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J. Math. Anal. Appl. 277 (2003) 164–179

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

On positive solutions for some semilinear periodic parabolic eigenvalue problems ☆

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Received 13 June 2001

Submitted by G. Komatsu

Abstract

For a bounded domain Ω in $\mathbb{R}^N$, $N \geqslant 2$, satisfying a weak regularity condition, we study existence of positive and T-periodic weak solutions for the periodic parabolic problem $Lu_\lambda = \lambda g(x,t,u_\lambda)$ in $\Omega \times \mathbb{R}$, $u_\lambda = 0$ on $\partial\Omega \times \mathbb{R}$. We characterize the set of positive eigenvalues with positive eigenfunctions associated, under the assumptions that g is a Caratheodory function such that $\xi \to g(x,t,\xi)/\xi$ is nonincreasing in $(0,\infty)$ a.e. $(x,t) \in \Omega \times \mathbb{R}$ satisfying some integrability conditions in (x,t) and

$$\int_0^T \operatorname*{esssup}_{x\in\Omega} \inf_{\xi>0} \frac{g(x,t,\xi)}{\xi}\, dt > 0.$$

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geqslant 2$, satisfying the following regularity condition: there exists $\rho_0 > 0$ and $\delta_0 \in (0,1)$ such that for all $x \in \partial\Omega$ and all $\rho \leqslant \rho_0$

$$\left|B_\rho(x) \cap \Omega\right| \leqslant (1-\delta_0)\left|B_\rho(x)\right|, \tag{1.1}$$

where $B_\rho(x)$ denotes the open ball in $\mathbb{R}^N$ centered at x with radius ρ and $|B_\rho(x)|$ denotes its Lebesgue measure.

☆ Partially supported by CONICET, Secyt-UNC and Agencia Cordoba Ciencia.

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PII: S0022-247X(02)00527-9

For $T > 0$ and $1 \leqslant p, q \leqslant \infty$, let $L^p(L^q)$ be the space of the measurable and T-periodic functions f on $\Omega \times \mathbb{R}$ (i.e., satisfying $f(x,t) = f(x,t+T)$ a.e. $(x,t) \in \Omega \times \mathbb{R}$) such that $\|f\|_{L^p(L^q)} < \infty$ where $\|f\|_{L^p(L^q)} = \|\|f(x,t)\|_{L^q(\Omega,dx)}\|_{L^p((0,T),dt)}$. Provided with this norm $L^p(L^q)$ is a Banach space. Similarly, let L_T^p be the Banach space of T-periodic functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$ such that $f|_{\Omega \times (0,T)} \in L^p(\Omega \times (0,T))$, equipped with its natural norm $\|f\|_{L_T^p} = \|f|_{\Omega \times (0,T)}\|_{L^p(\Omega \times (0,T))}$. Finally, let C_T be the space of continuous and T-periodic functions on $\bar{\Omega} \times \mathbb{R}$ provided with the L^∞ norm.

Let us fix for the whole paper $v, s \in \mathbb{R} \cup \{\infty\}$ such that $N/(2v) + 1/s < 1$, with $s > 2$. Let $\{a_{i,j}\}_{1 \leqslant i,j \leqslant N}$, $\{b_j\}_{1 \leqslant, j \leqslant N}$ be two families of T-periodic functions satisfying $a_{i,j} \in L_T^\infty$, $a_{i,j} = a_{j,i}$ for $1 \leqslant i, j \leqslant N$ and $b_j \in L^\infty(L^{2v})$. Assume that

$$\sum_{i,j} a_{i,j}(x,t)\xi_i\xi_j \geqslant \alpha_0|\xi|^2$$

for some $\alpha_0 > 0$ and all $(x,t) \in \Omega \times \mathbb{R}$, $\xi \in \mathbb{R}^N$. Let A be the $N \times N$ matrix whose i, j entry is $a_{i,j}$, let $b = (b_1, \ldots, b_N)$, let $c_0 \geqslant 0$ be a function in $L^s(L^v)$, and let L be the parabolic operator given by

$$Lu = u_t - \operatorname{div}(A\nabla u) + \langle b, \nabla u\rangle + c_0 u,$$

where $\langle\ ,\ \rangle$ denotes the standard inner product on $\mathbb{R}^N$.

Let

$$W = \left\{u \in L^2\big((0,T), H_0^1(\Omega)\big)\colon\ \frac{du}{dt} \in L^2\big((0,T), H^{-1}(\Omega)\big)\right\}.$$

Given $f \in L^2(L^p)$ with $p > 2N(N+2)^{-1}$, we say that u is a weak solution of the T-periodic problem $Lu = f$ in $\Omega \times \mathbb{R}$, $u = 0$ on $\partial\Omega \times \mathbb{R}$, if $u|_{\Omega \times (0,Tt)} \in W$, u is T-periodic in t and

$$\int_{\Omega \times (0,T)} \left[-u\frac{\partial h}{\partial t} + \langle A\nabla u, \nabla h\rangle + \langle b, \nabla u\rangle h + c_0 u h\right] = \int_{\Omega \times (0,T)} fh$$

for all $h \in C_c^\infty(\Omega \times (0,T))$. It is well known (see, e.g., [2,7]) that this problem has a unique T-periodic weak solution u with $u|_{\Omega \times (0,T)} \in L^2((0,T), H_0^1(\Omega))$.

Let us consider, in the above weak sense, existence of positive solutions for some nonlinear eigenvalue problems of the form

$$\begin{cases} Lu = \lambda g(x,t,u) & \text{in } \Omega \times \mathbb{R}, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}, \\ u \ T\text{-periodic in } t, \end{cases} \tag{1.2}$$

where g is a given function on $\Omega \times \mathbb{R} \times [0,\infty)$. The linear case $g(x,t,\xi) = m(x,t)\xi$ with $m \in L^s(L^v)$ is studied in [5]. For $m \in L^s(L^v)$, let

$$\tilde{m}(t) = \operatorname*{esssup}_{x \in \Omega} m(x,t) \tag{1.3}$$

and let

$$P(m) = \int_0^T \tilde{m}(t)\,dt. \tag{1.4}$$

It is proved in [5], Theorem 3.6, that for $m \in L^s(L^v)$, $P(m) > 0$ is a necessary and sufficient condition for the existence of a positive principal eigenvalue (with associated eigenfunctions in $L^r(L^p)$) for the problem

$$\begin{cases} Lu = \lambda m u & \text{in } \Omega \times \mathbb{R}, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}, \\ u \ T\text{-periodic in } t, & u > 0. \end{cases} \tag{1.5}$$

Moreover, such a positive eigenvalue (denoted by $\lambda_1(m)$) is unique and algebraically simple.

The condition $P(m) > 0$ perhaps needs some explanation. Observe that the case $\tilde{m} \notin L^1(0,T)$ is, a priori, possible. However, $P(m)$ is well defined. Indeed, since $\tilde{m}(t) \geqslant m(x,t)$ a.e. $(x,t) \in \Omega \times \mathbb{R}$ it holds that $\tilde{m}^-(t) \leqslant |m(x,t)|$ and so $P(m)$ is well defined (the value $+\infty$ is allowed).

Looking for nonlinear cases, if Ω is a $C^{2+\theta}$ bounded domain with $0 < \theta < 1$ and L is a parabolic operator (in nondivergence form) with Hölder continuous coefficients, it is well known (see, e.g., [6], Section 27) that if $g = g(x,t,\xi)$ is a concave function in ξ satisfying $g_\xi \in C^\theta(\bar{\Omega} \times \mathbb{R} \times [0,\infty))$ and $g(x,t,0) = 0$, then there exists a C^1 curve $\lambda \to u_\lambda$ of positive solutions for (1.2). In [6], these results follow from some global bifurcation theorems due to Rabinowitz (cf. [9]) and the implicit function theorem. On the other hand, analogous elliptic problems are studied for selfadjoint operators in [3] assuming that $g \in C^\theta(\bar{\Omega} \times [0,\infty))$ and that $\xi \to g(x,\xi)/\xi$ is nonincreasing. For the particular case $g(x,\xi) = g(\xi)$, results of similar nature are given in [10] under the assumption $g \in C([0,\infty))$ and, for $g \in C^\theta(\bar{\Omega} \times [0,\infty))$ and more general boundary conditions, in [11]. In order to relate these results to those in [6], observe that if $g(x,\xi)$ is concave in ξ and $g(x,0) \geqslant 0$, then $g(x,\xi)/\xi$ is nonincreasing.

