/r' = (p-1)/\rho_0 + 1/r$, or equivalently $2\delta(r) + (p-1)\delta(\rho_0) = n(p-1)/2$. If we put $\sigma_0 \equiv \delta(\rho_0) - \delta(r)$, we have $(p+1)\delta(r) + (p-1)\sigma_0 = n(p-1)/2$. Therefore, there exists ρ_0 satisfying $0 \le \sigma_0 < s$ if $(n/2 - s)(p-1)/(p+1) < \delta(r) \le n(p-1)/2(p+1)$. Since $B^s_{r,q} \subset B^{\sigma_0}_{r,2} \subset L^{\rho_0}$ provided that $0 \le \sigma_0 < s$, we have $\|u\|_{L^q(L^{\rho_0})} \le C\|u\|_{L^q(B^s_{r,q})} \le CR$, and consequently

$$\|\Phi(u)\|_{L^\infty(L^2)\cap L^q(L^r)} \le C\|\phi\|_{L^2} + CT^\kappa R^p.$$

Step 2. *The estimate of* $\partial_t\Phi(u)$ *in* Y. By Lemma 2.3,

$$\|\partial_t\Phi(u)\|_Y \le C\| - (\Delta/2)\phi + f(\phi)\|_{H^{s-2}} + C\|f'(u)\partial_t u\|_{\tilde{Y}}, \tag{3.1}$$

where

$$\tilde{Y} = B^{s/2-1}_{q',2}(L^{r'}) \cap \bigcap_{\pm}\left\{L^{\bar{q}(\theta_\pm)}(L^{\bar{r}(\theta_\pm)}) \cap L^{q_*(\theta_\pm)}(L^{r_*(\theta_\pm)})\right\}$$

and $0 < \theta_- < s/2 - 1 < \theta_+ < 1$. We put $s_0 = s - (n/2 - s)(p-1)$. By the assumption, we see that $s_0 > s - 2$. Using the Sobolev inequality and Lemma 2.4, we can show that

$$\|f(\phi)\|_{H^{s_0}} \le C\|\phi\|^{p-1}_{L^{2n/(n-2s)}}\|\phi\|_{H^s}, \tag{3.2}$$

thereby obtaining that the first term in the right-hand side of (3.1) is bounded by $C\left(1 + \|\phi\|^{p-1}_{H^s}\right)\|\phi\|_{H^s}$. Applying Lemma 2.2, we see that

$$\|f'(u)\partial_t u\|_{B^{s/2-1}_{q',2}(L^{r'})} \le CT^\kappa\|u\|^{p-1}_{L^q(L^{\rho_0})}\|\partial_t u\|_{B^{s/2-1}_{q,2}(L^r)} + CT^\kappa\|f'(u)\|_{B^{s/2-1}_{q/(p-1),2}(L^{\rho_2/(p-1)})}\|\partial_t u\|_{L^q(L^{\rho_1})}, \tag{3.3}$$

where κ and ρ_0 are the same as in Step 1, and ρ_1, ρ_2 satisfy $1/r' = 1/\rho_1 + (p-1)/\rho_2$, or equivalently $(p+1)\delta(r) + \sigma_1 + (p-1)\sigma_2 = n(p-1)/2$ with $\sigma_j \equiv \delta(\rho_j) - \delta(r)$, $j = 1, 2$. The first term in the right-hand side is bounded by $CT^\kappa R^p$ in the same way as in Step 1.

We estimate the second term separately in the cases $p \le 2$ and $p > 2$.

