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Expo. Math. 27 (2009) 1–36

EXPOSITIONES MATHEMATICAE

www.elsevier.de/exmath

A tropical toolkit

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Received 7 February 2008; received in revised form 3 April 2008

Abstract

We give an introduction to Tropical Geometry and prove some results in tropical intersection theory. The first part of this paper is an introduction to tropical geometry aimed at researchers in Algebraic Geometry from the point of view of degenerations of varieties using projective not-necessarily-normal toric varieties. The second part is a foundational account of tropical intersection theory with proofs of some new theorems relating it to classical intersection theory.

MSC 2000: 14A15; 14C17; 14C05

Keywords: Tropical Geometry; Tropical varieties; Intersection theory; Toric varieties; Degenerations

O monumento é bem moderno

Caetano Veloso [41]

1. Introduction

Tropical Geometry is an exciting new field of mathematics arising out of computer science. In the mathematical realm, it has been studied by Mikhalkin [24], Speyer [33], the Sturmfels school [30], Itenberg et al. [18], Gathmann and Markwig [15], and Nishinou and Siebert [27] among many others. It has found applications in the enumeration of curves [23], low-dimensional topology [40], algebraic dynamics [10], and the study of compactifications [17,39].

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0723-0869/$ - see front matter
doi:10.1016/j.exmath.2008.04.003

This paper is an introduction to tropical geometry from the point of view of degenerations of subvarieties of a toric variety. In this respect, its approach is close to that of the Sturmfels school.

In the first part of the paper, we use not-necessarily-normal projective toric varieties to introduce standard notions such as degenerations, the Gröbner and fiber fans, and tropical varieties. In the second part of the paper, we give a foundational account of tropical intersection theory. We define tropical intersection numbers, and show that tropical intersection theory computes classical intersection numbers under certain hypotheses, use tropical intersection theory to get data on deformation of subvarieties, and associate a tropical cycle to subvarieties. The two parts can be read independently.

We will express tropical geometry in the language of projective not-necessarily-normal toric schemes over a valuation ring (see [16, Chapter 5], for such toric varieties over fields). These toric schemes give toric degenerations. There are other constructions of toric degenerations analogous to different constructions of toric varieties. Analogous to the fan construction as in [11] is the approach of Speyer [33]. See also the paper of Nishinou and Siebert [27]. In [32], Speyer introduced a construction of toric degenerations paired with a map to projective space. The construction we use here has the advantage of being very immediate at the expense of some loss of generality by mandating projectivity and the loss of computability versus more constructive methods.

We have chosen in this paper to approach the material from the point of view of algebraic geometry and had to neglect the very beautiful combinatorial nature of this theory. We would like to suggest that the reader takes a look at [30] for a more down-to-earth introduction to tropical geometry. We also point out a number of references that are more combinatorial in nature and which relate to our approach. There is the wonderful book of Gelfand et al. [16], which gives a combinatorial description of the secondary polytope among many other beautiful results, the paper of Billera and Sturmfels on fiber polytopes [4] (see also the lovely book of Ziegler [42]), the book of Sturmfels on Convex Polytopes and Gröbner Bases [36] as well as the papers [20,35].

We should mention that since this paper first appeared in preprint form, there has emerged a synthetic approach to tropical intersection theory. The intersection theory of tropical fans was established by Gathmann et al. [14] and was extended to general tropical varieties in $\mathbb{R}^n$ by Allermann and Rau [1].

Many of the results from the first part of this paper are rephrased from Speyer's dissertation [33] and the general outlook is implicit in the work of Tevelev [39], which introduced the interplay between toric degenerations and tropical compactifications. Please see [8] for an explanation of the relationship between such work. We hope this piece will be helpful to other researchers.

2. Conventions

Let $\mathscr{R}$ be a ring with a valuation contained in a subgroup G of $(\mathbb{R}, +)$,

$$v : \mathscr{R}\backslash\{0\} \to G \subseteq \mathbb{R}.$$

Let $\mathbb{K}$ denote the field of fractions of $\mathscr{R}$, $\mathfrak{m}$ the maximal ideal $v^{-1}((0, \infty))$, and $\mathbf{k} = \mathscr{R}/\mathfrak{m}$.

There are two examples that will be most important:

(1) $\mathbb{K} = \mathbb{C}\{\{t\}\} = \bigcup_M \mathbb{C}((t^{1/M}))$, the field of formal Puiseux series, $v : \mathbb{K} \to \mathbb{Q}$, the order map and $\mathbf{k} = \mathbb{C}$.
(2) $\mathbb{K} = \mathbb{C}((t^{1/M}))$, the field of formal Laurent series in $t^{1/M}$, $v : \mathbb{K} \to 1/M\ \mathbb{Z}$, and $\mathbf{k} = \mathbb{C}$.

Note that the first choice of $\mathscr{R}$ has the disadvantage of not being Noetherian. This is not much of a hindrance because any variety defined over $\mathbb{K}$ in the first case can be defined over $\mathbb{K}$ in the second case for some M. This will be enough in practice.

In either case, given $x \in \mathbb{K}$, we may speak of the *leading term* of x. This is the non-zero complex coefficient of the lowest power of t occurring in the power-series expansion of x.

In either of these cases we have an inclusion $\mathbf{k} \hookrightarrow \mathscr{R}$ such that the composition

$$\mathbf{k} \hookrightarrow \mathscr{R} \to \mathscr{R}/\mathfrak{m} = \mathbf{k}$$

is the identity.

Also, for every $u \in G$, we have an element $t^u \in \mathbb{K}$ so that $v(t^u) = u$. These elements have the property that

$$t^{u_1} t^{u_2} = t^{u_1+u_2}.$$

The choice of a map $u \mapsto t^u$ as a section of v is perhaps unnatural. In [29], Payne introduced a formalism of tilted rings, which avoids the need for a section.

For an n-tuple, $w = (w_1, \ldots, w_n) \in G^n$, we may write t^w for $(t^{w_1}, \ldots, t^{w_n}) \in (\mathbb{K}^*)^n$. Similarly, we may write $v : (\mathbb{K}^*)^n \to G^n$ for the product of valuations.

For $g = (g_1, \ldots, g_n) \in (\mathbb{K}^*)^n$, $\chi = (\chi_1, \ldots, \chi_n) \in \mathbb{Z}^n$, we write g^χ for $g_1^{\chi_1} \ldots g_n^{\chi_n} \in \mathbb{K}^*$.

3. Polyhedral geometry

Here we review some notions from polyhedral geometry. Please see [42] for more details.

Let $\mathscr{A} \subset \mathbb{R}^n$ be a set of points. Let $P = \mathrm{Conv}(\mathscr{A})$ be their convex hull. For $v \in (\mathbb{R}^n)^\vee$, the face P_v of P is the set of points $x \in P$ that minimize the function $\langle x, v\rangle$. Let $\Gamma_v = \mathscr{A} \cap P_v$. The cone

$$C_{\Gamma_v} = \left\{ w \in (\mathbb{R}^n)^\vee \;\middle|\; \begin{array}{ll} \langle \chi_i, w\rangle = \langle \chi_j, w\rangle & \text{for } \chi_i, \chi_j \in \Gamma_v \\ \langle \chi, w\rangle < \langle \chi', w\rangle & \text{for } \chi \in \Gamma_v, \chi' \notin \Gamma_v \end{array} \right\}$$

is the normal cone to the face P_v. Observe that v is in the relative interior of C_{Γ_v}. The correspondence between P_v and C_{Γ_v} is inclusion reversing. The C_Γ's form a fan, $N(P)$, called the (inward) normal fan of P.

Two polytopes are said to be normally equivalent if they have the same normal fan.

A polyhedron in $\mathbb{R}^n$ is said to be *integral* with respect to a full-rank lattice $\Lambda \subset (\mathbb{R}^n)^\vee$ if it is the intersection of half-spaces defined by equations of the form $\{x | \langle x, w\rangle \geqslant a\}$ for $w \in \Lambda$, $a \in \mathbb{R}$. We will usually not note the lattice when it is understood.

Definition 3.1. A *polyhedral complex* in $\mathbb{R}^n$ is a finite collection $\mathscr{C}$ of polyhedra in $\mathbb{R}^n$ that contains the faces of any one of its members, and such that any non-empty intersection of two of its members is a common face.

A polyhedral complex is said to be *integral* if all of its members are integral polyhedra. The support $|\mathscr{C}|$ of a polyhedral complex $\mathscr{C}$ is the set-wise union of its polyhedra. We say that a polyhedral complex $\mathscr{C}$ is supported on a polyhedral complex $\mathscr{D}$ if $|\mathscr{C}| \subseteq |\mathscr{D}|$.

Definition 3.2. Given two integral polyhedral complexes, $\mathscr{C}$,$\mathscr{D}$ in $\mathbb{R}^n$, we say $\mathscr{C}$ is a refinement of $\mathscr{D}$ if every polyhedron in $\mathscr{D}$ is a union of polyhedra in $\mathscr{C}$.

It is well-known that for convex polytopes P and Q with normal fans $N(P)$, $N(Q)$ and $N(P)$ is a refinement of $N(Q)$ if and only λQ is a Minkowski summand of P for some $\lambda \in \mathbb{R}_{>0}$. See [3, Proposition 1.2].

Given a polyhedron P in a complex $\mathscr{C}$, we may construct a fan $\mathscr{F}$ called the *star* of P. Pick a point w in the relative interior of P. Let $\mathscr{D}$ be the set of all polyhedra in $\mathscr{C}$ containing P as a face. For every $Q \in \mathscr{D}$, let C_Q be the cone

$$C_Q = \{v \in \mathbb{R}^n | w + \varepsilon v \in Q \text{ for some } \varepsilon > 0\}.$$

These C_Q's give a fan $\mathscr{F}$. If P is a maximal polyhedron in $\mathscr{C}$, then its star is its affine span. Please note that this usage of star is non-standard.

Definition 3.3. Given n polytopes, $P_1, \ldots, P_n \subset \mathbb{R}^n$, their *mixed volume* is the coefficient of $\lambda_1\lambda_2, \ldots, \lambda_n$ in $\mathrm{Vol}(\lambda_1 P_1 + \cdots + \lambda_n P_n)$, which is a homogeneous polynomial of degree n in $\lambda_1, \ldots, \lambda_n$.

4. Toric schemes

4.1. Toric schemes over Spec $\mathscr{R}$

We take the point of view of [31] and use the language of toric schemes over Spec $\mathscr{R}$. We use the not-necessarily-normal projective toric varieties of [16].

For $T = (\mathbb{K}^*)^n$ a $\mathbb{K}$-torus, let $T^\wedge = \mathrm{Hom}(T, \mathbb{K}^*)$ be the character lattice and $T^\vee = \mathrm{Hom}(\mathbb{K}^*, T)$ be the one-parameter subgroup lattice. Let $T^\wedge_\mathbb{R} = \mathbb{R} \otimes T^\wedge$, $T^\vee_\mathbb{R} = \mathbb{R} \otimes T^\vee$, and $T^\vee_G = G \otimes T^\vee$.

A homomorphism of tori $T \to U$ induces homomorphisms $T^\vee \to U^\vee$ and $U^\wedge \to T^\wedge$.

Definition 4.1. Let $T = (\mathbb{K}^*)^n \hookrightarrow (\mathbb{K}^*)^{N+1}/(\mathbb{K}^*) \hookrightarrow PGl_{N+1}(\mathbb{K})$ be a composition of homomorphisms of groups where $(\mathbb{K}^*)^{N+1}/(\mathbb{K}^*)$ denotes the quotient by the diagonal subgroup and the last homomorphism is the diagonal inclusion. For $y \in \mathbb{P}^N_\mathbb{K}$, let T_y denote the stabilizer of y in T. The toric variety associated to (T, y) is the closure

$$Y = \overline{(T/T_y)y}.$$

Y lies in the fiber over the generic point in $\mathbb{P}^N_\mathscr{R} \to \mathrm{Spec}\,\mathscr{R}$. Let the toric scheme $\mathscr{Y}$ be the closure of Y in $\mathbb{P}^N_\mathscr{R}$, and let $Y_0 = \mathscr{Y} \times_{\mathrm{Spec}\,\mathscr{R}} \mathrm{Spec}\,\mathbf{k}$ be the special fiber.

Definition 4.2. If $y \in \mathbb{P}^n_\mathbf{k} \subset \mathbb{P}^n_\mathbb{K}$ for $\mathbf{k} \subset \mathbb{K}$ then the toric scheme is said to be *defined over* $\mathbf{k}$. Alternatively, it is obtained by base-change from a toric variety defined over $\mathbf{k}$ by the map $\mathrm{Spec}\,\mathbb{K} \to \mathrm{Spec}\,\mathbf{k}$ induced by the inclusion.

Example 4.3. Let $T = (\mathbb{K}^*)^2 \to (\mathbb{K}^*)^4/(\mathbb{K}^*)$ be the inclusion given by

$$(x_1, x_2) \mapsto (1, x_1, x_2, x_1x_2).$$

If $y = [1 : 1 : 1 : 1] \in \mathbb{P}^3_{\mathbb{K}}$ then

$$T \cdot y = \{[1 : x_1 : x_2 : x_1x_2] \mid x_1, x_2 \in \mathbb{K}^*\}.$$

The closure of the above is $\mathbb{P}^1 \times \mathbb{P}^1$ under the Segre embedding. This is defined over $\mathbf{k}$.

Definition 4.4. There is a natural map from $(\mathbb{K}^*)^n$ to Y given by

$$\begin{aligned}(\mathbb{K}^*)^n &\longrightarrow Y,\\ g &\longmapsto g \cdot y.\end{aligned}$$

The image of the map is called *the big open torus*. If the map is an open immersion, we say our toric variety is *immersive*.