Our aim in this paper is to show (see Theorem 3.7), following a different approach, that if $g : \Omega \times \mathbb{R} \times [0,\infty) \to \mathbb{R}$ satisfies the following conditions:

- (H1) $(x,t) \to g(x,t,\xi)$ is measurable for all $\xi \in [0,\infty)$ and T-periodic in t, $g(x,t,\cdot) \in C^1[0,\infty)$ a.e. $(x,t) \in \Omega \times \mathbb{R}$, and $\sup_{0 \leqslant \xi \leqslant \rho} |g_\xi(x,t,\xi)| \in L^s(L^v)$ for all $\rho > 0$;
- (H2) $\xi \to g(x,t,\xi)/\xi$ is nonincreasing in $(0,\infty)$ a.e. $(x,t) \in \Omega \times \mathbb{R}$;
- (H3) there exist $\delta > 0$ and $(x_0,t_0) \in \partial\Omega \times \mathbb{R}$ such that $\partial/\partial\xi(g(x,t,\xi)/\xi) < 0$ for all $\xi \in (0,\delta)$ a.e. $(x,t) \in B_\delta(x_0,t_0) \cap (\Omega \times \mathbb{R})$;
- (H4) the functions $\bar{m}(x,t) := \sup_{\xi > 0} g(x,t,\xi)/\xi$, $\underline{m}(x,t) := \inf_{\xi > 0} g(x,t,\xi)/\xi$ belong to $L^s(L^v)$;
- (H5) $\int_0^T \operatorname{esssup}_{x \in \Omega} \underline{m}(x,t)\, dt > 0$;

then

- (a) (1.2) has a positive solution $u_\lambda \in C_T$ if and only if

  $$\lambda_1(\bar{m}) < \lambda < \lambda_1(\underline{m});$$

- (b) u_λ can be chosen such that $\lambda \to u_\lambda$ is a C^1 map from $(\lambda_1(\bar{m}), \lambda_1(\underline{m}))$ into C_T, satisfying $\lim_{\lambda \to \lambda_1(\bar{m})^+} \|u_\lambda\|_\infty = 0$ and $\lim_{\lambda \to \lambda_1(\underline{m})^-} u_\lambda(x,t) = \infty$ for all $(x,t) \in \Omega \times \mathbb{R}$. Moreover, $u_\lambda(x,t) > 0$ for all $(x,t) \in \Omega \times \mathbb{R}$.

Moreover, we prove also (see Theorem 3.10) that for $\lambda \in (\lambda_1(\bar{m}), \lambda_1(\underline{m}))$, the existence of positive solutions for (1.2) remains true if (H3) is removed and (H1) is replaced by the assumption that g is a T-periodic Caratheodory function. Finally, a related maximum principle is presented in Theorem 3.9.

2. Some facts about linear problems with weight

Let us start with some comments about results concerning principal eigenvalues for periodic parabolic problems with weight contained in [5].

Remark 2.1. In [5], Lemma 2.1, it is shown that for s, v as in the introduction, there exist p, q, r, w such that $2 \leqslant q$, $r < \infty$, $r \leqslant s$, $p \leqslant w$, $2N(N+2)^{-1} < p < \infty$, $1/w = 1/q + 1/v$ and $N/2(1/p - 1/q) + 1/r < 1$. For such p, r it is proved in Theorem 3.6 that for $m \in L^s(L^v)$, the condition $P(m) > 0$ is necessary and sufficient for the existence of a unique positive principal eigenvalue $\lambda_1(m)$ for (1.5) with a positive eigenfunction associated in $L^r(L^p)$. The above conditions on p, q, r, w were imposed in order to apply results in [2] (namely Corollary 5.2, and so Theorems 5.1(a) and 4.4) without any regularity assumptions on Ω. However, we actually deal with domains satisfying condition (1.1) and thus $q = \infty$ is allowed in Theorem 4.4 in [2] (see [2], Remark 4.6(b)) and so also in Theorem 5.1(a) and Corollary 5.2. It follows that under condition (1.1) all results in [5] remain true taking there $q = \infty$, $w = v$, and p, r satisfying $2 \leqslant r < \infty$, $r < s$, $p < v$, $p < \infty$ and $N/(2p) + 1/r < 1$. We fix from now on p, r satisfying these conditions (since $N/(2v) + 1/s < 1$ such p, r exist).

Remark 2.2. For $f \in L^r(L^p)$, the (unique) solution u of the Dirichlet T-periodic problem $Lu = f$ belongs to C_T. Moreover, $L^{-1} : L^r(L^p) \to C_T$ is a compact operator. Indeed, taking into account (1.1) and Remark 4.6(b) in [2], we get that $u \in C_T$ and, as we said before, Theorem 5.1(a) in [2] remains true for $q = \infty$ and gives the compactness.

If X, Y are Banach spaces, let $B(X, Y)$ be the Banach space of the linear and bounded operators from X into Y. If $S \in B(X, Y)$ we will write $\|S\|_{X,Y}$ for its operator norm and if $S \in B(X, X)$ its norm will be denoted by $\|S\|_X$. For $R > 0$ and $f \in L^s(L^v)$ or $f \in C_T$ we will write $\bar{B}_R^{s,v}(f)$ or $\bar{B}_R^{C_T}(f)$ respectively for the closed balls centered at f and with radius R in the respective spaces.

Let $R, \Lambda \in (0, \infty)$. Recalling (1.1), we can take $q = \infty$ in [5], Proposition 2.4. An inspection of its proof shows that there exists $k_0 = k_0(R, \Lambda)$ such that for $k \geqslant k_0$ the operator $(L + \lambda(k - m))^{-1} : L^r(L^p) \to L_T^\infty$ is compact and positive. In fact, we have

Lemma 2.3. *Let $R, \Lambda \in (0, \infty)$ and let $k \geqslant k_0$ with k_0 as above. Then, for all $m \in \bar{B}_R^{s,v}(0)$, $\lambda \in [0, \Lambda]$ we have*

$$\left(L + \lambda(k - m)\right)^{-1}\left(L^r\left(L^p\right)\right) \subset C_T. \tag{2.1}$$

Moreover, $(L + \lambda(k - m))^{-1}|_{C_T} : C_T \to C_T$ is a compact operator.

Proof. As in Proposition 2.4 in [5], for $f \in L^r(L^p)$, the equation $(L + \lambda(k - m))u = f$ can be written as

$$\bigl(I - \lambda\bigl(L + \lambda(k + m^-)\bigr)^{-1} m^+\bigr)u = \bigl(L + \lambda(k + m^-)\bigr)^{-1} f. \tag{2.2}$$

Now, Remark 2.2 (applied to $L + \lambda(k + m^-)$ instead of L) gives that $((L + \lambda(k + m^-))^{-1} m^+)(L_T^\infty) \subset C_T$. Also, for k large enough

$$\bigl\|\lambda\bigl(L + \lambda(k + m^-)\bigr)^{-1} m^+\bigr\|_{C_T} \leqslant \bigl\|\lambda\bigl(L + \lambda(k + m^-)\bigr)^{-1} m^+\bigr\|_{L_T^\infty} < 1,$$

where the last inequality follows from Lemma 2.3 in [5] taking there $q = \infty$. Thus, for such a k, $(I - \lambda(L + \lambda(k + m^-))^{-1} m^+)^{-1}|_{C_T} : C_T \to C_T$ is a well defined and bounded operator and so (2.1) follows from (2.2). Since $(L + \lambda(k - m))^{-1} : L^r(L^p) \to L_T^\infty$ is compact, (2.1) gives the last assertion of the lemma. □

Lemma 2.3 implies that the principal eigenfunctions in $L^r(L^p)$ for problem (1.5) actually belong to C_T.

Remark 2.4. Let R, Λ, k_0, k, λ, m be as in Lemma 2.3. Then, the spectrum of

$$\bigl(L + \lambda(k - m)\bigr)^{-1}\big|_{C_T} : C_T \to C_T \tag{2.3}$$

agrees with the spectrum of

$$\bigl(L + \lambda(k - m)\bigr)^{-1} : L^r\bigl(L^p\bigr) \to L^r\bigl(L^p\bigr) \tag{2.4}$$

and, for a given eigenvalue, these operators have the same generalized eigenspaces. In particular, they have the same spectral radius $\rho_{k,\lambda,m}$. Moreover, since $\rho_{k,\lambda,m}$ is an algebraically simple eigenvalue for (2.4) (see [5, Remark 2.7]), the same is true for (2.3).

For $\lambda > 0$, $m \in L^s(L^v)$, let $\mu_m(\lambda)$ be defined by $\rho_{\lambda,k,m} = (\lambda k + \mu_m(\lambda))^{-1}$ (taking k large enough). Thus $\mu_m(\lambda)$ does not depend on k and can be characterized as the unique $\mu \in \mathbb{R}$ such that the problem

$$\begin{cases} Lu_{\lambda,m} = \lambda m u_{\lambda,m} + \mu u_{\lambda,m} & \text{in } \Omega \times \mathbb{R}, \\ u_{\lambda,m} = 0 & \text{on } \partial\Omega \times \mathbb{R}, \\ u_{\lambda,m}\ T\text{-periodic in } t, \end{cases} \tag{2.5}$$

has a positive solution $u_{\lambda,m}$ in $L^r(L^p)$, i.e., by Remark 2.4, in C_T. We recall that μ_m is real analytic, concave and $\mu_m(0) > 0$ (cf. [5], Lemma 3.2 and Remark 3.3).

Remark 2.5. Let Ω_0 be a bounded domain in $\mathbb{R}^N$ and let $\Gamma : \mathbb{R} \to \mathbb{R}^N$ be a C^2 and T-periodic curve. We set

$$B_{\Gamma,\Omega_0} = \bigl\{(x,t):\ x \in \Gamma(t) + \Omega_0,\ t \in (0,T)\bigr\}. \tag{2.6}$$

For $m \in L^s(L^v)$, let $P(m)$ be defined by (1.4). Observe that $P(m) > 0$ is equivalent to the following condition: there exist Ω_0 and Γ as above with $B_{\Gamma,\Omega_0} \subset \Omega \times \mathbb{R}$ and such that $\int_{B_{\Gamma,\Omega_0}} m > 0$. Indeed, clearly the existence of such a B_{Γ,Ω_0} implies $P(m) > 0$. Suppose now $P(m) > 0$. For $j \in \mathbb{N}$, let $m_j = \min\{j, m\}$ and let $\tilde{m}_j$, $\tilde{m}$ be defined by

(1.3). So $\tilde{m}_j(t) = \min\{j, \tilde{m}(t)\}$. Moreover, $\{\tilde{m}_j\}_{j\in\mathbb{N}}$ is a nondecreasing sequence that converges a.e. to $\tilde{m}$. Now, $m_j \in L^s(L^v)$ and so $\tilde{m}_j^- \in L^s(L^v)$. Then $\tilde{m}_j \in L^s(L^v)$. Also, $0 \leqslant \tilde{m}_j + \tilde{m}^- \leqslant \tilde{m}_{j+1} + \tilde{m}^-$, $j \in \mathbb{N}$. Thus $\lim_{j\to\infty} P(m_j) = P(m)$ and so $P(m_{j_0}) > 0$ for some j_0. Since m_{j_0} is bounded from above, Lemma 3.4 in [5] gives a bounded domain Ω_0 and $\Gamma \in C^2(\mathbb{R}, \mathbb{R}^N)$ such that $\int_{B_{\Gamma,\Omega_0}} m_{j_0} > 0$ and so $\int_{B_{\Gamma,\Omega_0}} m > 0$.