If $p \le 2$, let μ_1, μ_2 satisfy $1 < \mu_1 < s/2$, $(s-2)/2(p-1) < \mu_2 < 1$ with $\mu_1 + (p-1)\mu_2$ being sufficiently close to $s/2$. For such μ_1, μ_2, we choose ρ_1, ρ_2 such that $0 \le \sigma_j < s - 2\mu_j$, $j = 1, 2$. Such ρ_1, ρ_2 surely exist if

$$(p+1)\delta(r) \le \frac{n}{2}(p-1) < (p+1)\delta(r) + s - 2\mu_1 + (p-1)(s - 2\mu_2).$$

The left inequality is satisfied by virtue of the assumption for $\delta(r)$; the right inequality is also satisfied again by the assumption for $\delta(r)$, since the right-hand side is sufficiently close to $(p+1)\delta(r) + (p-1)s$ by our choice of μ_1, μ_2. Therefore it follows that $\|\partial_t u\|_{L^q(L^{\rho_1})} \le C\|u\|_{B^{\mu_1}_{q,2}(B^{\sigma_1}_{r,2})} \le CR$. On the other hand, by the inequality

$$|f'(u(t+\tau)) - f'(u(t))| \le C|u(t+\tau) - u(t)|^{p-1}$$

and the representation of the Besov norm (2.3), we have

$$\begin{aligned}\|f'(u)\|_{B^{s/2-1}_{q/(p-1),2}(L^{\rho_2/(p-1)})} &\le C\|f'(u)\|_{B^{(p-1)\mu_2}_{q/(p-1),2/(p-1)}(L^{\rho_2/(p-1)})}\\ &\le C\|u\|^{p-1}_{L^q(L^{\rho_2})} + C\left\{\int_{-\infty}^{\infty}\left(\tau^{-\mu_2}\|u(\cdot+\tau) - u\|_{L^q(I_\tau;L^{\rho_2})}\right)^2\frac{d\tau}{\tau}\right\}^{(p-1)/2}\\ &\le C\|u\|^{p-1}_{B^{\mu_2}_{q,2}(L^{\rho_2})} \le C\|u\|^{p-1}_{B^{\mu_2}_{q,2}(B^{\sigma_2}_{r,2})},\end{aligned} \tag{3.4}$$

where $I_\tau = \{t \in (0,T); t+\tau \in (0,T)\}$. Since $(L^r, B^s_{r,q})_{\sigma_j/s,2} = B^{\sigma_j}_{r,2}$ and $2\mu_j + \sigma_j < s$, we have by Lemma 2.1

$$\|u\|_{B^{\mu_j}_{q,2}(B^{\sigma_j}_{r,2})} \le C\|u\|^{1-\sigma_j/s}_{B^{s/2}_{q,2}(L^r)} \|u\|^{\sigma_j/s}_{L^q(B^s_{r,q})} \le CR.$$

Consequently we obtain $\|f'(u)\partial_t u\|_{B^{s/2-1}_{q',2}(L^{r'})} \le CT^\kappa R^p$.

If $p > 2$, we further decompose $(p-1)/\rho_2 = (p-2)/\rho_3 + 1/\rho_4$, or equivalently

$$(p+1)\delta(r) + \sigma_1 + (p-2)\sigma_3 + \sigma_4 = \frac{n}{2}(p-1)$$

with $\sigma_j \equiv \delta(\rho_j) - \delta(r), j = 3, 4$. We can choose ρ_3, ρ_4 such that $0 \le \sigma_3 < s$, $0 \le \sigma_4 < 2$. Then by the inequality

$$|f'(u(t+\tau)) - f'(u(t))| \le \int_0^1 d\lambda\, |f''(\lambda u(t+\tau) + (1-\lambda)u(t))|\, |u(t+\tau) - u(t)|,$$

we obtain $\|f'(u)\|_{B^{s/2-1}_{q/(p-1),2}(L^{\rho_2/(p-1)})} \le C\|u\|^{p-2}_{L^q(L^{\rho_3})}\|u\|_{B^{s/2-1}_{q,2}(L^{\rho_4})}$. Therefore, like in the case $p \le 2$, we obtain $\|f'(u)\partial_t u\|_{B^{s/2-1}_{q',2}(L^{r'})} \le CT^\kappa R^p$.