Now, we explain a method of defining toric schemes. Let $\mathscr{A} = \{\chi_1, \ldots, \chi_{N+1}\} \subset T^\wedge = \mathbb{Z}^n$ be a finite set. Let $a : \mathscr{A} \mapsto G$ be a function called a *height function*. Let $\mathbf{y} = (y_1, \ldots, y_{N+1}) \in (\mathbb{K}^*)^{N+1}$ be an element satisfying

$$v(y_i) = a(\chi_i).$$

The choice of $\mathscr{A}$ induces a homomorphism of groups

$$\begin{aligned}T = (\mathbb{K}^*)^n &\to (\mathbb{K}^*)^{N+1},\\ g = (g_1, \ldots, g_n) &\mapsto (g^{\chi_1}, \ldots, g^{\chi_{N+1}}).\end{aligned}$$

We may consider the map as a homomorphism $T \to (\mathbb{K}^*)^{N+1}/(\mathbb{K}^*)$, where the quotient is by the diagonal subgroup. Therefore, if $\mathbf{y} \in (\mathbb{K}^*)^{N+1}$,

$$g \cdot y = (g^{\chi_1} y_1, \ldots, g^{\chi_{N+1}} y_{N+1}).$$

One may ask how the toric variety depends on the choice of y. Let $\mathbf{y}, \mathbf{y}' \in (\mathbb{K}^*)^{N+1}$ satisfy

$$v(y_i) = v(y_i') = a(\chi_i).$$

Then $\mathbf{y}, \mathbf{y}'$ are related by multiplication by an element $g \in (\mathbb{K}^*)^{N+1}$ with $v(g) = 0$. This element lifts to an element of $(\mathbb{G}_m)^{N+1}_{\mathscr{R}}$. Therefore, the two choices of $\mathscr{Y}_{\mathscr{A},a}$ are related by an action of the diagonal torus in $\mathbb{P}^N_{\mathscr{R}}$. As a consequence, the special fibers are related by an action of the diagonal torus in $\mathbb{P}^N_{\mathbf{k}}$.

Let $\mathscr{Y}_{\mathscr{A},a}$ be the toric scheme associated to T and y. Note that if the integral affine span of $\mathscr{A}$ is $\mathbb{Z}^n$ then $Y_{\mathscr{A},a}$ is immersive.

It is a theorem that the normalization of Y is the toric variety associated to the normal fan of the polytope $\mathrm{Conv}(\mathscr{A})$. See [7] for details.

Definition 4.5. The *induced subdivision* of $\mathrm{Conv}(\mathscr{A})$ is given as follows. Let the *upper hull* of a be

$$\mathrm{UH} = \mathrm{Conv}(\{(\chi, b) | \chi \in \mathscr{A},\ b \geqslant a(\chi)\}).$$

The faces of UH project down to give a subdivision of $\mathrm{Conv}(\mathscr{A})$.

$\mathrm{Conv}(\mathscr{A})$ is called the *weight polytope* of Y, while the induced subdivision is called the *weight subdivision* of $\mathscr{Y}$.

Example 4.6. Let $\mathscr{A} = \{(0,0),(1,0),(0,1),(1,1)\}$ be the vertices of a lattice square. Let a be given by

$$a(0,0)=0,\ a(1,0)=0,\ a(0,1)=0,\ a(1,1)=1.$$

Choose $\mathbf{y}=(1,1,1,t^1)$. This induces the inclusion $T \hookrightarrow (\mathbb{K}^*)^4/(\mathbb{K}^*)$ given by

$$(x_1,x_2)\mapsto (x_1^0x_2^0, x_1^1x_2^0, x_1^0x_2^1, x_1^1x_1^1)=(1,x_1,x_2,x_1x_2)$$

as in Example 4.3. Therefore $\mathscr{Y}$ is the closure of the image of

$$(x_1,x_2)\mapsto [1:x_1:x_2:tx_1x_2].$$

The fiber over Spec $\mathbb{K}$ is isomorphic to the closure of

$$(x_1,x_2)\mapsto [1:x_1:x_2:x_1x_2],$$

which is $\mathbb{P}^1_{\mathbb{K}}\times\mathbb{P}^1_{\mathbb{K}}$ under the Segre embedding.

The special fiber can be seen as follows: taking the limit of (x_1,x_2) as $t\mapsto 0$, we get $[1:x_1:x_2:0]$ which is $\mathbb{P}^2$; taking the limit of $(t^{-1}x_1, t^{-1}x_2)$ as $t\mapsto 0$, we get $[0:x_1:x_2:x_1x_2]$, which is another $\mathbb{P}^2$. One sees that the special fiber is two copies of $\mathbb{P}^2$ joined along $\mathbb{P}^1$. We will show that this case is indicative of a general phenomenon in Lemma 4.19.

4.2. Recovering the weight subdivision

There is a way of working backwards from (T,y) to $\mathscr{A}$ and a subdivision of $\mathrm{Conv}(\mathscr{A})$.

Definition 4.7. Let V be a $\mathbb{K}$-vector space. A $\mathbf{k}$-weight decomposition is a vector space isomorphism defined over $\mathbf{k}\subset\mathbb{K}$

$$V\cong\bigoplus_{\chi\in\mathbb{Z}^n}V_\chi,$$

where H acts on V_χ with character χ.

Lemma 4.8. *Any $\mathbb{K}$-vector space V on which H acts linearly has a $\mathbf{k}$-weight decomposition.*

Proof. See [6, Propositions 8.4 and 8.11]. □

Lift $y\in\mathbb{P}^N_{\mathbb{K}}$ to $\mathbf{y}\in\mathbb{K}^{N+1}$. Write $\mathbf{y}=\sum_\chi \mathbf{y}_\chi$. Let $\mathscr{A}=\{\chi\in\mathbb{Z}^n | y_\chi\neq 0\}$. Then $\mathrm{Conv}(\mathscr{A})$ is called the weight polytope of Y. If $\dim V_\chi=1$, set $a_\chi=v(\mathbf{v}_\chi)$. Otherwise, write $\mathbf{v}_\chi=\mathbf{v}_1+\cdots+\mathbf{v}_n$ where $\mathbf{v}_i$ are vectors in a one-dimensional subspace on which H acts, and set $a_\chi=\min(v(\mathbf{v}_i))$. Take the subdivision of $\mathrm{Conv}(\mathscr{A})$ induced by a_χ, which is independent of the lift y.

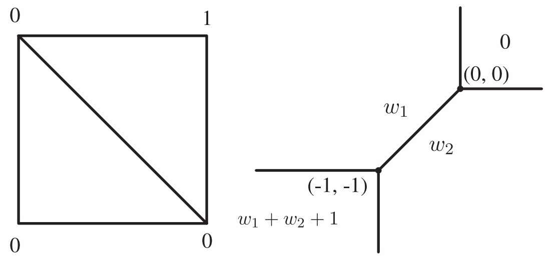

Fig. 1. The subdivision and its dual complex.

4.3. Dual complex

Consider the pairing

$$T^{\wedge}_{\mathbb{R}} \otimes T^{\vee}_{\mathbb{R}} \to \mathbb{R}$$

and the piecewise-linear function

$$F : T^{\vee}_{\mathbb{R}} \to \mathbb{R}$$

defined by

$$F(w) = \min_{\chi \in \mathscr{A}} (\langle \chi, w\rangle + a_\chi).$$

The domains of linearity of F give a polyhedral complex structure on $T^{\vee}_{\mathbb{R}}$. For $\Gamma \subset \mathscr{A}$, let

$$C_\Gamma = \left\{ w \in (\mathbb{R}^n)^\vee \;\middle|\; \begin{array}{ll} \langle \chi_i, w\rangle + a_{\chi_i} = \langle \chi_j, w\rangle + a_{\chi_j} & \text{for } \chi_i, \chi_j \in \Gamma \\ \langle \chi, w\rangle + a_\chi < \langle \chi', w\rangle + a_{\chi'} & \text{for } \chi \in \Gamma, \chi' \notin \Gamma \end{array} \right\}$$

If C_Γ is not empty, then Γ are points of $\mathscr{A}$ in a face of the weight subdivision. The C_Γ's fit together to form an integral polyhedral complex, the *dual complex*, which is dual to the weight subdivision. Note that if $a_\chi = 0$ for $\chi \in \mathscr{A}$, the weight subdivision becomes the weight polytope and the dual subdivision becomes the normal fan.

Example 4.9. Fig. 1 shows the weight subdivision and dual complex for Example 4.6. Here,

$$\begin{aligned} F(w) &= \min_{\chi \in \mathscr{A}} (\langle \chi, w\rangle + a_\chi) \\ &= \min(0, w_1, w_2, w_1 + w_2 + 1). \end{aligned}$$

The values of F on the dual complex are noted in the figure.

4.4. One-parameter families of points

Let us review the notion of specialization. For $y \in \mathbb{P}^N_{\mathbb{K}}$, we may take $\overline{y} \in \mathbb{P}^N_{\mathscr{R}}$, considered as a scheme over Spec $\mathscr{R}$. The *specialization* of y is

$$\hat{y} = \overline{y} \times_{\operatorname{Spec} \mathscr{R}} \operatorname{Spec} \mathbf{k} \in \mathbb{P}^N_{\mathbf{k}}.$$

We can compute the specialization by hand. Lift y to $\mathbf{y} \in \mathbb{K}^{N+1}\backslash\{0\}$ such that $\min(v(y_i))=0$. If $y_i \neq 0$, write $y_i = c_1 t^{b_i} + \ldots$, where the ellipsis denotes higher order terms. Let

$$S = \{i | b_i = 0\}.$$

Then $\hat{\mathbf{y}}$ satisfies

$$\hat{y}_i = \begin{cases} c_i & \text{if } i \in S \\ 0 & \text{else.} \end{cases}$$

Definition 4.10. Let $\mathscr{Y}$ be a toric scheme over $\mathscr{R}$. Let y be a point in Y. Given $g \in (\mathbb{K}^*)^n$, the family associated to (g, y) is the scheme over Spec $\mathscr{R}$ given by the closure of $g \cdot y$.

Definition 4.11. The limit of (g, y) is the point in Y_0 given by

$$\overline{g \cdot y} \times_{\mathrm{Spec}\, \mathscr{R}} \mathrm{Spec}\, \mathbf{k}.$$

Now, observe that

$$v((g \cdot y)_i) = \langle \chi_i, v(g) \rangle + a_{\chi_i},$$

where $v(y_i) = a_{\chi_i}$. Therefore, when we base-change to Spec $\mathbf{k}$, the only components of $g \cdot y_i$ that stay non-zero are the ones on which $\langle \chi_i, v(g) \rangle + a_{\chi_i}$ is minimized. Consequently, if $v(g) \in C_\Gamma$ for a cell Γ of the weight subdivision, and $\hat{y}$ is the limit of (g, y), then $\hat{y}_i \neq 0$ if and only if $\chi_i \in \Gamma$.

4.5. One-parameter families of subschemes

We will also consider degenerations of subschemes X of Y.

Definition 4.12. Let $w \in G^n$ and $g = t^w$. Consider the subscheme of $\mathscr{Y}$ given by $\overline{g \cdot X}$, the closure of $g \cdot X$. Define the *initial degeneration* of X to be the subscheme of Y_0 given by

$$\mathrm{in}_w(X) = \overline{g \cdot X} \times_{\mathrm{Spec}\, \mathscr{R}} \mathrm{Spec}\, \mathbf{k}.$$

Example 4.13. This definition specializes to the usual definition of the initial form of a polynomial. Let

$$f = x_1^2 x_2 + 7x_1x_2x_3 + 4x_3^3 \in \mathbb{K}[x_1, x_2, x_3],$$

and set $w = (3, 4)$. Let $X = V(f) \subset Y = \mathbb{P}^2_{\mathbb{K}}$. Then $t^w V(f)$, a subvariety of $\mathbb{P}^2_{\mathbb{K}}$, is $V(h)$ for

$$\begin{aligned} h &= (t^{-3}x_1)^2(t^{-4}x_2) + 7(t^{-3}x_1)(t^{-4}x_2)(x_3) + 4(x_3)^3 \\ &= t^{-10}x_1^2x_2 + 7t^{-7}x_1x_2x_3 + 4x_3^3 \\ &= t^{-10}(x_1^2x_2 + 7t^3x_1x_2x_3 + 4t^{10}x_3^3). \end{aligned}$$

Therefore,

$$\mathrm{in}_w(V(f)) = \overline{t^w \cdot V(f)} \times_{\mathrm{Spec}\,\mathscr{R}} \mathrm{Spec}\,\mathbf{k}$$

is cut out by

$$\mathrm{in}_w(f) = x_1^2 x_2.$$

Now that if $X = x$ is a point, then $\mathrm{in}_w(X) = \overline{t^w \cdot x} \times_{\mathrm{Spec}\,\mathscr{R}} \mathrm{Spec}\,\mathbf{k}$.

Every point of $\mathrm{in}_w(X)$ occurs as a limit of the form $\overline{g \cdot x} \times_{\mathrm{Spec}\,\mathscr{R}} \mathrm{Spec}\,\mathbf{k}$ for $x \in X$. This is the content of the *tropical lifting lemma*. This lemma was first announced without proof in [37]. A proposed proof was given in [34] but has been found to be incomplete. A proof using affinoid algebras was given by Draisma [9]. Jensen et al. provided an algorithm that finds a tropical lift in [19]. This algorithm uses some ideas from our proof and their paper is recommended as an exposition of our proof in terms of commutative algebra. In [29], by applying a projection argument to reduce to the hypersurface case, Payne gave a stronger version of tropical lifting that works over more general fields.

We first review the concept of relative dimension from [12, Chapter 20].