We will need the following result about perturbation of simple eigenvalues due to Crandall and Rabinowitz (see [1, Lemma 1.3]).

Lemma 2.6. *Let X be a real Banach space. Let T_0 be a bounded operator on X, and assume that r_0 is an algebraically simple eigenvalue of T_0. Then there exists $\delta > 0$ such that whenever $\|T - T_0\| < \delta$, there is a unique $r(T) \in \mathbb{R}$ satisfying $|r(T) - r_0| < \delta$ for which $r(T)I - T$ is singular. Moreover, the map $T \to r(T)$ is analytic and $r(T)$ is an algebraically simple eigenvalue of T. Finally, it can be chosen an eigenvector associated $v(T)$ such that also the map $T \to v(T)$ is also analytic.*

We have the following proposition.

Proposition 2.7. *The map $(\lambda, m) \to \mu_m(\lambda)$ is real analytic from $\mathbb{R} \times L^s(L^v)$ into $\mathbb{R}$. Moreover, a positive eigenfunction $u_{\lambda,m}$ for* (2.5) *can be chosen such that $(\lambda, m) \to u_{\lambda,m}$ is also real analytic from $\mathbb{R} \times L^s(L^v)$ into C_T.*

Proof. Let us show that $(\lambda, m) \to \mu_m(\lambda)$ is continuous. Let $\{(\lambda_j, m_j)\}_{j\in\mathbb{N}}$ be an arbitrary sequence that converges in $\mathbb{R} \times L^s(L^v)$ to some (λ_0, m_0). As in the proof of Lemma 3.2 in [5] we get that $\{\mu_{m_j}(\lambda_j)\}_{j\in\mathbb{N}}$ is bounded. After passing to some subsequence we can assume that $\mu_{m_j}(\lambda_j)$ converges to some $\mu \in \mathbb{R}$. Let $\tilde{u}_{\lambda_j,m_j}$ be a positive solution of (2.5) (taking there $\lambda = \lambda_j$, $m = m_j$) normalized by $\|\tilde{u}_{\lambda_j,m_j}\|_\infty = 1$. Remark 2.2 gives a subsequence $\tilde{u}_{\lambda_{j_k},m_{j_k}}$ that converges to some u satisfying $Lu = \lambda m u + \mu u$. Moreover, $u > 0$ and then $\mu = \mu_m(\lambda)$. So, $\{\mu_{m_j}(\lambda_j)\}_{j\in\mathbb{N}}$ has a subsequence that converges to $\mu_m(\lambda)$. This proves that $(\lambda, m) \to \mu_m(\lambda)$ is continuous.

Now, for $(\lambda_0, m_0) \in \mathbb{R} \times L^s(L^v)$, let $V_{\lambda_0,m_0} = (\lambda_0 - \delta, \lambda_0 + \delta) \times B_R^{s,v}(0)$ with δ, R positive and small enough, and let $k > k_0(R, \lambda_0 + \delta)$ with k_0 as in Lemma 2.3. Since $(\lambda, m) \to \mu_m(\lambda)$ is continuous, the same is true for $(\lambda, m) \to \rho_{\lambda,m}$ where $\rho_{\lambda,m}$ is the spectral radius (and so the algebraically simple positive principal eigenvalue) of $T_{\lambda,m} := (L + \lambda(k - m))^{-1} : C_T \to C_T$. Since $(\lambda, m) \to T_{\lambda,m}$ is real analytic (see the proof of Theorem 3.9 in [5]), Lemma 2.6 concludes the proof. □

Let

$$\mathcal{M} = \{m \in L^s(L^v)\colon\ P(m) > 0\} \tag{2.7}$$

with $P(m)$ defined by (1.4). By Remark 2.5 it is clear that $\mathcal{M}$ is an open set in $L^s(L^v)$.

Corollary 2.8. *For $m \in \mathcal{M}$ and $\lambda = \lambda_1(m)$, a positive eigenfunction u_m of problem* (1.5) *can be chosen such that $m \to u_m$ is real analytic from $\mathcal{M}$ into C_T.*

Proof. We know that $m \to \lambda_1(m)$ is real analytic (cf. [5, Theorem 3.9]). Let $u_{\lambda,m}$ be the eigenfunction for (2.5) provided by Proposition 2.7. Taking $u_m = u_{\lambda_1(m),m}$ the corollary follows. $\square$

For $\lambda > 0$, let

$$D_\lambda = \{m \in L^s(L^v)\colon\ \mu_m(\lambda) > 0\}. \tag{2.8}$$

By Proposition 2.7, D_λ is open in $L^s(L^v)$. Let us observe that for $\lambda > 0$ the condition $\mu_m(\lambda) > 0$ is equivalent to: $0 < \lambda < \lambda_1(m)$ if $\lambda_1(m)$ exists and to $\lambda > 0$ if the weight m has no positive principal eigenvalue.

Lemma 2.9. *Let $\lambda \in (0,\infty)$ and let $(m,h) \in D_\lambda \times L^r(L^p)$. Then the problem,*

$$\begin{cases} Lu = \lambda m u + h & \text{in } \Omega \times \mathbb{R}, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}, \\ u \ T\text{-periodic in } t, \end{cases} \tag{2.9}$$

has a unique solution $u \in C_T$. Moreover:

(a) *Let $S_\lambda(m,h)$ denote the solution operator for* (2.9). *Then $S_\lambda(m,\cdot) : L^r(L^p) \to C_T$ is compact, and if $h > 0$, then $S_\lambda(m,h)(x,t) > 0$ a.e. $(x,t) \in \Omega \times \mathbb{R}$.*
(b) *The operator $S_\lambda(.,h) : D_\lambda \to C_T$ is compact.*

Proof. For k large enough let $T = (L + \lambda(k-m))^{-1}$. Now, (2.9) is equivalent to

$$\left(\frac{1}{\lambda k} I - T\right) u = \frac{1}{\lambda k} T h. \tag{2.10}$$

Let $\rho(T)$ denote the spectral radius of $T|_{C_T}$. Since $\mu_m(\lambda) > 0$, we have $\rho(T) < 1/(\lambda k)$ and thus $(\frac{1}{\lambda k} I - T)^{-1} : C_T \to C_T$ is a well defined and bounded operator. Then, (2.10) is equivalent to $u = (\frac{1}{\lambda k} I - T)^{-1} \frac{1}{\lambda k} T h$ and so (2.9) has a unique solution $u \in C_T$. Also, the last formula together with Lemma 2.3 give the compactness of $S_\lambda(m,\cdot)$ and the positivity follows from Theorem 3.10 in [5].

To see (b), let $m \in D_\lambda$, let $\{m_j\}_{j\in\mathbb{N}}$ be a sequence in D_λ that converges weakly to m in $L^s(L^v)$ and let $u_j = S_{\lambda,h}(m_j)$. Then $\{u_j\}_{j\in\mathbb{N}}$ is bounded in C_T. Indeed, if for some subsequence $\lim_{k\to\infty} \|u_{j_k}\|_\infty = \infty$, from

$$L\left(\frac{u_{j_k}}{\|u_{j_k}\|_\infty}\right) = \lambda m \frac{u_{j_k}}{\|u_{j_k}\|_\infty} + \frac{h}{\|u_{j_k}\|_\infty}$$

and going to the limit, by Remark 2.2 we get that $\lambda = \lambda_1(m)$ contradicting that $m \in D_\lambda$. Now, since $\sup_j \|u_j\|_\infty < \infty$, from $Lu_j = \lambda m_j u_j + h$, the same compactness argument gives a subsequence u_{j_k} that converges to the solution of (2.9), i.e., to $S_{\lambda,h}(m)$. Since $\{m_j\}_{j\in\mathbb{N}}$ was arbitrary, this ends the proof. $\square$

Lemma 2.10. *Let $\mathcal{N}$ be the set of the functions $m \in L^s(L^v)$ such that $(L+m)^{-1}$: $L^r(L^p) \to C_T$ is a well defined and bounded operator. Then $\mathcal{N}$ is open in $L^s(L^v)$ and the map $m \to (L+m)^{-1}$ is continuous from $\mathcal{N}$ into $B(L^r(L^p), C_T)$.*

Proof. Let $m_0 \in \mathcal{N}$ and $m \in L^s(L^v)$. For $f \in L^r(L^p)$, the equation $Lu + mu = f$ is equivalent to $u = (L + m_0)^{-1}(m_0 - m)u + (L + m_0)^{-1} f$. Since

$$\left\|(L + m_0)^{-1}(m_0 - m)\right\|_{C_T} \leqslant \left\|(L + m_0)^{-1}\right\|_{L^r(L^p),C_T} \|m_0 - m\|_{L^s(L^v)}$$

it follows that for m close enough in $L^s(L^v)$ to m_0,

$$\left(I - (L + m_0)^{-1}(m_0 - m)\right)^{-1} : C_T \to C_T$$

is a well defined and bounded operator. Thus, for such m we have

$$u = \left(I - (L + m_0)^{-1}(m_0 - m)\right)^{-1}(L + m_0)^{-1} f, \tag{2.11}$$

but, (2.11) implies that for m close enough to m_0, $(L + m)^{-1}$ is a well defined and bounded operator from $L^r(L^p)$ into C_T. So $\mathcal{N}$ is open. Moreover, $\|(L + m)^{-1}\|_{L^r(L^p),C_T}$ remains bounded for m running on a small neighborhood of m_0. Since for such m

$$(L + m)^{-1} - (L + m_0)^{-1} = (L + m_0)^{-1}\left[\left(I - (m_0 - m)(L + m_0)^{-1}\right)^{-1} - I\right]$$

the lemma follows. □

For $\varepsilon > 0$, let $\Omega_\varepsilon = \{x \in \Omega\colon d(x, \partial\Omega) > \varepsilon\}$ and let $A_\varepsilon = \Omega - \Omega_\varepsilon$. We will need the following Harnack type inequality for the positive eigenfunctions of (1.5).