Step 3. *The estimate of $f'(u)\partial_t u$ in $L^{\bar q(\theta_\pm)}(L^{\bar r(\theta_\pm)}) \cap L^{q_*(\theta_\pm)}(L^{r_*(\theta_\pm)})$.* In this step we simply write $\bar q = \bar q(\theta_\pm)$, etc. Let $2 < \gamma_5, \gamma_6 < \infty$ and $\kappa > 0$ satisfy

$$1/\bar q = 1/\gamma_5 + (p-1)/\gamma_6 + \kappa.$$

If $1/\bar q \le p/q$, then we choose $\kappa > 0$ sufficiently small and γ_5, γ_6 such that

$$0 < \mu_5 - 1 \equiv 1/q - 1/\gamma_5 < s/2 - 1, \qquad 0 < \mu_6 \equiv 1/q - 1/\gamma_6;$$

if $1/\bar q > p/q$, then we choose $0 < \kappa < 1/\bar q - p/q$ and $\mu_5 - 1$, μ_6 to be sufficiently small positive numbers. Moreover, let ρ_5, ρ_6 satisfy $1/\bar r = 1/\rho_5 + (p-1)/\rho_6$, or equivalently

$$\delta(\bar r) = p\delta(r) - n(p-1)/2 + \sigma_5 + (p-1)\sigma_6$$

with $\sigma_j \equiv \delta(\rho_j) - \delta(r), j = 5, 6$. We choose ρ_5, ρ_6 such that $0 \le \sigma_5 < s - 2\mu_5$, $0 \le \sigma_6 < s - 2\mu_6$, which is possible if

$$0 \le \delta(\bar r) - p\delta(r) + \frac{n}{2}(p-1) < s - 2\mu_5 + (p-1)(s - 2\mu_6). \tag{3.5}$$

The left inequality of (3.5) is true since the middle of (3.5) is

$$(2\theta - 1)\delta(r) - p\delta(r) + \frac{n}{2}(p-1) > 2\theta\delta(r) > 0.$$

To check the right inequality of (3.5), we separately consider the cases $1/\bar q \le p/q$ and $1/\bar q > p/q$. If $1/\bar q \le p/q$, this is true for $\theta_\pm \sim s/2 - 1$ and sufficiently small κ since $\delta(\bar r) - p\delta(r) = 2(\kappa - \mu_5 - (p-1)\mu_6 + \theta)$ and therefore

$$\begin{aligned} s - 2\mu_5 + (p-1)(s-2\mu_6) - \left(\delta(\bar r) - p\delta(r) + \frac{n}{2}(p-1)\right) &= s - \theta - \left(\frac{n}{2} - s\right)(p-1) - 2\kappa \\ &\sim 2 - \left(\frac{n}{2} - s\right)(p-1) > 0. \end{aligned}$$

If $1/\bar q > p/q$, the right inequality of (3.5) is true since $\mu_5 - 1$ and μ_6 are sufficiently small and therefore

$$\begin{aligned} s - 2\mu_5 + (p-1)(s-2\mu_6) - \left(\delta(\bar r) - p\delta(r) + \frac{n}{2}(p-1)\right) &\sim s - 2 - \left(\frac{n}{2} - s\right)(p-1) + (p+3-s)\delta(r) \\ &> (s-2)\,(1 - \delta(r)) > 0. \end{aligned}$$

Therefore, Hölder's inequality, Sobolev's inequality and Lemma 2.1 yield

$$\|f'(u)\partial_t u\|_{L^{\bar q}(L^{\bar r})} \le CT^\kappa \|u\|^{p-1}_{L^{\gamma_6}(L^{\rho_6})}\|\partial_t u\|_{L^{\gamma_5}(L^{\rho_5})} \le CT^\kappa R^p. \tag{3.6}$$

We can analogously estimate $\|f'(u)\partial_t u\|_{L^{q_*}(L^{r_*})}$.