Definition 4.14. Let $p : Z \to S$ be a scheme over a regular base scheme S. For V, a closed integral subscheme of Z, let $T = \overline{p(V)}$. The *relative dimension* of V is

$$\mathrm{r}\,\dim\,\mathscr{V} = \mathrm{tr.\,deg}\,.(R(V)/R(T)) - \mathrm{codim}(T, S).$$

We will apply this definition for $T = \mathrm{Spec}\,\mathbb{C}[[t^{1/M}]]$. Note that a point in the special fiber is of relative dimension -1.

Lemma 4.15 (*Tropical lifting lemma*). *Let* $\mathbb{K} = \mathbb{C}\{\{t\}\}$. *If* $\tilde{x} \in \mathrm{in}_w(X)$ *then there exists* $x \in X$ *with*

$$\mathrm{in}_w(x) \equiv \overline{t^w \cdot x} \times_{\mathrm{Spec}\,\mathscr{R}} \mathrm{Spec}\,\mathbf{k} = \tilde{x}.$$

Proof. We treat X as a subscheme of $\mathbb{P}^N_{\mathbb{K}}$. If $\dim\, X = 0$, then the support of $t^w X$ is a union of closed $\mathbb{K}$-points. One such point specializes to $\tilde{x}$. The corresponding component has initial deformation supported on $\tilde{x}$ and gives the desired point in X. Therefore, we may suppose $\dim\, X = n > 0$.

Pick M sufficiently large so that X is defined over $\mathbb{F} = \mathbb{C}((t^{1/M}))$. Let $\mathscr{Q} = \mathbb{C}[[t^{1/M}]]$.

By replacing X by $t^w X$ we may suppose $w = 0$. Let $\overline{X}$ be the closure of X in $\mathscr{Y}$. Note that $\overline{X}$ is flat over $\mathrm{Spec}\,\mathscr{Q}$.

Let W_0 be a codimension n subvariety of $Y_0 \subset \mathbb{P}^N_k$ such that W_0 intersects

$$X_0 = \overline{X} \times_{\mathrm{Spec}\,\mathscr{Q}} \mathrm{Spec}\,\mathbf{k}$$

in a zero-dimensional subscheme containing $\tilde{x}$. Extend W_0 to a flat integral scheme $\mathscr{W} \to \mathrm{Spec}\,\mathscr{Q}$ so that $\mathscr{W} \times_{\mathrm{Spec}\,\mathscr{Q}} \mathrm{Spec}\,\mathbf{k} = W_0$ (for example, we may set $\mathscr{W} = W_0 \times_{\mathrm{Spec}\,\mathbf{k}} \mathrm{Spec}\,\mathscr{Q}$). Then, $\overline{X} \times_{\mathscr{Y}} \mathscr{W}$ is a scheme, all of whose components have non-negative relative dimension over $\mathrm{Spec}\,\mathscr{Q}$. The following equality holds for underlying sets:

$$(\overline{X} \times_{\mathscr{Y}} \mathscr{W}) \times_{\mathrm{Spec}\,\mathscr{Q}} \mathrm{Spec}\,\mathbf{k} = X_0 \times_{Y_0} W_0.$$

Since the scheme on the right is zero-dimensional, there are no components of $\overline{X}\times_{\mathscr{Y}}\mathscr{W}$ contained in the special fiber. Therefore, the induced reduced structure on $\overline{X}\times_{\mathscr{Y}}\mathscr{W}$ is flat, has relative dimension zero, and has a component of its limit supported on $\tilde{x}$. Let $W = \mathscr{W}\times_{\mathrm{Spec}\,\mathscr{Q}}\mathrm{Spec}\ \mathbb{F}$. By uniqueness of flat limits, the closure of the induced reduced structure on $X\times_Y W$ in $\mathscr{Y}$ is the induced reduced structure on $\overline{X}\times_{\mathscr{Y}}\mathscr{W}$.

Therefore, we may apply the zero-dimensional case to the induced reduced structure on $X\times_Y W$. $\square$

We will find the following corollary useful.

Corollary 4.16. *Under the hypotheses of the previous lemma and the equality of underlying sets $X = \overline{X\cap(\mathbb{K}^*)^n}$, we may suppose $x \in X\cap(\mathbb{K}^*)^n$.*

Proof. Produce $x \in X$ as above. If $x \in X\cap(\mathbb{K}^*)^n$ then we are done. Otherwise, there is a morphism

$$f : \mathrm{Spec}\ \mathbb{K}[[s]] \to X,$$

so that the generic point is sent to $X\cap(\mathbb{K}^*)^n$ while the closed point is sent to x. This morphism is defined over some $\mathbb{C}((t^{1/M}))$ and can be given as a base-change from

$$f : \mathrm{Spec}\ \mathbb{C}[[t^{1/M}]][[s]] \to X,$$

where we view X as defined over $\mathbb{C}[[t^{1/M}]]$. Therefore, we may extend the morphism to $f : \mathrm{Spec}\ \mathbb{C}[[t^{1/M}]][[s^{1/M}]] \to X$. Consider the diagonal morphism

$$i : \mathrm{Spec}\ \mathbb{C}[[u^{1/M}]] \to \mathrm{Spec}\ \mathbb{C}[[t^{1/M}]][[s^{1/M}]]$$

induced by

$$t^{1/M} \mapsto u^{1/M},\ s^{1/M} \mapsto u^{1/M}.$$

By restricting the composition $f\circ i$ to the generic point, Spec $\mathbb{C}((u^{1/M}))$, we find the desired $\mathbb{K}$-point. $\square$

4.6. Structure of $\mathscr{Y}_{\mathscr{A},a}$

$\mathscr{Y}_{\mathscr{A},a}$ has well-understood fibers over the generic and special point.

Definition 4.17. For Γ, a face of the weight polytope, let $Y^0(\Gamma) \subset Y$ be the set of all points $y \in Y \subseteq \mathbb{P}^N_{\mathbb{K}}$ so that their lifts $\mathbf{y} \in (\mathbb{K})^{N+1}\backslash\{0\}$ satisfy

$$\mathbf{y}_i \neq 0 \quad \text{if and only if } \chi_i \in \Gamma.$$

Definition 4.18. For Γ, a cell of the weight subdivision, let $Y_0^0(\Gamma) \subset Y_0 \subset \mathbb{P}^N_{\mathbf{k}}$ be the set of all points $y \in Y_0 \subseteq \mathbb{P}^N_{\mathbf{k}}$ so that their lifts $\mathbf{y} \in (\mathbf{k})^{N+1}\backslash\{0\}$ satisfy

$$\mathbf{y_i} \neq 0 \quad \text{if and only if } \chi_i \in \Gamma.$$

Proposition 4.19.

(1) *$Y = \mathscr{Y}_{\mathscr{A},a} \times_{\operatorname{Spec}\mathscr{R}} \operatorname{Spec}\mathbb{K}$ is the toric variety associated to $\mathscr{A}$. The non-empty faces of the weight polytope are in inclusion-preserving bijective correspondence with its torus orbits given by $\Gamma \mapsto Y^0(\Gamma)$.*
(2) *The scheme $Y_0 = \mathscr{Y}_{\mathscr{A},a} \times_{\operatorname{Spec}\mathscr{R}} \operatorname{Spec}\mathbf{k}$ is supported on the union of toric varieties associated to the top-dimensional cells of the weight subdivision such that the non-empty cells of the weight subdivision are in inclusion-preserving bijective correspondence with its torus orbits given by $\Gamma \mapsto Y_0^0(\Gamma)$.*

Proof. (1) is Proposition 1.9 of Chapter 5 of [16]. We give the proof of (2) which is directly analogous. Elements of $\mathscr{Y}_{\mathscr{A},a} \times_{\operatorname{Spec}\mathscr{R}} \operatorname{Spec}\mathbf{k}$ are of the form

$$\overline{g \cdot y} \times_{\operatorname{Spec}\mathscr{R}} \operatorname{Spec}\mathbf{k}$$

by Lemma 4.16. If $v(g) \in C_\Gamma$, the cell of the dual complex corresponding to Γ, then the limit $\overline{g \cdot y} \times_{\operatorname{Spec}\mathscr{R}} \operatorname{Spec}\mathbf{k}$ is in the orbit $Y_0^0(\Gamma)$.

Similarly if $w \in C_\Gamma$, by varying g with $v(g) = w$, we may make $\overline{g \cdot y} \times_{\operatorname{Spec}\mathscr{R}} \operatorname{Spec}\mathbf{k}$ be any point of $Y_0^0(\Gamma)$. □

Part (2) of the above lemma is simply not true at the level of scheme structure. As a counterexample, take $\mathscr{A} = \{0, 1, 2\}$, $a(0) = 0$, $a(1) = 1$, $a(2) = 0$. Then Y_0 is a double-line in $\mathbb{P}^2$. The corresponding subdivision is the single cell $[0, 2]$ whose toric variety is the reduced induced structure on Y_0. The construction of toric degenerations by fans as in [33] is better behaved in this respect.

In the case of Example 4.6, we see that Y_0 consists of two $\mathbb{P}^2$'s, five $\mathbb{P}^1$'s and four fixed-points.

It is instructive to phrase the above theorem in the language of the dual complex. Given two elements $g, g' \in (\mathbb{K}^*)^n$ with $v(g) = v(g')$, the limits of (g, y) and (g, y') are related by the action of an element of $(\mathbf{k}^*)^n$ and so lie in the same open torus orbit. Therefore, we may define an equivalence relation on G^n. Two elements $w, w' \in T_G^\vee$ are equivalent, written $w \sim_y w'$ if for $g, g' \in G$ satisfying $w = v(g)$ and $w' = v(g')$, the limits of (g, y) and (g', y) lie in the same open torus orbit.

Proposition 4.20. *$w \sim_y w'$ if and only if w and w' lie in the same cell in the dual complex associated to the toric scheme $\mathscr{Y}_{\mathscr{A},a}$.*

4.7. Invariant limits

The open orbits $Y^0(\Gamma)$ and $Y_0^0(\Gamma)$ are fixed point-wise by sub-tori in T.

Lemma 4.21. *Let Γ be a face of the weight polytope* (*resp. cell of the weight subdivision*). *Let $w \in C_\Gamma$ and $H \subset T$ be the sub-torus with $H_\mathbb{R}^\vee = \operatorname{Span}(C_\Gamma - w)$. Let $z \in Y^0(\Gamma)$* (*resp. $Y_0^0(\Gamma)$*)*. Then the maximal sub-torus fixing z is H.*

Proof. We give the proof for $Y_0^0(\Gamma)$. The proof for $Y_0(\Gamma)$ is similar.

Let $z \in Y_0^0(\Gamma)$. Lift z to $\mathbf{z} \in \mathbf{k}^{N+1} \backslash \{0\}$. Every $g \in H$ satisfies $g^{\chi_i} = g^{\chi_j}$ for $\chi_i, \chi_j \in \Gamma$. Let $g' = g^{\chi} \in \mathbf{k}^*$ for $\chi \in \Gamma$. Since $z_i \neq 0$ if and only if $\chi_i \in \Gamma$,

$$g \cdot \mathbf{z} = g'\mathbf{z},$$

which is another lift of z.

If $u \in T^\vee \backslash H^\vee$, there exists $\chi_i, \chi_j \in \Gamma$ such that

$$\langle \chi_i, u \rangle \neq \langle \chi_j, u \rangle.$$

It follows is that z is not fixed by the one-parameter subgroup corresponding to u. □

We may rephrase the above lemma.

Lemma 4.22. *Suppose $g \in T$ satisfies $v(g) \in C_\Gamma$. Then the limit of (g, y) in Y_0 is invariant under the torus H given by $H_{\mathbb{R}}^\vee = \mathrm{Span}(C_\Gamma - w)$.*

Proof. The limit of (g, y) lies in $Y_0^0(\Gamma)$. □

Suppose $v(g)$ lies in C_Γ, the cell of the dual complex dual to a cell Γ in the weight subdivision. We may make use of the map $\mathrm{Spec}\,\mathscr{R} \to \mathrm{Spec}\,\mathbf{k}$ to base-change the limit

$$\overline{g \cdot y} \times_{\mathrm{Spec}\,\mathscr{R}} \mathrm{Spec}\,\mathbf{k}$$

to

$$\hat{y} = (\overline{g \cdot y} \times_{\mathrm{Spec}\,\mathscr{R}} \mathrm{Spec}\,\mathbf{k}) \times_{\mathrm{Spec}\,\mathbf{k}} \mathrm{Spec}\,\mathscr{R}.$$

This just means that we should consider a limit point's coordinates as points in $\mathbb{K}$ rather than in $\mathbf{k}$ and take its closure.

Lemma 4.23. *The weight polytope of the toric scheme $\widehat{\mathscr{Y}} = \overline{(\mathbb{K}^*)^n \cdot \hat{y}}$ is $\mathrm{Conv}(\Gamma)$.*

Proof. Lift $\hat{y}$ to $\hat{\mathbf{y}} \in \mathbb{K}^{N+1} \backslash \{0\}$. The weights with which T acts on $\hat{\mathbf{y}}$ are $\chi \in \Gamma$. Therefore the weight polytope in $\mathrm{Conv}(\Gamma)$. □

The dual complex of $\widehat{\mathscr{Y}}$ is the normal fan of $\mathrm{Conv}(\Gamma)$. The normal fan of $\widehat{\mathscr{Y}}$ is the star of C_Γ, the cell of the dual complex dual to Γ.

Lemma 4.24. *Let $\hat{y} = \mathrm{in}_w(X)$. For $u \in T_G^\vee$,*

$$\mathrm{in}_u(\hat{y}) = \mathrm{in}_{w+\varepsilon u}(y)$$

for sufficiently small $\varepsilon > 0$.