Proposition 2.11. *Let $R, \Lambda \in (0, \infty)$. Then, for each $\varepsilon > 0$ there exists $c > 0$ such that if $m \in B_R^{s,v}$, $\lambda \in [0, \Lambda]$ and $u \in C_T$ is a positive solution of* (1.5), *then $\|u\|_\infty \leqslant c \operatorname{essinf}_{\Omega_\varepsilon \times (0,T)} u$.*

Proof. Let $1 < \tilde{s}, \tilde{v} < \infty$ be defined by $r^{-1} = s^{-1} + \tilde{s}^{-1}$, $p^{-1} = v^{-1} + \tilde{v}^{-1}$ and for $j = 1, 2$, let $\theta_j \in (0, 1)$ be defined by $\tilde{s}^{-1} = (1 - \theta_1)$ and $\tilde{v}^{-1} = 1 - \theta_2$. From (1.5) we have

$$\begin{aligned}
\|u\|_\infty &\leqslant \lambda \left\|L^{-1}\right\|_{L^r(L^p),C_T} \|m\|_{L^s(L^v)} \|u\|_{L^{\tilde{s}}(L^{\tilde{v}})} \leqslant c_1 \|u\|_{L^{\tilde{s}}(L^{\tilde{v}})} \\
&\leqslant c_1 \|u\|_{L^\infty(L^{\tilde{v}})}^{\theta_1} \|u\|_{L^1(L^{\tilde{v}})}^{1-\theta_1} \leqslant c_2 \|u\|_\infty^{\theta_1} \|u\|_{L_T^{\tilde{v}}}^{1-\theta_1} \\
&\leqslant c_2 \|u\|_\infty^{\theta_1} \left[\|u\|_\infty^{\theta_2} \|u\|_{L_T^1}^{1-\theta_2}\right]^{1-\theta_1} = c_2 \|u\|_\infty^{\theta_1+\theta_2-\theta_1\theta_2} \|u\|_{L_T^1}^{1-(\theta_1+\theta_2-\theta_1\theta_2)}
\end{aligned}$$

for some $c_1, c_2 > 0$. Since $1 - (\theta_1 + \theta_2 - \theta_1\theta_2) > 0$ we get

$$\|u\|_\infty \leqslant c_3 \|u\|_{L_T^1} \tag{2.12}$$

for some $c_3 > 0$. Now, $\|u\|_{L_T^1(A_\varepsilon \times (0,T))} \leqslant |A_\varepsilon| T \|u\|_\infty \leqslant c_3 T |A_\varepsilon| \|u\|_{L_T^1}$. Thus, if ε is small enough such that $c_3 T |A_\varepsilon| < 1/2$ we obtain

$$\|u\|_{L_T^1(\Omega_\varepsilon \times (0,T))} \geqslant \frac{1}{2} \|u\|_{L_T^1}. \tag{2.13}$$

From (2.12), (2.13), using Theorem 5.1 in [12], and taking into account the periodicity of u, it follows that $\|u\|_\infty \leqslant c \operatorname{essinf}_{\Omega_\varepsilon \times (0,T)} u$ for some $c > 0$, with c depending on ε, p, r, R, Λ, Ω and the operator L. □

Corollary 2.12. *Let $R, \Lambda \in (0,\infty)$. Then there exists $\Phi \in L_T^\infty$ with $\Phi(x,t) > 0$ for all $(x,t) \in \Omega \times \mathbb{R}$ such that if $m \in B_R^{s,v}$, $\lambda \in [0,\Lambda]$ and $u \in C_T$ is a positive solution of* (1.5), *then $u(x,t) \geqslant \|u\|_\infty \Phi(x,t)$ for all $(x,t) \in \Omega \times \mathbb{R}$.*

Proof. We can assume that $\|u\|_\infty = 1$. For $j \in \mathbb{Z}$, let

$$A_j = \{x \in \Omega \colon 2^{-j-1} < d(x, \partial\Omega) \leqslant 2^{-j}\}.$$

Thus $\Omega = \bigcup_{j\in\mathbb{Z}} A_j$. For j such that $A_j \neq \emptyset$, let c_j be the constant given by Proposition 2.11 taking $\varepsilon = 2^{-j-1}$. For $(x,t) \in A_j \times \mathbb{R}$ we set $\Phi(x,t) = 1/c_j$. So $\Phi(x,t) > 0$ for all (x,t). Now, Proposition 2.11 implies that $u(x,t) \geqslant \operatorname{essinf}_{\Omega_{2^{-j-1}} \times (0,T)} u \geqslant \frac{1}{c_j} = \Phi(x,t)$ for all $(x,t) \in A_j \times \mathbb{R}$. □

3. The main results

Let $g : \Omega \times \mathbb{R} \times [0,\infty) \to \mathbb{R}$ satisfying the following conditions:

(H1′) $(x,t) \to g(x,t,\xi)$ is measurable for all $\xi \in [0,\infty)$ and T-periodic in t, and $\xi \to g(x,t,\xi)$ is continuous in $[0,\infty)$ a.e. $(x,t) \in \Omega \times \mathbb{R}$.
(H2′) $\lim_{\xi\to 0^+} g(x,t,\xi)/\xi$ exists a.e. $(x,t) \in \Omega \times \mathbb{R}$.
(H3′) For all $\rho > 0$ $\inf_{0<\xi\leqslant\rho} g(x,t,\xi)/\xi \in L^s(L^v)$ and $\sup_{0<\xi} g(x,t,\xi)/\xi \in L^s(L^v)$.
(H4′) For all $\rho > 0$, $\int_0^T \operatorname{esssup}_{x\in\Omega} \inf_{0<\xi\leqslant\rho} g(x,t,\xi)/\xi > 0$.

For $u : \Omega \times \mathbb{R} \to [0,\infty)$, we set

$$m_u(x,t) = \begin{cases} \frac{g(x,t,u(x,t))}{u(x,t)} & \text{if } u(x,t) \neq 0, \\ \lim_{\xi\to 0^+} \frac{g(x,t,\xi)}{\xi} & \text{if } u(x,t) = 0. \end{cases} \tag{3.1}$$

Observe that if $u \in \bar{B}_\rho^{C_T}(0)$ then

$$\inf_{0<\xi\leqslant\rho} \frac{g(x,t,\xi)}{\xi} \leqslant m_u \leqslant \sup_{0<\xi\leqslant\rho} \frac{g(x,t,\xi)}{\xi}. \tag{3.2}$$

Let $g(u)$ be the Nemytskii operator defined by $g(u)(x,t) = g(x,t,u(x,t))$. If g satisfies (H1′)–(H4′), from the Lebesgue dominated convergence theorem it follows easily that $u \to g(u)$ and $u \to m_u$ are continuous maps from C_T into $L^s(L^v)$.

Let $\mathcal{M}$ be defined by (2.7) and for $m \in \mathcal{M}$, let $\Phi_1^{(m)}$ denote the positive principal eigenfunction associated to $\lambda_1(m)$, normalized by $\|\Phi_1^{(m)}\|_\infty = 1$. Corollary 2.8 implies that $m \to \Phi_1^{(m)}$ is continuous from $\mathcal{M}$ into C_T. It follows that $u \to \Phi_1^{(m_u)}$ is a continuous map from C_T into C_T.

Proposition 3.1. *Let $g : \Omega \times \mathbb{R} \times [0,\infty) \to \mathbb{R}$ satisfying* (H1′)–(H4′). *Then, for each $\rho > 0$,* (1.2) *has a positive eigenvalue with a positive and T-periodic eigenfunction associated $u_\rho \in C_T$ satisfying $\|u_\rho\|_\infty = \rho$. Moreover, $u_\rho(x,t) > 0$ for all $(x,t) \in \Omega \times \mathbb{R}$.*

Proof. We extend $g(x,t,\cdot)$ to the whole real line defining $g(x,t,\xi) = -g(x,t,-\xi)$ for $\xi < 0$. By (H2′) and (H3′) $\partial g/\partial\xi|_{\xi=0} \in L^s(L^v)$. Let $\rho > 0$. From (H3′), (H4′) and Remark 2.5, it follows that there exists a domain Ω_0 and a T periodic curve $\Gamma \in C^2(\mathbb{R}, \mathbb{R}^N)$ such that $B_{\Gamma,\Omega_0} \subset \Omega \times \mathbb{R}$ and $\int_{B_{\Gamma,\Omega_0}} \inf_{0<\xi\leqslant\rho} g(x,t,\xi)/\xi > 0$. For $w \in \bar{B}_\rho^{C_T}(0)$, let m_w be defined as in (3.1). Thus, (3.2) and (H3′) imply that $m_w \in L^s(L^v)$ and $\int_{B_{\Gamma,\Omega_0}} m_w > 0$ (i.e., $P(m_w) > 0$). So, there exists $\lambda_1(m_w)$.