Step 4. *The estimate of $f(u)$ in $L^\infty(H^{s-2}) \cap L^q(B^{s-2}_{r,q})$.* We estimate $f(u)$ in $L^\infty(H^{s-2})$. The estimate in $L^q(B^{s-2}_{r,q})$ is similar. Let μ_7, μ_8 satisfy $1/q < \mu_j < 1$ with $\kappa \equiv \mu_7 + (p-1)\mu_8 - p/q$ being sufficiently small. Let ρ_7, ρ_8 satisfy $1/2 = 1/\rho_7 + (p-1)/\rho_8$, or equivalently

$$p\delta(r) + \sigma_7 + (p-1)\sigma_8 - s + 2 - n(p-1)/2 = 0$$

with $\sigma_7 \equiv \delta(\rho_7) - \delta(r) + s - 2$, $\sigma_8 \equiv \delta(\rho_8) - \delta(r)$. We choose ρ_7, ρ_8 such that $s - 2 \le \sigma_7 < s - 2\mu_7$, $0 \le \sigma_8 < s - 2\mu_8$, which is possible if

$$p\delta(r) - \frac{n}{2}(p-1) \le 0 < p\delta(r) - 2\mu_7 - 2(p-1)\mu_8 + 2 - \left(\frac{n}{2} - s\right)(p-1).$$

The left inequality holds by the assumption, and the right inequality holds since the right-hand side is equal to $2 - (n/2 - s)(p-1) - 2\kappa$, which is positive if $\kappa > 0$ is sufficiently small. Therefore

$$\begin{aligned}\|f(u)\|_{L^\infty(H^{s-2})} &\le C\|u\|^{p-1}_{L^\infty(L^{\rho_8})}\|u\|_{L^\infty(B^{s-2}_{\rho_7,2})}\\ &\le CT^\kappa\|u\|^{p-1}_{B^{\mu_8}_{q,2}(B^{\sigma_8}_{r,2})}\|u\|_{B^{\mu_7}_{q,2}(B^{\sigma_7}_{r,2})} \le CT^\kappa R^p.\end{aligned}$$

Step 5. In view of Steps 1–4, we have proved

$$\|\Phi(u)\|_{X_0} \le C\left(1 + \|\phi\|^{p-1}_{H^s}\right)\|\phi\|_{H^s} + CT^\kappa R^p$$

for $u \in \mathscr{B}$. Like in Step 1, we obtain

$$\|\Phi(u) - \Phi(v)\|_{L^q(L^r)} \le CT^\kappa R^{p-1}\|u - v\|_{L^q(L^r)} \tag{3.7}$$

for $u, v \in \mathscr{B}$. Therefore, for sufficiently large $R > 0$ and sufficiently small $T > 0$, Φ is a contraction mapping from $\mathscr{B}$ to itself, which implies the unique existence of the solution to (1.3) in X_0. We should also show the continuity of u in H^s. To this end, it suffices to show that $f(u) \in C(H^{s-2})$ since we immediately obtain $u \in C^1(H^{s-2})$ by Lemma 2.3 and the previous steps. By the estimate (3.2) with ϕ replaced by u, we can show that $\|f(u)\|_{L^\infty(H^{s_0})} \le CT^\kappa R^p$. On the other hand, we can easily prove $f(u) \in C(L^2)$. Indeed, by the Hölder and the Sobolev inequalities, we see that

$$\|f(u(t+h)) - f(u(t))\|_{L^2} \le C\|u\|^{p-1}_{L^\infty(H^s)}\|u(t+h) - u(t)\|_{H^{s-s_0}} \to 0$$

as $h \to 0$. Since $s_0 > s-2$, we obtain $f(u) \in C(H^{s-2})$ by interpolation.