Proof. Let $w \in C_\Gamma$ for Γ, a cell of the weight subdivision. Then the weight polytope of $\hat{y}$ is $\mathrm{Conv}(\Gamma)$. Therefore, u is in a cone of the normal fan of Γ dual to some face $\Gamma' \subseteq \Gamma$. It follows that the coordinates of $\mathrm{in}_u(\hat{y})$ in $\mathbb{P}^N$ are non-zero only for $\chi_i \in \Gamma'$ and in that case are equal to the leading terms of the coordinate of $t^w y$. Now, C_Γ is a face of $C_{\Gamma'}$ and we may pick small $\varepsilon > 0$ such that $w + \varepsilon u \in C_{\Gamma'}$. Therefore, $\mathrm{in}_{w+\varepsilon u}(y) = \mathrm{in}_u(\hat{y})$. □

4.8. Naturality of dual complexes

Lemma 4.25. *Given a proper surjective $(\mathbb{K}^*)^n$-equivariant morphism of n-dimensional toric schemes, $f : \mathscr{X} \to \mathscr{Y}$ then the dual complex of $\mathscr{X}$ is a refinement of that of $\mathscr{Y}$. The normal fan to the weight polytope of X is a refinement of that of Y.*

Proof. Let $x \in \mathbb{P}^N_{\mathbb{K}}$ so that $\mathscr{X} = \overline{T \cdot x}$ and $\mathscr{Y} = \overline{T \cdot f(x)}$ for (possibly different) diagonal actions of T on $\mathbb{P}^N_{\mathbb{K}}, \mathbb{P}^{N'}_{\mathbb{K}}$.

Now let C_Γ be a k-dimensional cell in the dual complex of $\mathscr{X}$. We must show that $f^\vee(C_\Gamma)$ is in the relative interior of a cell in the dual complex of $\mathscr{Y}$ of dimension at least k. If $g \in T$ satisfies $v(g) \in C_\Gamma$, then the limit, $\hat{x}$ of (g, x) is invariant under the k-dimensional torus H with $H^\vee_{\mathbb{R}} = \mathrm{Span}(C_\Gamma - v(g))$. Since f is equivariant, $f(\hat{x})$ is the limit of $(g, f(x))$. Furthermore if $v(g) \in C_{\Gamma'}$, a cell in the dual complex of $\mathscr{Y}$ then $f(\hat{x})$ is invariant under an l-dimensional torus H' with $H'^\vee_{\mathbb{R}} = \mathrm{Span}(C_{\Gamma'} - v(g))$. Since $f(\hat{x})$ is also invariant under H, then $l > k$.

To prove the statement for the weight polytope, we may set $\mathscr{X} = X \times \mathrm{Spec}\,\mathbb{K}[[s]]$, $\mathscr{Y} = Y \times \mathrm{Spec}\,\mathbb{K}[[s]]$ where s is an algebraic indeterminate. Consider the valuation $v : \mathbb{K}[[s]] \to \mathbb{Z}$ given by $v(s) = 1$, $v(\mathbb{K}^*) = 0$. Then the weight subdivision of $\mathscr{X}$ and $\mathscr{Y}$ are exactly the weight polytopes of X and Y and the same argument applies. $\square$

4.9. Equivariant inclusions

In this section we consider a projection of integral polytopes $p : P \to Q$, where $P = \mathrm{Conv}(\mathscr{A})$.

Definition 4.26. Given a finite set $\mathscr{A} \subseteq \mathbb{Z}^n$ and a function

$$a : \mathscr{A} \to \mathbb{R},$$

a projection $p : \mathbb{Z}^n \to \mathbb{Z}^m$, let $\mathscr{B} = p(\mathscr{A})$ and define *the image height function*

$$b : \mathscr{B} \to \mathbb{R}$$

by

$$b(\psi) = \min(\{a(\chi) | \chi \in p^{-1}(\psi)\}).$$

The associated subdivision is the *image subdivision*.

Note that the image subdivision is dependent on the height function and not only on the original subdivision. Weight polytopes and weight subdivisions are contravariant.

Lemma 4.27. *Let $i : T \hookrightarrow U$ be an injective homomorphism of tori, so*

$$\overline{T \cdot v} \hookrightarrow \overline{U \cdot v} \hookrightarrow \mathbb{P}^n$$

is a chain of equivariant inclusions. Then the induced projection

$$i^\wedge : U^\wedge \to T^\wedge$$

takes the weight polytope and the weight subdivision of $\overline{U \cdot v}$ *to those of* $\overline{T \cdot v}$.

Proof. The proof is straightforward. □

5. Degenerations

5.1. Moduli spaces

Tropical geometry is, in a certain sense, a method of parameterizing degenerations of subvarieties of a toric variety. There are two useful spaces for parameterizing degenerations, the Chow variety and the Hilbert scheme. Points in these moduli spaces correspond to cycles or to subschemes. This is useful because limits of points in the moduli space correspond to limits of cycles and subschemes. This allows us to apply the machinery developed in the previous section to limits of subvarieties.

Let $Y \subseteq \mathbb{P}^N$ be a projective toric variety whose torus action extends to one on $\mathbb{P}^N$. Recall that k-dimensional algebraic cycles of Y are finite formal sums of k-dimensional subvarieties of Y with integer coefficients. Consider a subvariety $X \subset Y$, with degree d in Y and Hilbert polynomial P. There are two projective parameter spaces that one can construct, $\mathrm{Chow}_d(Y)$ and $\mathrm{Hilb}_P(Y)$ that each have a point corresponding to X. Points in $\mathrm{Chow}_d(Y)$ correspond to certain cycles in Y of degree d. We denote the point (called the *Chow form*) in $\mathrm{Chow}_d(Y)$ corresponding to X by R_X. $\mathrm{Chow}_d(Y)$ is constructed as a closed subscheme of $\mathrm{Chow}_d(\mathbb{P}^N)$, which is a projective scheme. Points in $\mathrm{Hilb}_P(Y)$ correspond to closed subschemes of Y with Hilbert polynomial P. The point $[X]$ in $\mathrm{Hilb}_P(Y)$ corresponding to X is called the *Hilbert point*. Similarly, $\mathrm{Hilb}_P(Y)$ is a closed subscheme of $\mathrm{Hilb}_P(\mathbb{P}^N)$ which is projective.

See [22] for an in-depth construction of both varieties. See also [16] for a discussion of the Chow variety. We will break from the usage in [22] and use Chow to denote the un-normalized Chow variety which is there called Chow′. Note that the Hilbert scheme can be constructed over an arbitrary Noetherian scheme S while there are restrictions on the base-scheme of the Chow variety.

Let us review some useful properties of the Chow varieties and Hilbert schemes.

Property 5.1. *The torus action on Y induces a group action on* Chow_d and Hilb_P, *which extends to an action on the ambient projective space.*

Because the torus T acts on Y, for $g \in T$, $g \cdot X$ is a subvariety of Y of degree d and Hilbert polynomial P. Therefore, $R_{g \cdot X} \in \mathrm{Chow}_d(Y)$ and $[g \cdot X] \in \mathrm{Hilb}_P(Y)$. This induces T-actions on $\mathrm{Chow}_d(Y)$ and $\mathrm{Hilb}_P(Y)$ given by

$$\begin{array}{cc} T \times \mathrm{Chow}_d(Y) \to \mathrm{Chow}_d(Y), & T \times \mathrm{Hilb}_P(Y) \to \mathrm{Hilb}_P(Y), \\ (g, R_X) \mapsto R_{g \cdot X}, & (g, [X]) \mapsto [g \cdot X]. \end{array}$$

Property 5.2. *There is a natural equivariant morphism* $FC : \mathrm{Hilb}_P \to \mathrm{Chow}_d$ (*see* [26, 5.4]) *called the* fundamental class map *that takes a scheme to its underlying cycle.*

A subscheme X of Y has an underlying cycle. Therefore, one may define a map

$$\begin{aligned} FC : \mathrm{Hilb}_P &\to \mathrm{Chow}_d, \\ [X] &\mapsto R_X. \end{aligned}$$

This map is equivariant with respect to the above T-actions.

Property 5.3. *The Hilbert scheme possesses a universal flat family* $\mathrm{Univ}_P \to \mathrm{Hilb}_P$.

This universal family Univ_P is a subscheme of $Y \times \mathrm{Hilb}_P(Y)$. The fiber over the Hilbert point $[X]$ is the subscheme X. In particular, if $\mathrm{Spec}\,\mathbb{K} \to \mathrm{Hilb}_P(Y)$ is the $\mathbb{K}$-point $[X]$ then $\mathrm{Univ}_P \times_{\mathrm{Hilb}_P(Y)} \mathrm{Spec}\,\mathbb{K} = X$.

The Chow variety does not usually have a universal flat family.

Property 5.4. *The Hilbert scheme is natural under base-change. If* $Y \to S$ *is projective then* $\mathrm{Hilb}_P(Y/S)$ *parameterizes* S*-subschemes of* Y *with Hilbert polynomial* P*. If* $Z \to S$ *is a morphism then*

$$\mathrm{Hilb}_P(Y \times_S Z/Z) = \mathrm{Hilb}_P(Y/S) \times_S Z.$$

The Chow variety does not have this property.

The Hilbert scheme with its universal flat family and naturality properties is a much better behaved moduli space. This makes it more useful for our purposes. However, there are very beautiful combinatorial structures associated with the Chow variety. See [16] for details.

Now, we may use the Hilbert scheme to relate deformations of subschemes to limits of the form (g, y). Let X be a subscheme of a toric variety Y. Let $g \in T$ and $w = v(g)$. By uniqueness of flat limits, the $\mathrm{Spec}\,\mathscr{R}$-point $\overline{g \cdot [X]}$ is the Hilbert point of $\overline{g \cdot X}$ in $\mathrm{Hilb}_P(\mathscr{Y})$. Therefore, the specialization of $g \cdot [X]$,

$$\overline{g \cdot [X]} \times_{\mathrm{Spec}\,\mathscr{R}} \mathrm{Spec}\,\mathbf{k} \in \mathrm{Hilb}_P(\mathscr{Y}) \times_{\mathrm{Spec}\,\mathscr{R}} \mathrm{Spec}\,\mathbf{k} = \mathrm{Hilb}_P(Y_0)$$

is the Hilbert point, $[\overline{g \cdot X} \times_{\mathrm{Spec}\,\mathscr{R}} \mathrm{Spec}\,\mathbf{k}]$. We may pull back the universal family by $\mathrm{Spec}\,\mathscr{R} \to \mathrm{Hilb}_P(\mathscr{Y})$ to get a scheme $\mathscr{U}$ over $\mathrm{Spec}\,\mathscr{R}$. Its special fiber is $\overline{g \cdot X} \times_{\mathrm{Spec}\,\mathscr{R}} \mathrm{Spec}\,\mathbf{k}$. If $g = t^w$, then the special fiber is the initial degeneration $\mathrm{in}_w(X)$.

5.2. Associated toric schemes

Let Y be a toric scheme in $\mathbb{P}^N_{\mathbb{K}}$ with a torus T. Let X be a subvariety of Y. We may take the Hilbert point $[X] \in \mathrm{Hilb}_P(Y)$ or the Chow form $R_X \in \mathrm{Chow}_d(Y)$ and consider the two toric schemes, called the Hilbert and Chow images, respectively

$$\mathrm{HI} = \overline{T/T_X \cdot [X]} \subseteq \mathrm{Hilb}_P(Y), \quad \mathrm{CI} = \overline{T/T_X \cdot R_X} \subseteq \mathrm{Chow}_d(Y),$$

where T_X denotes the stabilizer of $[X]$ or R_X.

Definition 5.5. The subdivisions (in $(T/T_X)^\wedge \subseteq T^\wedge$) associated to the Hilbert and Chow images are called the *state subdivision* and the *secondary subdivision*, respectively. The dual polyhedral complexes (in $(T/T_X)^\vee$) are called the *Gröbner complex* and the *Chow complex*. In the case where X and Y are defined over $\mathbf{k}$, these notions become the *state polytope, fiber polytope, the Gröbner fan, and the fiber fan*, respectively.

In the case where X is also a toric subvariety in Y, the name fiber polytope is standard. Otherwise our usage is somewhat non-standard.

Now we may apply Proposition 4.20 to the Gröbner complex.

Proposition 5.6. *Two points w, w' lie in the same cell in the Gröbner complex if and only if* $\mathrm{in}_w(X)$ *and* $\mathrm{in}'_w(X)$ *are related by a $T_{\mathbf{k}}$-action.*

In the case where X is defined over $\mathbf{k}$, this proposition is close to the usual definition of the Gröbner fan. The usual definition, however, is a refinement of our definition. This is because the initial ideals in the standard definition are sensitive to embedded primes associated to the irrelevant ideal. Our definition is not. The definition we give is based on that of [2].

We may also apply Lemma 4.22 to the Gröbner complex.

Lemma 5.7. *If $w \in G^n$ is in the relative interior of a k-dimensional cell of the Gröbner complex of X then the closed subscheme* $\mathrm{in}_w(X)$ *is invariant under a k-dimensional torus.*

Proof. By Lemma 4.22 the Hilbert point of $\mathrm{in}_w(X)$ is invariant under a k-dimensional torus. Therefore, the closed subscheme $\mathrm{in}_w(X)$ is invariant under the same torus. $\square$

Lemma 5.8. *For $u \in T_G^\vee$,*

$$\mathrm{in}_u(\mathrm{in}_w(X)) = \mathrm{in}_{w+\varepsilon u}(X)$$

for $\varepsilon > 0$ sufficiently small.

Proof. This is Lemma 4.24 applied to the Hilbert point $[X]$. $\square$

There is a natural projection $p : T_{\mathbb{R}}^\vee \to (T/T_X)_{\mathbb{R}}^\vee$. We may abuse notation and use the term Gröbner or Chow complex to also denote the appropriate complex's inverse image under p.