Let $T : \bar{B}_\rho^{C_T}(0) \to \bar{B}_\rho^{C_T}(0)$ be defined by $T(w) = \rho\Phi_1^{(m_w)}$. Then T is a compact map. Indeed, T is continuous. Now, let $\{w_j\}_{j\in\mathbb{N}}$ be a sequence in $\bar{B}_\rho^{C_T}$. From (3.2) we have $\|m_{w_j}\|_{L^s(L^v)} \leqslant c$ for some $c > 0$ and all j. Moreover, (3.2) and Proposition 3.1 in [5] imply that $\{\lambda_1(m_{w_j})\}_{j\in\mathbb{N}}$ is bounded. So $\{\rho\lambda_1(m_{w_j})m_{w_j}\Phi_1^{(m_{w_j})}\}_{j\in\mathbb{N}}$ is bounded in $L^r(L^p)$, and hence the compactness of T follows from Remark 2.2.

Now, Schauder's fixed point theorem (e.g., [4, Corollary 11.2]) gives a fixed point $u_\rho \in \bar{B}_\rho^{C_T}$ for T. Then u_ρ is positive, $\|u_\rho\|_\infty = \rho$ and $Lu_\rho = \lambda_1(m_{u_\rho})g(x,t,u_\rho)$. Finally, since u_ρ satisfies $Lu_\rho = \lambda_1(m_{u_\rho})m_{u_\rho}u_\rho$, Corollary 2.12 says that $u_\rho(x,t) > 0$ for all $(x,t) \in \Omega \times \mathbb{R}$. □

Proposition 3.2. *Let* $g : \Omega \times \mathbb{R} \times [0,\infty) \to \mathbb{R}$ *satisfying* (H1′)–(H3′) *and* (H4). *Let* h *be a nonnegative and nonzero function in* $L^r(L^p)$. *Then for all* $0 < \lambda < \lambda_1(\bar{m})$ *the problem,*

$$\begin{cases} Lu = \lambda g(x,t,u) + h(x,t) & \text{in } \Omega \times \mathbb{R}, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}, \\ u \ T\text{-periodic in } t, \end{cases} \tag{3.3}$$

has a positive solution $u_\lambda \in C_T$. *Moreover,* $u_\lambda(x,t) > 0$ *a.e.* $(x,t) \in \Omega \times \mathbb{R}$.

Proof. We extend g as in the proof of Proposition 3.1. Let D_λ be defined by (2.8) and for $m \in D_\lambda$, let $S_{\lambda,h}(m)$ be the solution of (2.9). For $w \in C_T$, let m_w be defined by (3.1). Since $0 < \lambda < \lambda_1(\bar{m})$ and since $m_w \leqslant \bar{m}$, the comparison principle stated in [5, Remark 3.7] gives $0 < \kappa < \lambda_1(m_w)$ and so $m_w \in D_\lambda$ for all $w \in C_T$. Moreover, there exists $R > 0$ such that

$$\left\| S_{\lambda,h}(m_w) \right\|_\infty \leqslant R \tag{3.4}$$

for all $w \in C_T$. Indeed, if not, let $\{w_j\}_{j\in\mathbb{N}}$ be a sequence in C_T such that

$$\lim_{j\to\infty} \left\| S_{\lambda,h}(m_{w_j}) \right\|_\infty = \infty$$

and let $u_{w_j} = S_{\lambda,h}(m_{w_j})$. Since $\{m_{w_j}\}_{j\in\mathbb{N}}$ is bounded in $L^s(L^v)$ we can assume, after passing to some subsequence, that m_{w_j} converges weakly in $L^s(L^v)$ to some m. From

$$L\left(\frac{u_{w_j}}{\|u_{w_j}\|_\infty}\right) = \lambda m_{w_j} \frac{u_{w_j}}{\|u_{w_j}\|_\infty} + \frac{h}{\|u_{w_j}\|_\infty}$$

and Remark 2.2, we get that $Lu = \lambda mu$ has a positive solution. But $m \leqslant \bar{m}$ and so (by [5, Remark 3.7]) we get $\lambda < \lambda_1(\bar{m}) \leqslant \lambda_1(m)$. Contradiction.

For $w \in C_T$ we set $\tilde{S}(w) = S_{\lambda,h}(m_w)$. Then, since $w \to m_w$ is continuous from C_T into $L^s(L^v)$, it follows from Lemma 2.9(b) that $\tilde{S} : C_T \to C_T$ is a compact map. Now, let

R satisfying (3.4). We have $\tilde{S}(\bar{B}_R^{C_T}(0)) \subset \bar{B}_R^{C_T}(0)$ and so, the Schauder theorem gives a positive solution for (3.3). The last assertion follows from Lemma 2.9(a). $\square$

For g satisfying (H1), we extend g to a function $\tilde{g}: \Omega \times \mathbb{R} \times (-1, \infty) \to \mathbb{R}$ defined by

$$\tilde{g}(x, t, \xi) = \xi \lim_{\xi \to 0^+} \frac{g(x, t, \xi)}{\xi}, \quad \xi \in (-1, 0). \tag{3.5}$$

Definition 3.3. Let V be the open subset of C_T defined by $V = \{v \in C_T\colon\ v(x,t) > -1,\ (x,t) \in \bar{\Omega} \times \mathbb{R}\}$ and let

$$D = \left\{(\lambda, u) \in (0, \infty) \times C_T\colon\ u \in V \text{ and } \tilde{g}_\xi(u) \in D_\lambda\right\}$$

with D_λ given by (2.8). We recall that the condition $\tilde{g}_\xi(u) \in D_\lambda$ is equivalent to: $0 < \lambda < \lambda_1(\tilde{g}_\xi(u))$ if $\lambda_1(\tilde{g}_\xi(u))$ exists and $\lambda > 0$ if $\lambda_1(\tilde{g}_\xi(u))$ does not exist.

Let $F: D \to C_T$ be defined by

$$F(\lambda, u) = \left(L - \lambda \tilde{g}_\xi(u)\right)^{-1} \tilde{g}(u).$$

Note that by Lemma 2.9 F is well defined.

Lemma 3.4. *Suppose that g satisfies* (H1). *Then D is open in $\mathbb{R} \times C_T$.*

Proof. We proceed by contradiction. Suppose that $(\lambda, u) \in D$ and that $\{(\lambda_j, u_j)\}_{j \in \mathbb{N}}$ is a sequence in $\mathbb{R} \times C_T$ such that $\lim_{j\to\infty}(\lambda_j, u_j) = (\lambda, u)$ and $(\lambda_j, u_j) \notin D$ for all j. Clearly $u_j \in V$ for j large enough. Let $R = 1 + \|u\|_\infty$. Thus there exists j_0 such that $|u_j(x,t)| \leqslant R$ for all $(x,t) \in \bar{\Omega} \times \mathbb{R}$, $j \geqslant j_0$.

Suppose first that $\lambda_1(\tilde{g}_\xi(u))$ exists. Then $\int_{B_{\Gamma,\Omega_0}} \tilde{g}_\xi(u) > 0$ for some B_{Γ,Ω_0} as in Remark 2.5. Since $\tilde{g}_\xi(u_j)$ converges to $\tilde{g}_\xi(u)$ we have, enlarging j_0 if necessary, that $\int_{B_{\Gamma,\Omega_0}} \tilde{g}_\xi(u_j) > 0$ for $j \geqslant j_0$. Thus there exists $\lambda_1(\tilde{g}_\xi(u_j))$ for such a j. Moreover, $\lim_{j\to\infty} \lambda_1(\tilde{g}_\xi(u_j)) = \lambda_1(\tilde{g}_\xi(u)) > \lambda$ (the inequality because $(\lambda, u) \in D$). Now, since $\lambda_j \to \lambda$ we have $\lambda_j < \lambda_1(\tilde{g}_\xi(u_j))$ and so $(\lambda_j, u_j) \in D$ for j large enough. Contradiction.