Step 6. *Continuity of the flow mapping*. Let $\phi_m \to \phi$ in H^s and let u_m be the solution to (1.3) with ϕ replaced by ϕ_m. We shall show that $u_m \to u$ in X_0. We may assume that $\|u_m\|_{X_0} \le R$. In the same way as in the proof of (3.7), we can easily show that $\|u_m - u\|_{L^\infty(L^2)\cap L^q(L^r)} \le C\|\phi_m - \phi\|_{L^2} \to 0$. To prove the continuous dependence in full strength, we remark that $\partial_t u_m(0) = -(\Delta/2)\phi_m + f(\phi_m) \to \partial_t u(0)$ in H^{s-2}. This can be proved in the same way as in Step 5. We also remark that $\|u_m - u\|_{L^q(B^\sigma_{r,\rho})} \to 0$ if $\sigma < s$, since $(L^r, B^s_{r,q})_{\sigma/s,2} = B^\sigma_{r,2}$ and $\|u_m - u\|_{L^q(L^r)} \to 0$. Moreover $\|u_m - u\|_{B^\mu_{q,2}(B^\sigma_{r,\rho})} \to 0$ if $\sigma + 2\mu < s$ by Lemma 2.1 since $\{u_m\}$ is bounded in $B^{s/2}_{q,2}(L^r)$. Using Lemma 2.3 we have

$$\begin{aligned}\|\partial_t(u_m - u)\|_Y &\le C\|\partial_t(u_m - u)(0)\|_{H^{s-2}} + C\|f'(u_m)\partial_t u_m - f'(u)\partial_t u\|_{\tilde{Y}}\\ &\le C\|\partial_t(u_m - u)(0)\|_{H^{s-2}} + C\|f'(u_m)(\partial_t u_m - \partial_t u)\|_{\tilde{Y}} + C\|(f'(u_m) - f'(u))\partial_t u\|_{\tilde{Y}}.\end{aligned} \tag{3.8}$$

In the same way as in the previous steps, the middle term in the right-hand side is estimated by $CT^\kappa R^{p-1}\|\partial_t(u_m - u)\|_Y$, which is absorbed in the left-hand side. Therefore, in order to prove $\|\partial_t(u_m - u)\|_Y \to 0$, we have only to show that the last term in the right-hand side of (3.8) tends to zero. In what follows, we only consider the case $p \le 2$; we only need a slight modification in the case $p > 2$. We estimate $\|\left(f'(u_m) - f'(u)\right)\partial_t u\|_{\tilde{Y}}$ by analogy with (3.3) and (3.6) with $f'(u)$ replaced by $f'(u_m) - f'(u)$. Since

$$\|f'(u_m) - f'(u)\|_{L^{q/(p-1)}(L^{\rho_0/(p-1)})} \le C\|u_m - u\|^{p-1}_{L^q(B^{\sigma_0}_{r,2})}$$

and

$$\|f'(u_m) - f'(u)\|_{L^{\gamma_6/(p-1)}(L^{\rho_6/(p-1)})} \le C\|u_m - u\|^{p-1}_{B^{\mu_6}_{q,2}(B^{\sigma_6}_{r,2})}$$

tend to zero in view of the remark above, it suffices to show that $f'(u_m) \to f'(u)$ in $B^{s/2-1}_{q/(p-1),2}(L^{\rho_2/(p-1)})$. The estimate (3.4) shows that $\|f'(u_m)\|_{B^{\mu_2(p-1)}_{q/(p-1),2/(p-1)}(L^{\rho_2/(p-1)})} \le CR^{p-1}$. On the other hand, again by the remark above, we see that

$$\|f'(u_m) - f'(u)\|_{L^{q/(p-1)}(L^{\rho_2/(p-1)})} \le C\|u_m - u\|^{p-1}_{L^q(B^{\sigma_2}_{r,2})} \to 0.$$

Therefore, we obtain that $f'(u_m) \to f'(u)$ in $B^{s/2-1}_{q/(p-1),2}(L^{\rho_2/(p-1)})$ by interpolation. We finally check that $f(u_m) \to f(u)$ in $L^\infty(H^{s-2}) \cap L^q(B^{s-2}_{r,q})$. Like in Step 5, we can show that $\|f(u)\|_{L^\infty(H^{s_0})\cap L^q(B^{s_0}_{r,q})} \le CR^p$ and

$$\|f(u_m) - f(u)\|_{L^\infty(L^2)\cap L^q(L^r)} \le CR^{p-1}\|u_m - u\|_{L^\infty(H^{s-s_0})\cap L^q(B^{s-s_0}_{r,2})} \to 0,$$

thereby proving the assertion. □

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