Example 5.9. Let Y be a toric variety defined over $\mathbf{k}$ given by a set of exponents $\mathscr{A} \subset \mathbb{Z}^n$. Let X be a hypersurface defined in Y by

$$f(x) \equiv \sum_{\omega \in \mathscr{A}} a_\omega x^\omega = 0,$$

where $a_\omega \in \mathbb{K}$ and x^ω are coordinates on $Y \subset \mathbb{P}^{|\mathscr{A}|-1}$. We may treat $[a_\omega]$ as coordinates on a projective space $(\mathbb{P}^{|\mathscr{A}|-1})^\vee$. The torus T acts on $(\mathbb{P}^{|\mathscr{A}|-1})^\vee$ by

$$\begin{aligned} T \times (\mathbb{P}^{|\mathscr{A}|-1})^\vee &\to (\mathbb{P}^{|\mathscr{A}|-1})^\vee, \\ (g, [a_\omega]) &\mapsto [g^{-\omega} a_\omega]. \end{aligned}$$

Then the equation

$$\sum_{\omega\in\mathscr{A}} a_\omega x^\omega = 0$$

cuts out a universal hypersurface $\mathscr{U} \subset Y \times (\mathbb{P}^{|\mathscr{A}|-1})^\vee$ over $(\mathbb{P}^{|\mathscr{A}|-1})^\vee$. This universal family is flat and therefore defines a T-equivariant morphism $(\mathbb{P}^{|\mathscr{A}|-1})^\vee \to \mathrm{Hilb}_P(Y)$. The image of this morphism contains the Hilbert point of X. Therefore, the Hilbert image, $T \cdot [X]$, is isomorphic to Y but with the opposite torus action. The state polytope, which is the weight polytope of the Hilbert image, is $-\mathrm{Conv}(\mathscr{A})$. The Gröbner fan is the normal fan, $N(-\mathrm{Conv}(\mathscr{A}))$.

For a down-to-earth exposition of this example, see [36, Proposition 2.8].

Example 5.10. Suppose that Y is a toric variety defined over $\mathbf{k}$. Let X be a reduced $\mathbb{K}$-point contained in an open torus orbit $Y^0(\Gamma)$. The Hilbert scheme parameterizes reduced points in Y. Therefore, the Hilbert image is $Y(\Gamma)$, the closure of $Y^0(\Gamma)$. The weights on the Hilbert point of X are $\chi \in \Gamma$, while the height function is $a(\chi) = \langle \chi, v(X)\rangle$. It follows that the piecewise-linear function F whose domains of linearity are the cells of the dual complex is

$$F(w) = \min_{\chi\in\Gamma}\langle \chi, w + v(X)\rangle.$$

In particular if x lies in the big open torus of Y then the Gröbner complex is just the normal fan of $\mathrm{Conv}(\Gamma)$ translated by $-v(X)$.

Let us examine initial deformations if X is a point in the big open torus in a toric variety Y. If $w = -v(X)$ then $t^w X$ has valuation 0 and so $\mathrm{in}_w(X)$ is a point in the big open torus of Y_0. Otherwise, $\mathrm{in}_w(X)$ lies in some torus orbit. In fact, if $w + v(X) \in C_\Gamma$ for a face Γ of Y's polytope, then $\mathrm{in}_w(X)$ is a point in $Y^0(\Gamma)$. This is in agreement with the proof of Proposition 4.19.

Example 5.11. Let Y be a toric variety defined over $\mathbf{k}$. Let X be the scheme-theoretic image of a map $\mathrm{Spec}\,\mathbf{k}[\varepsilon]/\varepsilon^2 \to Y$. We visualize X as a point in Y with a tangent vector anchored at it. Suppose the image lies in the big open torus and that the vector is chosen generically. Let us find the weight polytope of HI. By Proposition 4.19, it suffices to find the vertices, which correspond to the torus-fixed points in HI. The torus-fixed points in HI are schemes S consisting of a fixed point p of Y together with a projectivized tangent vector pointing along a one-dimensional orbit E containing p. By the genericity condition, all choice of (p, E) with $p \in E$ are possible. We must find the weights corresponding to these fixed points.

Let us first work out the case where $Y = \mathbb{P}^n$. If $\mathrm{HI} \subset \mathbb{P}^N$ and $\mathbf{y} \in \mathbf{k}^{N+1}\backslash\{0\}$ is a vector corresponding to a torus fixed point Q, then the vertex of the weight polytope of HI corresponds to the character of the action of $T = (\mathbf{k}^*)^n$ on $\mathbf{y}$. Because the embedding of HI is given by the composition of the embedding of the Hilbert scheme into a Grassmannian with the Plücker embedding into $\mathbb{P}^N$, the action of T on $\mathbf{y}$ is the same as the action of T on $\wedge^{\mathrm{top}}(\Gamma(\mathscr{O}_Q(l)))$, where l is a sufficiently large positive integer. Now, a torus fixed-point of HI consists of a pair (p, E). Suppose p is given by the point $X_i = \delta_{ir}$ in homogeneous coordinates. Let $x_j = X_i/X_r$ be inhomogeneous coordinates on $X_r \neq 0$. Then the fixed point Q is given as the image of an affine morphism

$$\begin{aligned}
&\mathbb{A}^n \leftarrow \operatorname{Spec} \mathbf{k}[\varepsilon]/\varepsilon^2,\\
&\mathbf{k}[x_1, \ldots, \hat{x}_r, \ldots, x_{n+1}] \to \mathbf{k}[\varepsilon]/\varepsilon^2,\\
&x_i \to c\delta_{is}\varepsilon,
\end{aligned}$$

where $c \in \mathbf{k}$ is some constant. In other words, the tangent vector points along the x_s-axis. The vector space $\mathcal{O}_Q(l)$ is spanned by two monomials, X_r^l and $X_r^{l-1}X_s$. They have characters le_r and $(l-1)e_r + e_s$, respectively where e_i are the standard unit basis vectors of $T^\wedge$. Therefore, $\wedge^{\mathrm{top}}(\Gamma(\mathcal{O}_Q(l)))$ has character $(2l-2)e_r + (e_r + e_s)$. Let Δ^{n-1} be the unit simplex in $T^\wedge$ and Γ the convex hull of the mid-points of 2Δ. Then the state polytope of X, which is the weight polytope of HI, is $(2l-2)\Delta + \Gamma$.

For a general toric variety $Y \subseteq \mathbb{P}^n$, we note that the Hilbert scheme $\mathrm{Hilb}_P(Y)$ is constructed as a subscheme of $\mathrm{Hilb}_P(\mathbb{P}^n)$. Let U be the torus of $\mathbb{P}^n$, T the torus of Y, $i : T \to U$ the homomorphism of tori, and $i^\vee : U^\vee \to T^\vee$ the induced projection. If Q is a T-fixed point of $\mathrm{Hilb}_P(Y)$, then Q is a U-fixed point and the character of the corresponding vertex in $U^\vee$ pulls back by $i^\vee$ to the appropriate character in $T^\vee$. Therefore, if $\Gamma = \mathrm{Conv}(\mathscr{A})$ is the polytope corresponding to Y and Δ the convex hull of the mid-points of the edges of 2Γ, the state polytope of X is $(2l-2)\Gamma + \Delta$ by Lemma 4.27. See [28] for a computation of the related case of the Gröbner fan of generic point configurations in affine space.

The Chow image in this case is isomorphic to Y as its points correspond to points of Y with multiplicity 2. The fiber polytope is Γ. Because the fiber polytope, P is a Minkowski summand of the state polytope $(2l-2)\Gamma + \Delta$, the Gröbner fan is a refinement of the fiber fan. This is an example of a general fact.

Proposition 5.12. *The Gröbner complex is a refinement of the fiber complex.*

Proof. The fundamental class map FC : HI $\to$ CI satisfies the hypotheses of Lemma 4.25. $\square$

For a combinatorial commutative algebra proof of the above, see [35].

6. Tropical varieties

6.1. Intersection of sub-tori

Before we give the definition of tropical varieties, we must digress to consider the intersection two sub-tori in $(\mathbf{k}^*)^n$. Let

$$H_1 = (\mathbf{k}^*)^{m_1}, \quad H_2 = (\mathbf{k}^*)^{m_2} \hookrightarrow T = (\mathbf{k}^*)^n$$

be two injective homomorphisms with $m_1 + m_2 = n$ such that images under the induced maps $H_i^\vee \to T^\vee$ are transversal. Let $y_1, y_2 \in (\mathbf{k}^*)^n$. Let $V_i = H_i \cdot y_i$. We compute the intersection of V_1 and V_2.

The inclusions $H_1, H_2 \hookrightarrow (\mathbf{k}^*)^n$ correspond to surjections $T^\wedge \to H_i^\wedge$. Let M_i be the kernel of the surjections. We may also write M_i as $H_i^\perp$.

Proposition 6.1. *The number of intersection points, $|V_1 \cap V_2|$ is equal to $[T^\wedge : M_1 + M_2]$, the lattice index of $M_1 + M_2$ in $T^\wedge$.*

Proof. The following argument is adapted from [37, pp. 32–33]. Pick bases for M_1 and M_2. V_i is cut out by the equations

$$x^{\mathbf{a}} = y_1^{\mathbf{a}}, \quad x^{\mathbf{b}} = y_2^{\mathbf{b}}$$

for $x \in (\mathbf{k}^*)^n$, where $\mathbf{a}$ ranges over the basis for M_1 and $\mathbf{b}$ ranges over a basis for M_2. We write the basis vectors as row vectors and concatenate them to form an $n \times n$-matrix.

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}.$$

Put this matrix in Hermitian normal form $UA = R$ where $U \in \mathrm{SL}_n(\mathbb{Z})$, and R is an upper triangular invertible matrix. Therefore, the entries of R are

$$R = \begin{bmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ 0 & r_{22} & \dots & r_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & r_{nn} \end{bmatrix}.$$

Finding intersection points of V_1 and V_2 amounts to solving the system

$$x_1^{r_{i1}} x_2^{r_{i2}}, \dots, x_n^{r_{in}} = c_i$$

for certain $c_i \in \mathbf{k}^*$. There are $r_{11}r_{22}, \dots, r_{nn} = |\det(A)| = [T^\wedge : M_1 + M_2]$ solutions. $\square$

The definition of tropical intersection numbers in [24] requires that the above lattice index be equal to $[\mathbb{Z}^n : M_1^\perp + M_2^\perp]$ where $M_i^\perp$ is the perpendicular lattice to M_i. For the sake of completeness, we include a proof with simplifications by Frédéric Bihan that the lattice indexes are equal.

Lemma 6.2. *Let L and M be saturated lattices in $\mathbb{Z}^n$ of complementary rank so that $L + M$ has rank n. Then*

$$[\mathbb{Z}^n : L + M] = [\mathbb{Z}^n : L^\perp + M^\perp],$$

where

$$L^\perp = \ker((\mathbb{Z}^n)^\vee \to L^\vee),$$
$$M^\perp = \ker((\mathbb{Z}^n)^\vee \to M^\vee).$$

Proof. Let $k = \mathrm{rank}(L)$. Let $Q = \{q_1, \ldots, q_k\}$ be a basis for $M^\perp$ and $R = \{r_1, \ldots, r_k\}$ be a basis for L.

We first claim that

$$[Z^n : L + M] = |\det([q_i(r_j)]_{i,j=1,\ldots,k})|.$$

Since M is saturated, we may pick a basis $E = \{e_1, \ldots, e_n\}$ for $\mathbb{Z}^n$ so that $\{e_{k+1}, \ldots, e_n\}$ is a basis for M. Let $F = \{f_1, \ldots, f_k\}$ be a basis for L, and form the $n \times n$-matrix A whose column vectors are the coordinates of $f_1, \ldots, f_k, e_{k+1}, \ldots, e_n$, with respect to the basis E. $[\mathbb{Z}^n : L + M] = |\det(A)|$. The matrix A is block lower-triangular with respect to blocks of size $k \times k$ and $(n-k) \times (n-k)$ centered at the diagonal. The lower right $(n-k) \times (n-k)$ block is the identity matrix. Therefore,

$$|\det(A)| = |\det([a_{ij}]_{i,j=1,\ldots,k})| = |\det([e_i^\vee(f_j)]_{i,j=1,\ldots,k})|.$$

The determinant on the right is invariant under change of basis for L and $M^\perp$. The claim is proven.

Similarly, $[\mathbb{Z}^n : L^\perp + M^\perp]$ is the absolute value of the determinant of the $k \times k$-matrix formed by letting a basis of $(L^\perp)^\perp$ act on a basis of $M^\perp$. Since L is saturated, $(L^\perp)^\perp = L$, so R is a basis of $(L^\perp)^\perp$. Therefore,

$$[\mathbb{Z}^n : L^\perp + M^\perp] = |\det([r_i(q_j)]_{i,j=1,\ldots,k})|.$$

It follows that the lattice indexes, $[\mathbb{Z}^n : L + M]$,$[\mathbb{Z}^n : L^\perp + M^\perp]$ are equal to the absolute values of determinants of transposed matrices. Therefore, they are equal. □

6.2. Definition of trop

Let $\mathscr{Y}$ be an immersive toric scheme defined over $\mathbf{k}$ so $\mathscr{Y} = Y_0 \times_{\mathrm{Spec}\,\mathbf{k}} \mathrm{Spec}\,\mathscr{R}$. Let X be some subvariety of $\mathscr{Y}$ that intersects the big open torus. Let HI be the Hilbert image of X. Its complex is the Gröbner complex.