Suppose now that $\lambda_1(\tilde{g}_\xi(u))$ does not exist. Let

$$J_e = \left\{j \in \mathbb{N}\colon\ \lambda_1\left(\tilde{g}_\xi(u_j)\right) \text{ exists}\right\}.$$

If J_e is finite then $(\lambda_j, u_j) \in D$ for j large enough. We claim that if J_e is not finite then $\{\lambda_1(\tilde{g}_\xi(u_j))\colon\ j \in J_e\}$ is unbounded. To see this we proceed by contradiction. Let $w_j \in C_T$ be a positive eigenfunction associated to the weight $\tilde{g}_\xi(u_j)$ normalized by $\|w_j\|_\infty = 1$. Since we have assumed that $\lambda_1(\tilde{g}_\xi(u_j)) \leqslant c$ for some c and all $j \in J_e$, (H1) and Hölder's inequality give that $\|\lambda_1(\tilde{g}_\xi(u_j))\tilde{g}_\xi(u_j)w_j\|_{L^r(L^p)} \leqslant c'$ for some c' and all $j \in J_e$. Then there exists a subsequence $\lambda_1(\tilde{g}_\xi(u_{j_k}))\tilde{g}_\xi(u_{j_k})w_{j_k}$ that is weakly convergent to some $f \in L^r(L^p)$ and so Theorem 5.1(a) in [2] (applied with $q = \infty$) implies that w_{j_k} converges in the L^∞ norm to some w. Thus $w \in C_T$ and $f = \lambda_1(\tilde{g}_\xi(u))\tilde{g}_\xi(u)w$. Moreover, $w > 0$ and $Lw = \lambda_1(\tilde{g}_\xi(u))\tilde{g}_\xi(u)w$. Contradiction. So our claim is proved. Thus, for some subsequence u_{j_k} with $j_k \in J_e$ we have

$$\lim_{k\to\infty} \lambda_1\left(\tilde{g}_\xi(u_{j_k})\right) = \infty.$$

Since λ_j converges we have $\lambda_j < \lambda_1(\tilde{g}_\xi(u_{j_k}))$ for k large enough and then $(\lambda_{j_k}, u_{j_k}) \in D$ for such k. Contradiction. □

Lemma 3.5. *Suppose that g satisfies* (H1). *Then $F : D \to C_T$ is a continuous map. Moreover, for each $(\lambda_0, u_0) \in D$ there exists a neighborhood $U_\delta = (\lambda_0 - \delta, \lambda_0 + \delta) \times B_\delta(u_0)$ such that $F|_{U_\delta}$ is a compact map.*

Proof. From (H1), the map $(\lambda, u) \to -\lambda\tilde{g}_\xi(u)$ is continuous from D into $L^s(L^v)$. So, by Lemma 2.10 $(\lambda, u) \to (L - \lambda\tilde{g}_\xi(u))^{-1}$ is continuous from D into $B(L^r(L^p), C_T)$. Then, for δ small enough, $\|(L - \lambda\tilde{g}_\xi(u))^{-1}\|_{L^r(L^p), C_T}$ remains bounded for (λ, u) running on U_δ. Since $(\lambda, u) \to \tilde{g}(u)$ is also continuous we get that F is continuous and so, for a smaller δ if necessary, $F(U_\delta)$ is bounded in C_T. For such a δ, let $\{(\lambda_j, u_j)\}_{j\in\mathbb{N}}$ be a sequence in U_δ and let $w_j = F(\lambda_j, u_j)$. Now,

$$Lw_j = \lambda_j\tilde{g}_\xi(u_j)w_j + \tilde{g}(u_j). \tag{3.6}$$

Since $F(U_\delta)$ is bounded in C_T we have $\|w_j\|_\infty \leqslant c$ and then, by (H1), the sequence of the $L^s(L^v)$ norms of the right member of (3.6) is bounded. Thus, the same is true for its $L^r(L^p)$ norms and so Remark 2.2 gives the compactness assertion. □

Remark 3.6. Lemma 3.5 allows us to apply an extension to Banach spaces of Peano's theorem about local existence of solutions for initial value problems (as stated, e.g., in [8, Chapter 6, Theorem 3.6]) in order to obtain that, for $(\lambda, u) \in D$, there exists a neighborhood $U_{\lambda,u} = (\lambda - \varepsilon, \lambda + \varepsilon) \times \bar{B}_\varepsilon^{C_T}(u)$ and $\delta > 0$ such that for all $(\tilde{\lambda}, \tilde{u}) \in U_{\lambda,u}$ a solution for the initial value problem,

$$\begin{cases} \frac{du_\lambda}{d\lambda} = F(\lambda, u_\lambda), \\ u_{\tilde{\lambda}} = \tilde{u}, \end{cases}$$

is defined for $\lambda \in (\tilde{\lambda} - \delta, \tilde{\lambda} + \delta)$.

Theorem 3.7. *Let $g : \Omega \times \mathbb{R} \times [0, \infty) \to \mathbb{R}$ satisfying* (H1)–(H5). *Then* (1.2) *has a positive solution $u_\lambda \in C_T$ if and only if $\lambda_1(\bar{m}) < \lambda < \lambda_1(\underline{m})$. Moreover, u_λ can be chosen such that $\lambda \to u_\lambda$ is a C^1 map from $(\lambda_1(\bar{m}), \lambda_1(\underline{m}))$ into C_T and $u_\lambda(x, t) > 0$ for all $(x, t) \in \Omega \times \mathbb{R}$. We also have that $\lim_{\lambda\to\lambda_1(\bar{m})^+} \|u_\lambda\|_\infty = 0$ and $\lim_{\lambda\to\lambda_1(\underline{m})^-} u_\lambda(x, t) = \infty$ for all $(x, t) \in \Omega \times \mathbb{R}$.*

Proof. If (λ, u_λ) solves (1.2) with $u_\lambda > 0$, let $m_{u_\lambda} : \Omega \times \mathbb{R} \to \mathbb{R}$ be defined by (3.1). By (H4), $m_{u_\lambda} \in L^s(L^v)$. Since $Lu_\lambda = \lambda m_{u_\lambda} u_\lambda$ we have $\lambda = \lambda_1(m_{u_\lambda})$. Now, $\underline{m} \leqslant m_{u_\lambda} \leqslant \bar{m}$. Moreover, it is easy to see using (H3) that the strict inequalities hold in a subset of positive measure and so [5, Remark 3.7] gives that $\lambda_1(\bar{m}) < \lambda < \lambda_1(\underline{m})$.

To prove the remaining assertions of the theorem, we start with the solution (λ_0, u_0) of (1.2) given by Proposition 3.1. Since $u_0(x, t) > 0$ for all $(x, t) \in \Omega \times \mathbb{R}$, we have that $\lambda_0 = \lambda_1(g(u_0)/u_0)$. Also, (H2) implies that $g_\xi(u_0) \leqslant g(u_0)/u_0$. Moreover, if $\delta > 0$ and $(x_0, t_0) \in \partial\Omega \times \mathbb{R}$ are given by (H3) we have the strict inequality a.e. $(x, t) \in B_\delta(x_0, t_0) \cap (\Omega \times \mathbb{R})$. So, by [5, Remark 3.7], we have $\lambda_0 < \lambda_1(g_\xi(u_0))$ if $\lambda_1(g_\xi(u_0))$

exists and then $(\lambda_0, u_0) \in D$. Taking into account Remark 3.6 we have a local solution for the Cauchy initial value problem,

$$\begin{cases} \frac{du_\lambda}{d\lambda} = F(\lambda, u_\lambda), \\ u_{\lambda_0} = u_0. \end{cases} \tag{3.7}$$

Consider a maximal solution (i.e., with maximal connected domain) for this problem and let $I = (\alpha, \beta)$ be its domain. Observe that F is continuous and so $\lambda \to u_\lambda$ is continuously differentiable from I into C_T. Now, $du_\lambda/d\lambda = F(\lambda, u_\lambda)$ can be read $(L - \lambda\tilde{g}_\xi(u_\lambda))du_\lambda/d\lambda = \tilde{g}(u_\lambda)$ and so, in a distributional sense, we have $L(du_\lambda/d\lambda) = \tilde{g}(u_\lambda) + \lambda\tilde{g}_\xi(u_\lambda)du_\lambda/d\lambda$, i.e., $d/d\lambda\,(Lu_\lambda) = d/d\lambda\,(\lambda\tilde{g}(u_\lambda))$. Hence, $Lu_\lambda - \lambda\tilde{g}(u_\lambda)$ does not depend on λ. Since it is zero for $\lambda = \lambda_0$ we have $Lu_\lambda = \lambda\tilde{g}(u_\lambda)$ for all $\lambda \in I$.

Let us divide the rest of the proof in three steps.

Step 1. There exists an open interval I_0 around λ_0 such that $u_\lambda(x,t) > 0$ for all $(x,t) \in \Omega \times \mathbb{R}$, $\lambda \in I_0$.

Let $\tilde{m}_{u_\lambda}$ be defined by (3.1) with $\tilde{g}$ in place of g. For $\lambda \in I$, (H5) implies that $P(\tilde{m}_{u_\lambda}) > 0$ and so $\lambda_1(\tilde{m}_{u_\lambda})$ exists. Clearly we have $\lambda_0 = \lambda_1(\tilde{m}_{u_{\lambda_0}})$. Now, since $\lambda \to \tilde{m}_{u_\lambda}$ is continuous, Theorem 3.9 in [5] gives that $\lambda \to \lambda_1(\tilde{m}_{u_\lambda})$ is also continuous. So, given $\varepsilon > 0$ there exists $\delta > 0$ such that $\lambda_1(\tilde{m}_{u_\lambda})^{-1} \in (1/\lambda_0 - \varepsilon, 1/\lambda_0 + \varepsilon)$ for $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$. On the other hand, $\lambda_1(\tilde{m}_{u_\lambda})^{-1} \in \sigma(L^{-1}M_\lambda)$, where M_λ denotes the operator multiplication by $\tilde{m}_{u_\lambda}$ and where $\sigma(L^{-1}M_\lambda)$ denotes the spectrum of $L^{-1}M_\lambda : C_T \to C_T$. Since $\lambda^{-1} \in \sigma(L^{-1}M_\lambda)$, taking $\varepsilon > 0$ small enough, the Crandall–Rabinowitz lemma implies that $\lambda = \lambda_1(\tilde{m}_{u_\lambda})$ for λ close enough to λ_0 and so $u_\lambda > 0$ for such λ. Moreover, Corollary 2.12 says that $u_\lambda(x,t) > 0$ for all $(x,t) \in \Omega \times \mathbb{R}$.