Definition 6.3. The tropical variety of X, $\mathrm{Trop}(X) \subset G^n$ is given by all $w \in G^n$ so that $\mathrm{in}_w(X)$ intersects the big open torus, $(\mathbf{k}^*)^n \subset Y_0$.

By Proposition 5.6, if w and w' are in the same cell of the Gröbner complex, then $\mathrm{in}_w(X)$ is related to $\mathrm{in}'_w(X)$ by an action of $(\mathbf{k}^*)^n$. If $\mathrm{in}_w(X)$ intersects the big open torus, so does $\mathrm{in}'_w(X)$. Therefore, the tropical variety is a union of cells of the Gröbner complex. We may put a integral polyhedral complex structure on $\mathrm{Trop}(Y)$ to make it a subcomplex of the Gröbner complex.

The tropical variety is usually given by the image under the valuation map. We show that these definitions are equivalent.

Consider the isomorphism between the big open torus of Y and $(\mathbb{K}^*)^n$ given by $g \mapsto g \cdot y$. This allows us to define a valuation map $v : X \cap (\mathbb{K}^*)^n \to G^n$

Lemma 6.4. $\mathrm{Trop}(X)$ *is equal to the image* $-v(X)$.

Proof. $-v(X) \subseteq \operatorname{Trop}(X)$: Let $g \in X \cap (\mathbb{K}^*)^n$. It suffices to show that the degeneration $\overline{g^{-1} \cdot X} \times_{\operatorname{Spec} \mathscr{R}} \operatorname{Spec} \mathbf{k}$ intersects the big open torus in Y_0. But,

$$1 = \overline{(g^{-1} \cdot g)} \times_{\operatorname{Spec} \mathscr{R}} \operatorname{Spec} \mathbf{k} \in \overline{g^{-1} \cdot X} \times_{\operatorname{Spec} \mathscr{R}} \operatorname{Spec} \mathbf{k}$$

is a point in the big open torus.

$\operatorname{Trop}(X) \subseteq -v(X)$: If $w \in \operatorname{Trop}(X)$, then

$$\overline{t^w \cdot X} \times_{\operatorname{Spec} \mathscr{R}} \operatorname{Spec} \mathbf{k} \cap (\mathbf{k}^*)^n$$

is non-empty. Let $\tilde{x}$ be a closed point of the above. Then Lemma 4.15 produces a point $x \in X$ with $\operatorname{in}_w(x) = \tilde{x}$. It follows that $-v(x) = w$. $\square$

Example 6.5. Let $H \subset T$ be a sub-torus and $x \in T$. Let $X = H \cdot x$. Then $\operatorname{Trop}(X)$ is $-H_G^\vee - v(x)$.

Example 6.6. Let us revisit Example 5.9. The Hilbert image is the toric variety associated to $-\mathscr{A}$. We have the morphism $(\mathbb{P}^{|\mathscr{A}|-1})^\vee \to \mathrm{HI}$. The hypersurface in Y corresponding to $[a_\omega] \in \mathrm{HI}$,

$$\sum_{\omega \in \mathscr{A}} a_\omega x^\omega = 0$$

is disjoint from the big open torus if and only exactly one a_ω is not zero. Such points correspond to the torus fixed points of HI or alternatively, the top-dimensional cones of the Gröbner fan. Therefore the tropical variety of the hypersurface $V(f)$ is the union of the positive codimension cones of $N(-\operatorname{Conv}(\mathscr{A}))$.

Let us relate the tropical variety of $\operatorname{in}_w(X)$ to that of X.

Lemma 6.7. *Let w be a point in a cell τ of the tropical variety*, $\operatorname{Trop}(X)$. *Then* $\operatorname{Trop}(\operatorname{in}_w(X))$ *is the star of τ in* $\operatorname{Trop}(X)$.

Proof. Recall that by Lemma 5.8, $\operatorname{in}_u(\operatorname{in}_w(X)) = \operatorname{in}_{w+\varepsilon u}(X)$ for sufficiently small ε. Therefore, $\operatorname{in}_u(\operatorname{in}_w(X))$ intersects the open torus if and only $w + \varepsilon u \in \operatorname{Trop}(X)$. $\square$

The dimension of X and the dimension of $\operatorname{Trop}(X)$ are related. We give a proof adapted from [37]. We begin with the case where $\operatorname{Trop}(X)$ is zero-dimensional.

Lemma 6.8. *If $X \subseteq (\mathbb{K}^*)^n$ is a variety with* $\dim(\operatorname{Trop}(X)) = 0$ *then X is zero-dimensional.*

Proof. Suppose X is positive dimensional. Choose a coordinate projection $p : (\mathbb{K}^*)^n \to \mathbb{K}^*$ so that $p(X)$ is an infinite set. By Chevalley's theorem [25], $p(X)$ is a finite union of locally closed sets and, since it is infinite, it must be an open set. Therefore, $\operatorname{Trop}(X)$ is bigger than a point. $\square$

We can reduce the general case to the above lemma.

Proposition 6.9. *If $X \cap (\mathbb{K}^*)^n$ is purely d-dimensional, so is* $\mathrm{Trop}(X)$.

Proof. Suppose dim $\mathrm{Trop}(X) = k$. Let w be an element of the relative interior of a top-dimensional cell of $\mathrm{Trop}(X)$. Then w is in the relative interior of a k-dimensional cell C_Γ of the Gröbner complex. By Lemma 4.22, $\mathrm{in}_w(X)$ is invariant under a k-dimensional torus, U. The initial degeneration $\mathrm{in}_w(X)$ intersects the open torus so if $x \in \mathrm{in}_w(X) \cap (\mathbf{k}^*)^n$, the k-dimensional variety $U \cdot x$ is a subset of $\mathrm{in}_w(X)$. Since $\mathrm{in}_w(X)$ is a flat deformation of X, it is also d-dimensional. Therefore $k \leqslant d$. By Lemma 6.7, the tropical variety of $\mathrm{in}_w(X)$ is the k-dimensional subspace $\mathrm{Span}(C_\Gamma - w)$.

Now, we show $d = k$. Let W be a variety of the form $H \cdot z$ where $H \subset (\mathbf{k}^*)^n$ is an $(n-k)$-dimensional torus with $H^\vee$ is transverse to $\mathrm{Trop}(\mathrm{in}_w(X))$. Now, by the Kleiman–Bertini theorem [21], there is a choice of z so that $\mathrm{in}_w(X) \cap W$ is empty or of dimension $d - k$. By Proposition 6.1, $U \cdot x$ and W must intersect, so $\mathrm{in}_w(X) \cap W$ is non-empty. But, $\mathrm{Trop}(\mathrm{in}_w(X) \cap W) \subseteq \mathrm{Trop}(\mathrm{in}_w(X)) \cap \mathrm{Trop}(W)$ which is a point. Therefore, $\mathrm{in}_w(X) \cap W$ is a $d - k$ dimensional scheme whose tropicalization is a point. By the above lemma $d = k$. $\square$

6.3. Multiplicities

Let X be an m-dimensional subvariety of a toric variety Y. If w is in the relative interior of an m-dimensional cell C_Γ of $\mathrm{Trop}(X)$, then $\mathrm{in}_w(X) \cap (\mathbf{k}^*)^n$ is a subscheme invariant under an m-dimensional torus H with $H_{\mathbb{R}}^\vee = \mathrm{Span}(C_\Gamma - w)$. Therefore, $\mathrm{in}_w(X) \cap (\mathbf{k}^*)^n$ is supported on $\coprod_i (H \cdot p_i)$ where p_i are points in $(\mathbf{k}^*)^n$. This allows us to define multiplicities on $\mathrm{Trop}(X)$.

Definition 6.10. Given a top-dimensional cell σ of $\mathrm{Trop}(X)$, let w be a point in the relative interior of σ. Decompose the underlying cycle of $\mathrm{in}_w(X) \cap (\mathbf{k}^*)^n$ as

$$[\mathrm{in}_w(X) \cap (\mathbf{k}^*)^n] = \sum m_i [H \cdot p_i]$$

for $H \cong (\mathbf{k}^*)^m \subset T_{\mathbf{k}}$, $p_i \in (\mathbf{k}^*)^n$. The *multiplicity* m_σ is

$$m_\sigma = \sum_i m_i.$$

This multiplicities are also called weights.

$\mathrm{Trop}(X)$ obeys the following *balancing condition* first given in [33, Theorem 2.5.1].

Definition 6.11. An integrally weighted m-dimensional integral polyhedral complex is said to be *balanced* if the following holds. Let τ be an $(m-1)$-dimensional cell of $\mathrm{Trop}(X)$ and $\sigma_1, \ldots, \sigma_l$ be the m-dimensional cells adjacent to τ. Let $w \in \tau^\circ$, $V = \mathrm{Span}(\tau - w)$, and λ the projection $\lambda : T^\vee \to T^\vee / V$. Let $p_j = \lambda(\sigma_j - w)$. Note that p_j is an interval adjacent to 0, and let $v_j \in T^\vee / V$ be the primitive integer vector along $\mathrm{Span}_+(p_j)$. Then

$$\sum_{j=1}^{l} m_{\sigma_j} v_j = 0.$$

We will give a proof that the balancing condition is satisfied in Theorem 8.13. The following relates the multiplicities on $\mathrm{Trop}(\mathrm{in}_w(X))$ to those on $\mathrm{Trop}(X)$.

Lemma 6.12. *Let $w \in \tau^\circ$ be a point in the relative interior of a cell of* $\mathrm{Trop}(X)$. *Let $\sigma_1, \ldots, \sigma_l$ be the top-dimensional cells in* $\mathrm{Trop}(X)$ *containing τ. Then the multiplicities of the cones $\overline{\sigma}_1, \ldots, \overline{\sigma}_l$ in* $\mathrm{Trop}(\mathrm{in}_w(X))$ *corresponding to $\sigma_1, \ldots, \sigma_k$ are $m_{\sigma_1}, \ldots, m_{\sigma_l}$.*

Proof. Let $u \in \overline{\sigma}_i$. Then $\mathrm{in}_u(\mathrm{in}_w(X)) = \mathrm{in}_{w+\varepsilon u}(X)$ by Lemma 4.24. By shrinking ε further if necessary, we may suppose $w + \varepsilon u \in \sigma_i$. Therefore, the degeneration $\mathrm{in}_u(\mathrm{in}_w(X))$ used to compute $m_{\overline{\sigma}_i}$ is the same as the degeneration $\mathrm{in}_{w+\varepsilon u}(X)$ used to compute m_{σ_i}. □

7. Intersection theory motivation: Bezout vs. bernstein

Let us consider two curves in $(\mathbb{C}^*)^2$ cut out by polynomials $f(x, y)$ and $g(x, y)$. Suppose they have no component in common. We would like to bound the number of intersection points in $(\mathbb{C}^*)^2$ counted with multiplicity. The Bernstein bound will motivate tropical intersection theory.

7.1. Bezout bound

We first consider the Bezout bound. We compactify $(\mathbb{C}^*)^2$ to the projective plane $\mathbb{P}^2$. The intersection number is given by topology and is equal to $\deg(f)\deg(g)$. This intersection bound is rigid in that it is invariant under deformations of f and g. Unfortunately, the bound is not the best because we introduced new intersections on the coordinate hyperplanes by compactifying.

Let us make this concrete by picking polynomials (all borrowed from [37]). Let

$$f(x, y) = a_1 + a_2 x + a_3 xy + a_4 y,$$

$$g(x, y) = b_1 + b_2 x^2 y + b_3 x y^2.$$

To consider these polynomials on $\mathbb{P}^2$, we must homogenize them to

$$F(X, Y, Z) = a_1 Z^2 + a_2 XZ + a_3 XY + a_4 YZ,$$

$$G(X, Y, Z) = b_1 Z^3 + b_2 X^2 Y + b_3 X Y^2.$$

Then the Bezout bound is $2 \times 3 = 6$. Notice that both curves contain the points $[1 : 0 : 0]$ and $[0 : 1 : 0]$. This leads Bezout's theorem to over-count the number of intersections by 2. It is impossible to remove these additional intersection points by an action of $(\mathbb{C}^*)^2$ since these points are fixed under the torus action.

7.2. Bernstein bound

Another approach is offered by Bernstein's theorem:

Theorem 7.1. *Given Laurent polynomials*

$$f_1, \ldots, f_n \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$$

with finitely many common zeroes in $(\mathbb{C}^*)^n$, *let* Δ_i *be the Newton polytopes of* f_i. *The number of common zeroes is bounded by the mixed volume of the* Δ_i*'s.* □

Bernstein's theorem can be conceptualized in the above case as follows. One can compactify $(\mathbb{C}^*)^2$ to a nonsingular toric variety so that the closure of the curves cut out by $f=0$ and by $g=0$ does not intersect any torus fixed points. For instance, one may take the toric variety whose fan is the normal fan to the Minkowski sum of the Newton polygons of f and g. One may apply a $(\mathbb{C}^*)^2$-action to $\overline{\{f=0\}}$ to ensure that there are no intersections outside of $(\mathbb{C}^*)^2$. By refining the fan further, we may suppose that the toric variety is smooth. Then one can bound the number of intersection points by the topological intersection number of the two curves. This reproduces the Bernstein bound.

8. Intersection theory

Henceforth, we will be using tropical varieties $Y(\Delta)$ defined by a fan Δ as in [11].