Step 2. $u_\lambda > 0$ for all $\lambda \in I$.

Consider the maximal open subinterval J of I containing λ_0 such that $u_\lambda(x,t) > 0$ for all $(x,t) \in \Omega \times \mathbb{R}$, $\lambda \in J$. We will prove that $J = I$. Let

$$\lambda^+ = \sup\bigl\{\lambda \in I:\ u_\eta(x,t) > 0 \text{ for } (x,t) \in \Omega \times \mathbb{R}, \eta \in [\lambda_0, \lambda)\bigr\},$$
$$\lambda^- = \inf\bigl\{\lambda \in I:\ u_\eta(x,t) > 0 \text{ for } (x,t) \in \Omega \times \mathbb{R}, \eta \in (\lambda, \lambda_0]\bigr\}.$$

It is enough to prove that $\lambda^- = \alpha$, $\lambda^+ = \beta$. Let us show that $\lambda^+ = \beta$. We proceed by contradiction. Suppose $\lambda^+ < \beta$. We already know that $\lambda^+ > \lambda_0$. We claim that this implies that $\lambda^+ \in J$. Indeed, let Φ be the function provided by Corollary 2.12 taking there $\Lambda = \lambda^+$ and $R = \|\underline{m}\|_{L^s(L^v)} + \|\bar{m}\|_{L^s(L^v)}$. Now, since $Lu_\lambda = \lambda m_{u_\lambda} u_\lambda$ we get that $u_\lambda \geqslant \|u_\lambda\|_\infty \Phi$ for all $\lambda \in [\lambda_0, \lambda^+)$. Suppose first that $\|u_\lambda\|_\infty \geqslant c$ for some $c > 0$ and all $\lambda \in [\lambda_0, \lambda^+)$. Then $u_{\lambda^+} \geqslant c\Phi > 0$ and so $\lambda^+ \in J$. If there is not such a c, then we have $\lim_{j\to\infty} \|u_{\lambda_j}\|_\infty = 0$ for some sequence $\{\lambda_j\}_{j\in\mathbb{N}} \subset [\lambda_0, \lambda^+)$. After passing to a subsequence we can assume that $\lambda_j \to \tilde{\lambda}$ for some $\tilde{\lambda} \in [\lambda_0, \lambda^+]$. Then $u_{\tilde{\lambda}} = 0$ and so $\tilde{\lambda} = \lambda^+$. On the other hand, (H2) implies that $m_{u_{\lambda_j}}$ converges to $\bar{m}$ in $L^s(L^v)$ and then $\lambda_1(m_{u_{\lambda_j}}) \to \lambda_1(\bar{m})$. But $Lu_{\lambda_j} = \lambda_j m_{u_{\lambda_j}} u_{\lambda_j}$ with $u_{\lambda j} > 0$ and so $\lambda_j = \lambda_1(m_{u_{\lambda_j}})$. Thus, $\lambda_1(\bar{m}) = \lambda^+ < \lambda_0$. Contradiction. Thus we have proved that $\lambda^+ \in J$. Now, reasoning as in the proof of the existence of I_0 but now with λ^+ and u_{λ^+} instead of λ_0 and u_0, respectively, we find an interval around λ^+ where each u_λ is positive, contradicting the definition of λ^{+}. So $\lambda^+ = \beta$.

On the other side, clearly $\lambda^- < \lambda_0$. As above, if $\|u_\lambda\|_\infty \geqslant c$ for some $c > 0$ and all $\lambda \in (\lambda^-, \lambda_0]$, we would have $\lambda^- \in J$ and this leads to a contradiction with the definition of λ^-. Thus $\lim_{j\to\infty} \|u_{\lambda_j}\|_\infty = 0$ for some sequence $\{\lambda_j\}_{j\in\mathbb{N}} \subset (\lambda^-, \lambda_0]$. The dominated convergence theorem implies that $m_{u_{\lambda_j}}$ converges in $L^s(L^v)$ to $\bar{m}$, and so $\lambda_j = \lambda_1(m_{u_{\lambda_j}}) \to \lambda_1(\bar{m})$. Since $\alpha \leqslant \lambda^- \leqslant \lambda_j$ we conclude that $\lambda^- = \alpha$.

Step 3. $I = (\lambda_1(\bar{m}), \lambda_1(\underline{m}))$.

Let $\{\lambda_j\}_{j\in\mathbb{N}}$ be a sequence in I such that $\lambda_j \to \beta$. Then, as above, we have $\inf_j \|u_{\lambda_j}\|_\infty > 0$ (if not, we get $\lambda_1(\bar{m}) = \beta$ which contradicts the fact $\lambda_1(\bar{m}) < \beta$). Suppose first that $c_1 \leqslant \|u_{\lambda_j}\|_\infty \leqslant c_2$ for some $c_1, c_2 > 0$ and all j. Then $\|\lambda_j m_{u_{\lambda_j}} u_{\lambda_j}\|_{L^r(L^p)} \leqslant c$ for all j, and thus Remark 2.2 gives a subsequence $u_{\lambda_{j_k}}$ convergent to some $u > 0$ that satisfies $Lu = \beta m_u u$. So $\beta = \lambda_1(m_u)$. Moreover, $(\beta, u) \in D$. In fact, this is true if $g_\xi(u)$ has no positive principal eigenvalue. If $\lambda_1(g_\xi(u))$ exists, by (H2) and (H3) we have $g_\xi(u) \leqslant m_u$ with strict inequality on a subset of positive measure, then $\beta < \lambda_1(g_\xi(u))$ and so $(\beta, u) \in D$. Thus, by Remark 3.6, there exists a neighborhood $U_{\beta,u} = (\beta - \varepsilon, \beta + \varepsilon) \times \bar{B}_\varepsilon^{C_T}(u)$ and $\delta > 0$ such that for all $(\tilde{\beta}, \tilde{u}) \in U_{\beta,u}$ there exists $\lambda \to u_\lambda$ defined for $\lambda \in (\beta - \delta, \beta + \delta)$ that solves the Cauchy problem $du_\lambda/d\lambda = F(\lambda, u_\lambda)$ with initial value $u_{\tilde{\beta}} = \tilde{u}$. Taking $(\tilde{\beta}, \tilde{u}) = (\lambda_{j_k}, u_{\lambda_{j_k}})$ with k large enough, we get a contradiction with the maximality of I. Then we have proved that $\lim_{j\to\infty} \|u_{\lambda_j}\|_\infty = \infty$ and so, by Corollary 2.12, we have $\lim_{j\to\infty} u_{\lambda_j}(x,t) = \infty$ for each $(x,t) \in \Omega \times \mathbb{R}$. So $\lim_{j\to\infty} m_{u_{\lambda_j}} = \underline{m}$ in $L^s(L^v)$ and then $\beta = \lambda_1(\underline{m})$. A similar argument gives that if $\{\lambda_j\}_{j\in\mathbb{N}}$ is a sequence such that $\lambda_j \to \alpha$, then necessarily $\|u_{\lambda_j}\|_\infty \to 0$ and so $\alpha = \lambda_1(\bar{m})$. $\square$

As an immediate consequence we have

Corollary 3.8. *Let* $g : \Omega \times \mathbb{R} \times [0, \infty) \to \mathbb{R}$ *satisfying* (H1)–(H5). *Then the semilinear periodic parabolic problem,*

$$\begin{cases} Lu = g(x,t,u) & \text{in } \Omega \times \mathbb{R}, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}, \\ u \ T\text{-periodic in } t, \end{cases}$$

has a positive solution $u \in C_T$ *if and only if* $\lambda_1(\bar{m}) < 1 < \lambda_1(\underline{m})$. *Moreover,* $u(x,t) > 0$ *for all* $(x,t) \in \Omega \times \mathbb{R}$.

We have also the following related maximum principle.

Theorem 3.9. *Let* $g : \Omega \times \mathbb{R} \times [0, \infty) \to \mathbb{R}$ *satisfying* (H1)–(H5) *and let h be a nonnegative and nonzero function in* $L^r(L^p)$. *Then, for all* $0 < \lambda < \lambda_1(\bar{m})$, (3.3) *has a positive solution* $u_\lambda \in C_T$ *satisfying that* $\lambda \to u_\lambda$ *is a* C^1 *map from* $(0, \lambda_1(\bar{m}))$ *into* C_T *and* $u_\lambda(x,t) > 0$ *a.e.* $(x,t) \in \Omega \times \mathbb{R}$.

Proof. We start with a solution (given by Proposition 3.2) (λ_0, u_0) of the Dirichlet problem $Lu_0 = \lambda_0 g(x,t,u_0) + h$ with $0 < \lambda_0 < \lambda_1(\bar{m})$ and $u_0 > 0$. Let $\tilde{g}$ be defined by (3.5). Since $\bar{m} \geqslant m_{u_0} > g_\xi(u_0)$ we have $(\lambda_0, u_0) \in D$. As in Theorem 3.7, consider a maximal solution $\lambda \to u_\lambda$ for (3.7) defined on some interval $I = (\alpha, \beta) \subset (0, \lambda_1(\bar{m}))$ with $\lambda_0 \in I$. As there,

$\lambda \to u_\lambda$ is continuously differentiable from I into C_T and $Lu_\lambda = \lambda\tilde{g}(x,t,u_\lambda) + h$ for all $\lambda \in I$, i.e. $Lu_\lambda = \lambda\tilde{m}_{u_\lambda}u_\lambda + h$, where $\tilde{m}_{u_\lambda}$ is defined by (3.1) with $\tilde{g}$ in place of g. Since $0 < \lambda < \lambda_1(\bar{m})$ we have $0 < \lambda < \lambda_1(\tilde{m}_{u_\lambda})$ and so Lemma 2.9(a) implies that $u_\lambda(x,t) > 0$, a.e. $(x,t) \in \Omega \times \mathbb{R}$.

To prove that $I = (0, \lambda_1(\bar{m}))$ we proceed by contradiction. Suppose that $\beta < \lambda_1(\bar{m})$. Consider a sequence $\{\lambda_j\}_{j\in\mathbb{N}} \subset I$ such that $\lambda_j \to \beta$. From Remark 2.2 and since $h > 0$ it is easy to see that $\inf_j \|u_{\lambda_j}\|_\infty > 0$. If $\|u_{\lambda_j}\|_\infty \leqslant c$ for some $c > 0$ and all j, a compactness argument gives a solution $u > 0$ of the problem $Lu = \beta m_u u + h$, i.e. of $Lu = \beta g(x,t,u) + h$. But $\lambda_1(m_u) \geqslant \lambda_1(\bar{m}) > \beta$ and so $(\beta, u) \in D$. Then, as in the end of the proof of Theorem 3.7, recalling Remark 3.6 we get a contradiction with the maximality of I. Thus $\lim_{k\to\infty} \|u_{\lambda_{j_k}}\|_\infty = \infty$ for some subsequence $u_{\lambda_{j_k}}$, but, if this is the case, from $Lu_\lambda/\|u_\lambda\| = \lambda m_{u_\lambda}u_\lambda/\|u_\lambda\| + h/\|u_\lambda\|$, Remark 2.2 gives a positive solution w of $Lw = \beta m w$ where m is the weak limit of a suitable subsequence of $m_{u_{\lambda_j}}$. So $\beta = \lambda_1(m) \geqslant \lambda_1(\bar{m})$, contradicting $\beta < \lambda_1(\bar{m})$. Therefore $\beta = \lambda_1(\bar{m})$.

Suppose now that $\alpha > 0$. Let $\{\lambda_j\}_{j\in\mathbb{N}} \subset I$ such that $\lambda_j \to \alpha$. Proceeding as above, we obtain that $\inf_j \|u_{\lambda_j}\|_\infty > 0$. Moreover, if $\|u_{\lambda_j}\|_\infty \leqslant c$ for some $c > 0$, then $Lu = \alpha g(x,t,u) + h$ for some $u > 0$. Hence, since $\alpha < \beta = \lambda_1(\bar{m}) \leqslant \lambda_1(m_u)$, we have that $(\alpha, u) \in D$ and this leads to a contradiction. So, $\lim_{k\to\infty} \|u_{\lambda_{j_k}}\|_\infty = \infty$ for some subsequence $u_{\lambda_{j_k}}$. But then, reasoning as above, by Remark 2.2 we have a positive solution u of $Lu = \alpha m u$ where m is the weak limit of some subsequence of m_{j_k}. Thus $\alpha = \lambda_1(m) \geqslant \lambda_1(\bar{m}) = \beta$, contradiction. □

Theorem 3.10. *Let $g : \Omega \times \mathbb{R} \times [0,\infty) \to \mathbb{R}$ satisfying* (H1′), (H2), (H4) *and* (H5). *Then* (1.2) *has a positive solution $u_\lambda \in C_T$ for all $\lambda \in (\lambda_1(\bar{m}), \lambda_1(\underline{m}))$. Moreover, if* (1.2) *has a positive solution for some $\lambda > 0$, then $\lambda \in [\lambda_1(\bar{m}), \lambda_1(\underline{m})]$.*

Proof. The second assertion follows as in Theorem 3.7. In order to prove the first one, we first prove that the theorem holds if (H1′) is replaced by (H1). To see this, let $\psi_0 \in C^1([0,\infty))$ satisfying $\psi_0'(\xi) < 0$ for all ξ, $\psi_0(0) = 1$ and $\lim_{\xi\to\infty} \psi_0(\xi) = 0$ and for $0 < \varepsilon < 1$, let $g_\varepsilon(x,t,\xi) = g(x,t,\xi) + \varepsilon\xi\psi_0(\xi)$. Thus, for each ε, g_ε satisfies (H1)–(H5). Let $\underline{m}_\varepsilon = \lim_{\xi\to\infty} g_\varepsilon(x,t,\xi)/\xi$, $\bar{m}_\varepsilon = \lim_{\xi\to 0} g_\varepsilon(x,t,\xi)/\xi$. Observe that $\underline{m}_\varepsilon$ and $\bar{m}_\varepsilon$ converge in $L^s(L^v)$ to $\underline{m}$ and $\bar{m}$ as ε tends to zero, and therefore $\lim_{\varepsilon\to 0} \lambda_1(\underline{m}_\varepsilon) = \lambda_1(\underline{m})$ and $\lim_{\varepsilon\to 0} \lambda_1(\bar{m}_\varepsilon) = \lambda_1(\bar{m})$. Now, $\lambda_1(\bar{m}) < \lambda < \lambda_1(\underline{m})$ implies that $\lambda_1(\bar{m}_\varepsilon) < \lambda < \lambda_1(\underline{m}_\varepsilon)$ for ε small enough. Thus, by Theorem 3.7 we have a positive solution $u_\lambda^{(\varepsilon)}$ for (1.2) with g_ε in place of g, i.e.,

$$Lu_\lambda^{(\varepsilon)} = \lambda g_\varepsilon\bigl(x,t,u_\lambda^{(\varepsilon)}\bigr) = \lambda\left[\frac{g(x,t,u_\lambda^{(\varepsilon)})}{u_\lambda^{(\varepsilon)}} + \varepsilon\psi_0\bigl(u_\lambda^{(\varepsilon)}\bigr)\right]u_\lambda^{(\varepsilon)}.$$

Let $m_{\varepsilon,\lambda}$ be the expression inside the brackets. Then the norms $\|m_{\varepsilon,\lambda}\|_{L^s(L^v)}$ have an upper bound independent of ε. Let $\{\varepsilon_j\}_{j\in\mathbb{N}}$ be a sequence that converges to zero. We claim that $\|u_\lambda^{(\varepsilon_j)}\|_\infty \leqslant c$ for some $c > 0$ and all j. In fact, if not, we would have for some subsequence that $\lim_{k\to\infty} \|u_\lambda^{(\varepsilon_{j_k})}\|_\infty = \infty$, and so, by Corollary 2.12, $\lim_{k\to\infty} u_\lambda^{(\varepsilon_{j_k})}(x,t) = \infty$ for all (x,t) and consequently $\lim_{k\to\infty} m_{\varepsilon_{j_k},\lambda} = \underline{m}$ in $L^s(L^v)$. Let $w_k = u_\lambda^{(\varepsilon_{j_k})}/\|u_\lambda^{(\varepsilon_{j_k})}\|_\infty$. From

$Lw_k = \lambda m_{\varepsilon_{j_k},\lambda} w_k$, using Remark 2.2 we get easily that $\lambda = \lambda_1(\underline{m})$. Contradiction. Thus, if $\varepsilon_j \to 0$, we have $\|u_\lambda^{(\varepsilon_j)}\|_\infty \leqslant c$ with c independent of ε. Also, if for some subsequence $\lim_{k\to\infty} \|u_\lambda^{(\varepsilon_{j_k})}\|_\infty = 0$ we would get $\lambda = \lambda_1(\bar{m})$. From these facts and the compactness of L^{-1} we obtain (going to the limit as ε goes to 0) that (1.2) has a positive solution.

Finally, suppose that g satisfies the hypothesis of the theorem. Let $\tilde{g}$ be defined by (3.5). We set $\psi(x,t,\xi) = \lim_{\xi\to 0^+} \tilde{g}(x,t,\xi)/\xi$ if $-1 < \xi \leqslant 0$ and $\psi(x,t,\xi) = g(x,t,\xi)/\xi$ if $\xi > 0$. Let $\phi \in C^\infty(\mathbb{R})$ with $\operatorname{supp}(\phi) \subset [-1,1]$, $0 \leqslant \phi \leqslant 1$ and $\int_{\mathbb{R}} \phi = 1$. Also, for $\varepsilon > 0$, let $\phi_\varepsilon \in C^\infty(\mathbb{R})$ be defined by $\phi_\varepsilon(\xi) = \frac{1}{\varepsilon}\phi(\xi/\varepsilon)$ and let $\tilde{g}_\varepsilon(x,t,\xi) = \xi(\psi(x,t,\cdot) * \phi_\varepsilon)(\xi)$. It is easy to check that for ε small enough $\tilde{g}_\varepsilon|_{[0,\infty)}$ satisfies (H1), (H2), (H4) and (H5). Thus, for $\lambda \in (\lambda_1(\bar{m}), \lambda_1(\underline{m}))$ we have (by the first part of the proof) a positive solution $u_\lambda^{(\varepsilon)}$ for $Lu_\lambda^{(\varepsilon)} = \lambda \tilde{g}_\varepsilon|_{[0,\infty)}(x,t,u_\lambda^{(\varepsilon)})$. Now, similar arguments as we have used above give the theorem. □

Remark 3.11. Let us mention that all our results remain true for the stationary case, i.e., for semilinear elliptic problems, replacing $L^s(L^v)$ by $L^r(\Omega)$, $r > N/2$.

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