8.1. Intersection theory over discrete valuation rings

Let us first review some notions of intersection theory from [12]. Let Y be a scheme. A k-cycle on Y is a finite formal sum, $\sum n_i[V_i]$ where the V_i's are k-dimensional subvarieties of Y and the n_i's are integers. k-cycles form a group under formal addition. There is a notion of rational equivalence on cycles, and the Chow group, $A_k(Y)$ is the group of cycles defined up to rational equivalence. This group is analogous to homology. If Y is complete, there is a natural degree map $\deg : A_0(Y) \to \mathbb{Z}$ given by

$$\sum m_i[p_i] \mapsto \sum m_i.$$

For any proper morphism $f : X \to Y$, there is an induced push-forward homomorphism

$$f_* : A_k(X) \to A_k(Y).$$

This push-forward homorphism commutes with degree. If X is a disjoint union $X = \bigsqcup X_i$, then we have $A_k(X) = \bigoplus A_k(X_i)$. If Y is a smooth n-dimensional variety, there is an intersection product

$$A_k(Y) \otimes A_l(Y) \to A_{k+l-n}(Y).$$

If V and W are varieties in Y of dimension k and l, respectively, then the intersection product factors through a refined intersection product

$$A_k(Y) \otimes A_l(Y) \to A_{k+l-n}(V \cap W) \xrightarrow{i_*} A_{k+l-n}(Y),$$

where $i : V \cap W \to Y$. There is also Chow cohomology $A^k(Y)$, which is defined operationally.

Intersection theory can also be defined over discrete valuation rings. The reference is [12, Chapter 20]. We will state the results for $\mathscr{R} = \mathbb{C}[[t^{1/M}]]$, but they are true for more general choices of $\mathscr{R}$. In practice, however, given varieties defined over $\mathbb{C}\{\{t\}\}$, we may find a sufficiently large M so that they are defined over $\mathbb{C}((t^{1/M}))$ and apply the results for the corresponding choice of $\mathscr{R}$. Let $p : \mathscr{Y} \to \operatorname{Spec} \mathscr{R}$ be a scheme over $\operatorname{Spec} \mathscr{R}$. Let $Y = \mathscr{Y} \times_{\operatorname{Spec} \mathscr{R}} \operatorname{Spec} \mathbb{K}$, $Y_0 = \mathscr{Y} \times_{\operatorname{Spec} \mathscr{R}} \operatorname{Spec} \mathbf{k}$.

Many results from intersection theory including the existence of degree and refined intersection product remain true in this case using relative dimension over $\operatorname{Spec} \mathscr{R}$ in place of absolute dimension. The new feature in this situation is the *specialization map*

$$s : A_k(Y/\mathbb{K}) \to A_k(Y_0/\mathbf{k})$$

which is the Chow-theoretic analog of $X \to (\overline{X}) \times_{\operatorname{Spec} \mathscr{R}} \operatorname{Spec} \mathbf{k}$.

Proposition 8.1. *If $\mathscr{Y}$ is smooth over* $\operatorname{Spec} \mathscr{R}$ *then the specialization map is a ring homomorphism. Moreover it commutes with refined intersection product.*

Proof. See [12, Corollary 20.3 and Example 20.3.2]. □

8.2. Transversal intersections

Let $V^k, W^l \subset Y^n$ be varieties of dimensions k and l where $k + l = n$. Let Y be a smooth toric variety over $\operatorname{Spec} \mathbb{K}$.

Definition 8.2. V^k and W^l are said to *intersect properly* if $V \times_Y W$ is a zero-dimensional scheme.

Definition 8.3. Two tropical varieties $\operatorname{Trop}(V)$, $\operatorname{Trop}(W)$ are said to *intersect transversally* if they intersect in the relative interior of transversal top-dimensional cells.

Note that it is not sufficient that V and W intersect transversally for $\operatorname{Trop}(V)$ and $\operatorname{Trop}(W)$ to intersect transversally. In fact, V and W can be disjoint while their tropicalizations intersect (or even coincide, for example, $x + y = 1$ and $x + y = 2$ in $(\mathbb{K}^*)^2$). However, the transversal intersection lemma of [5] does give a condition for V and W to intersect:

Lemma 8.4. *If* $\operatorname{Trop}(V)$ *and* $\operatorname{Trop}(W)$ *intersect transversally at $w \in \mathbb{R}^n$, then $w \in$* $\operatorname{Trop}(V \cap W)$.

Proof. Since w is in a top-dimensional cell of $\operatorname{Trop}(V)$ and of $\operatorname{Trop}(W)$ then

$$\operatorname{supp}(\operatorname{in}_w(V)) = H_1 \cdot V_\sigma,$$

$$\operatorname{supp}(\operatorname{in}_w(W)) = H_2 \cdot W_\tau,$$

where supp denotes underlying sets, V_σ, W_τ are finite sets of points, and H_1, H_2 are sub-tori of dimension k and l, respectively. By Proposition 6.1,

$$(\operatorname{in}_w(V) \times_{Y_0} \operatorname{in}_w(W)) \cap (\mathbf{k}^*)^n$$

is non-empty and zero-dimensional. Let z be a closed point of $(\mathrm{in}_w(V)\times_{Y_0}\mathrm{in}_w(W))\cap(\mathbf{k}^*)^n$. Now let $\mathscr{V}=\overline{t^w\cdot V}$, $\mathscr{W}=\overline{t^w\cdot W}$. Let $\mathscr{Z}$ be a maximal irreducible component of $\mathscr{V}\times_{\mathscr{Y}}\mathscr{W}$ containing z. Therefore, $(\mathscr{Z}\times_{\mathrm{Spec}\,\mathscr{R}}\mathrm{Spec}\,\mathbf{k})\cap(\mathbf{k}^*)^n$ is non-empty and zero-dimensional.

We claim $\mathscr{Z}$ is not contained in the fiber over $\mathrm{Spec}\,\mathbf{k}$. Since $\mathscr{V}$ and $\mathscr{W}$ have relative dimension k and l, respectively, each top-dimensional irreducible component $\mathscr{V}\times_{\mathscr{Y}}\mathscr{W}$ must have relative dimension at least 0 and therefore cannot be contained in the special fiber as a zero-dimensional subscheme.

$Z=\mathscr{Z}\times_{\mathrm{Spec}\,\mathscr{R}}\mathrm{Spec}\,\mathbb{K}\subset t^wV\times_Y t^wW$ is non-empty and $z\in\mathrm{in}_w(t^{-w}Z)\subseteq\mathrm{in}_w(V\times_Y W)$. Therefore $V\times_Y W$ must have a point of valuation $-w$. □

Lemma 8.5. *If all intersections of* $\mathrm{Trop}(V)$ *and* $\mathrm{Trop}(W)$ *are transversal, then* $V\cap(\mathbb{K}^*)^n$ *and* $W\cap(\mathbb{K}^*)^n$ *intersect properly.*

Proof. Let Z be the intersection of the two varieties with the reduced induced structure. Then $\mathrm{Trop}(Z)=\mathrm{Trop}(V)\cap\mathrm{Trop}(W)$ is zero-dimensional. Lemma 6.8 shows that every component of Z is zero-dimensional. □

8.3. Intersection of tropicalizations

We will define an intersection number for transversal tropical varieties of complementary dimensions.

Let Y be an n-dimensional smooth toric variety defined over $\mathbf{k}$. Let $V^k, W^l\subseteq Y$ be varieties of complementary dimensions such that $\mathrm{Trop}(V)$ and $\mathrm{Trop}(W)$ intersect tropically transversely. Let $x\in\mathrm{Trop}(V)\cap\mathrm{Trop}(W)$ such that x is contained in top-dimensional cells σ_x,τ_x of $\mathrm{Trop}(V)$ and $\mathrm{Trop}(W)$, respectively. Translate $\mathrm{Trop}(V)$ and $\mathrm{Trop}(W)$ so that x is at the origin. We have inclusions $\mathbb{R}\sigma_x,\mathbb{R}\tau_x\hookrightarrow T^\vee_{\mathbb{R}}$, which induce projections $T^\wedge_{\mathbb{R}}\to(\mathbb{R}\sigma_x)^\vee$ and $T^\wedge_{\mathbb{R}}\to(\mathbb{R}\tau_x)^\vee$. Let M_x and N_x be the lattices defined by

$$M_x=\ker(T^\wedge_{\mathbb{R}}\to(\mathbb{R}\sigma_x)^\vee)\cap T^\wedge,$$

$$N_x=\ker(T^\wedge_{\mathbb{R}}\to(\mathbb{R}\tau_x)^\vee)\cap T^\wedge.$$

Let m_x,n_x be the multiplicities of σ_x and τ_x in $\mathrm{Trop}(V)$ and $\mathrm{Trop}(W)$, respectively, and define the *tropical intersection number* to be

$$\deg(\mathrm{Trop}(V)\cdot\mathrm{Trop}(W))=\sum_{x\in\mathrm{Trop}(V)\cap\mathrm{Trop}(W)} m_xn_x[T^\wedge:M_x+N_x].$$

This definition is analogous to the definition in classical intersection theory. Here, m_x,n_x are analogous to the multiplicities of subvarieties in cycles and the lattice index is analogous to the length of a zero-dimensional component of the intersection.

Definition 8.6. V and W *intersect in the interior* if the support of $V\times_Y W$ is contained in the big open torus T of Y.

Theorem 8.7. *If V and W intersect tropically transversally and in the interior then the tropical intersection number of* $\mathrm{Trop}(V)$ *and* $\mathrm{Trop}(W)$ *is equal to the classical intersection number.*

Proof. Let us replace $\mathbb{K}$ by a field $\mathbb{C}((t^{1/M}))$ over which V and W are defined. First note that $\mathrm{Trop}(V \cap W) = \mathrm{Trop}(V) \cap \mathrm{Trop}(W)$ by the transverse intersection lemma. It follows that $V \cap W$ is zero-dimensional. Decompose this intersection into a disjoint union

$$V \times_Y W = \coprod_{x \in \mathrm{Trop}(V) \cap \mathrm{Trop}(W)} Z_x,$$

where $v(Z_x) = -x$. Now, the refined intersection product is

$$V \cdot W \in A_0(V \cap W) = \bigoplus A_0(Z_x)$$

and the intersection number is the degree of the intersection product. If

$$\pi_x : A_0(V \cap W) \to A_0(Z_x)$$

is the projection onto the summand, then

$$\deg(V \cdot W) = \sum_{x \in \mathrm{Trop}(V) \cap \mathrm{Trop}(W)} \deg(\pi_x(V \cdot W)).$$

Let $w \in \mathrm{Trop}(V) \cap \mathrm{Trop}(W)$ and

$$\mathscr{V} = \overline{t^w \cdot V} \subseteq \mathscr{Y},$$

$$\mathscr{W} = \overline{t^w \cdot W} \subseteq \mathscr{Y}.$$

Note that $\mathscr{V}$ and $\mathscr{W}$ are flat over $\mathscr{R}$.

Decompose the intersection of $\mathscr{V}$ and $\mathscr{W}$ as

$$\mathscr{V} \times_{\mathscr{Y}} \mathscr{W} = \coprod_{x \in \mathrm{Trop}(V) \cap \mathrm{Trop}(W)} \mathscr{Z}_x,$$

where

$$\mathscr{Z}_x \times_{\mathrm{Spec}\,\mathscr{R}} \mathrm{Spec}\,\mathbb{K} = t^w \cdot Z_x.$$

The zero-dimensional scheme $(\mathscr{Z}_x)_0 = \mathscr{Z}_x \times_{\mathrm{Spec}\,\mathscr{R}} \mathrm{Spec}\,\mathbf{k}$ is contained in $(\mathbf{k}^*)^n$ only if $x = w$. Otherwise, it is disjoint from $(\mathbf{k}^*)^n$. Let $(\mathscr{V} \times_{\mathscr{Y}} \mathscr{W})_0 = (\mathscr{V} \times_{\mathscr{Y}} \mathscr{W}) \times_{\mathrm{Spec}\,\mathscr{R}} \mathrm{Spec}\,\mathbf{k}$. Since $\mathscr{Z}_w$ is proper over $\mathrm{Spec}\,\mathscr{R}$, by [12, Proposition 20.3 and Corollary 20.3], the images of $[t^w V] \otimes [t^w Y]$ under the following compositions are equal

$$A_k(Y) \otimes A_l(Y) \to A_0(t^w(V \cap W)) \xrightarrow{\pi_w} A_0(t^w Z_w) \xrightarrow{s} A_0((\mathscr{Z}_w)_0) \xrightarrow{\deg} \mathbb{Z},$$

$$A_k(Y) \otimes A_l(Y) \xrightarrow{s \otimes s} A_k(Y_0) \otimes A_l(Y_0) \to A_0((\mathscr{V} \times_{\mathscr{Y}} \mathscr{W})_0) \xrightarrow{\pi_w} A_0((\mathscr{Z}_w)_0) \xrightarrow{\deg} \mathbb{Z}.$$

But the second composition is just the degree of the intersection of the tori $\mathrm{in}_w(V)$ and $\mathrm{in}_w(W)$. Their intersection number is $m_w n_w [T^\wedge : M_w + N_w]$ by Proposition 6.1. Summing over $w \in \mathrm{Trop}(V) \cap \mathrm{Trop}(W)$, we get the result. $\square$

8.4. Transversality

Lemma 8.8. *If V and W intersect all torus orbits properly then there exists $\lambda \in (\mathbf{k}^*)^n$, such that $\lambda \cdot V$ intersects W properly and in the interior.*

Proof. By the Kleiman–Bertini theorem [21] applied to each orbit closure $V(\sigma)$, there exists a non-empty open set $U \subset (\mathbb{K}^*)^n$ such that for all $\lambda \in U$, $\lambda \cdot V$ intersects W properly and in the interior. It suffices to show that $U \cap (\mathbf{k}^*)^n$ is non-empty.

Suppose $U \cap (\mathbf{k}^*)^n$ is empty. Let $f \in \mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be a Laurent polynomial over $\mathbb{K}$ so that $(\mathbb{K}^*)^n \backslash V(f) \subseteq U$. Then $V(f)$ contains all $\mathbf{k}$-points. By clearing denominators, we may suppose $f \in R = \mathbb{C}[[t^{1/M}]]$ for some M where $t^{1/M}$ does not divide f. Since $f = 0$ on $(\mathbf{k}^*)^n$, $f|_{t^{1/M}=0} = 0$. It follows that $t^{1/M}$ divides f. This gives a contradiction. $\square$

Note that $\lambda \cdot V$ and V have the same tropical variety.

8.5. Balancing condition

In this section, we prove that if X is an m-dimensional subvariety of a toric variety Y, then $\mathrm{Trop}(X)$ satisfies the balancing condition. The strategy of the proof is that a well-defined tropical intersection number between $\mathrm{Trop}(X)$ and $\mathrm{Trop}(H \cdot z)$ for H a sub-torus and $z \in T$ guarantees that $\mathrm{Trop}(X)$ is balanced.

We need the following technical lemma.

Lemma 8.9. *Let $x \in (\mathbb{K}^*)^n$ and σ be a cone in Δ. Then $v(x) \in \sigma^\circ$ if and only if $\mathrm{in}_0(x) \in \mathcal{O}_\sigma$, the open torus corresponding to σ.*

Proof. Consider the toric chart $U_\sigma = \mathrm{Spec}\, \mathbb{K}[\sigma^\vee \cap T^\wedge] \supset (\mathbb{K}^*)^n$. The torus orbit $\mathcal{O}_\sigma$ is cut out by the ideal I_σ which is the kernel of the projection

$$\mathbb{K}[\sigma^\vee \cap T^\wedge] \to \mathbb{K}[\sigma^\perp \cap T^\wedge].$$

A monomial $m \in I_\sigma$ is of the form x^u for u satisfying $\langle u, y\rangle > 0$ for all $y \in \sigma^\circ$. Since $v(x) \in \sigma^\circ$, $v(m(x)) > 0$ for every monomial $m \in I_\sigma$ while $v(m(x)) = 0$ for every $m \in \mathbb{K}[\sigma^\perp \cap T^\wedge]$.

Suppose $v(x) \in \sigma^\circ$. If $m \in \mathbb{C}[\sigma^\vee \cap T^\wedge]$ is a monomial, $m(x)|_{t=0} = m(\mathrm{in}_0(x))$. Therefore, for $f \in I_\sigma$, $v(f(x)) > 0$ so under the specialization $t = 0$, $f(x)$ goes to 0. On the other hand, for every $m \in \mathbb{C}[\sigma^\perp \cap T^\wedge]$, $m(x)$ goes to its leading term, which is non-zero. It follows that $\mathrm{in}_0(x) \in \mathcal{O}_\sigma$.

Now, suppose $\mathrm{in}_0(x) \in \mathcal{O}_\sigma$. For any monomial $m = x^u \in I_\sigma$, we have $m(x)|_{t=0} = m(\mathrm{in}_0(x)) = 0$. Therefore, $v(m(x)) > 0$, which implies $\langle u, v(x)\rangle > 0$. For $u \in \sigma^\perp$, $m = x^u$ is non-zero on $\mathrm{in}_0(x)$. It follows that $\langle u, v(x)\rangle = v(m(x)) = 0$ and so $v(x) \in \sigma^\circ$. $\square$

We need the following Lemma of Tevelev.

Lemma 8.10 (*Tevelev [39], Lemma 2.2*). *Let $Y(\Delta)$ be a complete toric variety given by a fan Δ. Let $X \subset Y(\Delta)$ be a subvariety defined over* $\mathbf{k}$. *Then $-\mathrm{Trop}(X)$ intersects a cone σ in the fan Δ in its relative interior if and only if $\overline{X}$ intersects $\mathcal{O}_\sigma$.*

Proof. Write $X_{\mathbf{k}}$ for X and $X_{\mathbb{K}}$ for $X \times_{\operatorname{Spec} \mathbf{k}} \operatorname{Spec} \mathbb{K}$. Observe that $X_{\mathbf{k}} = \mathrm{in}_0(X_{\mathbb{K}})$.

Suppose $-\mathrm{Trop}(X) \cap \sigma^\circ$ is non-empty. Then there exists $x \in X_{\mathbb{K}}$ with $v(x) \in \sigma^\circ$. Therefore, $\mathrm{in}_0(x) \in \mathcal{O}_\sigma$.

Now suppose $\overline{X} \cap \mathcal{O}_\sigma$ is non-empty. Then by Corollary 4.16, there exists $x \in V_{\mathbb{K}} \cap (\mathbb{K}^*)^n$ with $\mathrm{in}_0(x) \in \mathcal{O}_\sigma$. It follows that $v(x) \in \sigma^\circ$. □

Definition 8.11. A subvariety $X \subset Y$ of dimension l is said to *intersect orbits properly* if

(1) for σ a cone in Δ with $\dim \sigma > l$, X is disjoint from $\mathcal{O}_\sigma$,
(2) for σ a cone in Δ with $\dim \sigma = l$, $X \cap \mathcal{O}_\sigma$ is a 0-dimensional scheme.

By replacing Δ with a finer fan so that $-\mathrm{Trop}(X)$ is supported on a union of cones of dimension at most l, we may always ensure that X intersects orbits properly.

We first prove that curves defined over $\mathbf{k}$ are balanced.

Lemma 8.12. *Let X be a curve defined over* $\mathbf{k}$ *in a complete toric variety $Y(\Delta)$. Then* $\mathrm{Trop}(X)$ *is balanced.*

Proof. By refining Δ, we may suppose that X intersects torus orbits properly and that Y is smooth. $\mathrm{Trop}(X)$ consists of rays $\rho_1, \ldots, \rho_l$ weighted with multiplicities $m_1, \ldots, m_l$. Let v_i be the primitive integer vector along ρ_i. It suffices to show that

$$\sum_{j=1}^{l} m_j \langle u, v_j \rangle = 0$$

for any u in $T^\wedge$. Let $H \subset T$ be the sub-torus so that $H^\vee = u^\perp$. Let $W_y = H \cdot y$ for $y \in (\mathbf{k}^*)^n$. By refining Δ further, we may suppose that W_y intersects torus orbits properly. By replacing W_y by $\lambda \cdot W_y$, we may suppose that W_y intersects X is the interior.

Since for $w, w' \in T_G^\vee$, $t^w W$ and $t^{w'} W$ are related by the T-action, they are linearly equivalent. Therefore, by Lemma 8.8 and Theorem 8.7, the tropical intersection number $\deg(\mathrm{Trop}(X) \cdot \mathrm{Trop}(t^w W))$ is independent of w.

We may suppose without loss of generality that u is primitive. Pick $w \in T^\vee$ such that $\langle u, w \rangle > 0$ and $y \in (\mathbf{k}^*)^n$. Then $\mathrm{Trop}(t^w W_y) = -w - H_{\mathbb{R}}^\vee$ with some multiplicity n_W. Then $\rho_j \cap \mathrm{Trop}(t^w W)$ is non-empty if and only if $\langle u, v_j \rangle < 0$. The multiplicity of such an intersection is

$$m_j n_W [T^\wedge : (\mathbb{Z}u) + v_j^\perp] = m_j n_W |\langle u, v_j \rangle|.$$

Therefore,

$$\deg(\mathrm{Trop}(W) \cdot \mathrm{Trop}(X)) = \sum_{j : \langle u, v_j \rangle < 0} m_j n_W |\langle u, v_j \rangle|.$$

Replacing w by $-w$, we see

$$-\sum_{j : \langle u, v_j \rangle < 0} m_j n_W \langle u, v_j \rangle = \sum_{j : \langle u, v_j \rangle > 0} m_j n_W \langle u, v_j \rangle$$

from which the conclusion follows. □

Theorem 8.13. $\mathrm{Trop}(X)$ *satisfies the balancing condition.*

Proof. Let τ be some $(m-1)$-dimensional cell of $\mathrm{Trop}(X)$ and $\sigma_1, \ldots, \sigma_l$, the adjacent m-dimensional cells. Let w be a point in the relative interior of τ. $\mathrm{in}_w(X)$ is a subscheme that is invariant under an $(m-1)$-dimensional torus. $\mathrm{Trop}(\mathrm{in}_w(X))$, the star of τ consists of the linear subspace $\overline{\tau} = \mathrm{Span}(\tau - w)$ and the cones $\overline{\sigma_i} = \mathrm{Span}^+(\sigma_i - w) + \overline{\tau}$. The multiplicities of the σ's in $\mathrm{Trop}(\mathrm{in}_w(X))$ are the same as those of the corresponding cells in $\mathrm{Trop}(X)$ by Lemma 6.12.

Let V be the union of the components of $\mathrm{in}_w(X)$ that intersect the big open torus. Then, $\mathrm{Trop}(V) = \mathrm{Trop}(\mathrm{in}_w(X))$ and by refining Δ, we may ensure V intersects the torus orbits properly. Let K be the $(m-1)$-dimensional invariant torus of V, and $p : T \to T/K$ be the quotient map. The image of $\mathrm{Trop}(V)$ under that map is a one-dimensional integral polyhedral complex with one vertex and l rays $\mathbb{R}_+ v'_1, \ldots, \mathbb{R}_+ v'_l$ emanating from it where v'_i is a primitive integer vector. For $u \in (T/K)^\wedge$, let $H \subset (T/K)^\vee$ be the $(n-m-1)$-dimensional torus with $H^\vee = u^\perp$. Now let $H' \subset T$ be a $(n-m-1)$-dimensional torus with $p(H') = H$. Pick $w \in T_G^\vee$ such that $\langle u, p^\vee(w)\rangle > 0$. For $y \in (\mathbf{k}^*)^n$, let $W_y = H' \cdot y$. Then $\overline{\sigma}_j$ intersects $\mathrm{Trop}(t^w W_y)$ if and only if $\langle u, v_j \rangle < 0$. The intersection multiplicity in that case is

$$\begin{aligned} m_{\overline{\sigma}_j} n_W [T^\wedge : (H')^\wedge + (\ker(T^\wedge \to (\mathbb{R}\overline{\sigma}_j)^\vee) \cap T^\wedge)] &= m_{\overline{\sigma}_j} n_W [(T/K)^\wedge : \mathbb{Z}u + v_j^\perp] \\ &= m_{\overline{\sigma}_j} n_W |\langle u, v_j \rangle|. \end{aligned}$$

The argument now proceeds as in the case of curves. $\square$

We should mention that the above argument can be simplified by using the theorem that tropicalization is natural under monomial morphisms as proved by Sturmfels and Tevelev [38].

9. Tropical cycles and the cohomology of toric varieties

In this section, we work over a field $\mathbb{K} \supset \mathbf{k} = \mathbb{C}$. $\mathbb{K}$ may be the field of the Puiseux series or the complex numbers.

9.1. Minkowski weights

In [13], Fulton and Sturmfels gave a description of Chow cohomology of a complete toric variety in terms of the fan. This description is closely related to the balancing condition for tropical varieties.

Consider a complete toric variety Y given by a complete n-dimensional fan Δ. The Chow cohomology of Y is given by Minkowski weights. Let $\Delta^{(k)}$ be the set of all cones of codimension k. For a cone $\sigma \in \Delta^{(k)}$, $\tau \in \Delta^{(k+1)}$, $\tau \subset \sigma$, let N_σ be the lattice span of σ and let $n_{\sigma,\tau} \in \sigma$ be an integer vector whose image generates the one-dimensional lattice N_σ / N_τ.

Definition 9.1. A *Minkowski weight of codimension k* is a function

$$c : \Delta^{(k)} \to \mathbb{Z},$$

so that for every $\tau \in \Delta^{(k+1)}$ and every element $u \in \tau^{\perp} \cap \mathbb{Z}^n$,

$$\sum_{\sigma \in \Delta^{(k)} | \sigma \supset \tau} c(\sigma) < u, n_{\sigma,\tau} > = 0.$$

As a consequence of showing $A^k(Y) = \mathrm{Hom}(A_k(Y), \mathbb{Z})$, it is proven in [13] that the Chow cohomology group $A^k(Y)$ is canonically isomorphic to the group of Minkowski weights of codimension k.

We can view a Minkowski weight as an integrally weighted integral fan,

$$\bigcup_{c(\sigma) \neq 0} \sigma,$$

where the cone σ is weighted by $c(\sigma)$. There is a formula for the cup-product in terms of Minkowski weights. If we view Minkowski weights c and d of complementary dimension as fans, then their tropical intersection number (after translating one fan to ensure that they are tropically transverse) is equal to the degree of their cup product evaluated on the fundamental class of Y, $\deg((c \cup d) \cap [Y])$.

If $X \subset Y$ is a codimension k subvariety defined over $\mathbf{k}$, the function taking a cone in $\mathrm{Trop}(X)$ to its multiplicity satisfies the balancing condition, which is exactly the Minkowski weight condition.

9.2. Associated cocycles

If Y is smooth, to every algebraic cycle X of codimension k in Y, we may associate a Minkowski weight of codimension k by Poincare duality. We will do this explicitly using toric geometry.

Lemma 9.2. *Let $Y(\Delta)$ be a smooth toric variety over* $\mathbf{k}$. *Let X be a codimension k algebraic cycle. Define a function*

$$c : \Delta^{(k)} \to \mathbb{Z},$$

$$c : \sigma \mapsto \deg([X] \cdot [V(\sigma)]).$$

Then c is a Minkowski weight and $c \cap [Y(\Delta)] = [X]$.

Proof. $[X]$ has a Poincaré-dual d satisfying $d \cap \alpha = \deg([X] \cdot \alpha)$ for $\alpha \in A_k(Y)$. For all k-dimensional torus orbits, $V(\sigma)$, we have

$$c(\sigma) = \deg([X] \cdot [V(\sigma)]) = d \cap V(\sigma)$$

Since $A_*(Y)$ is generated by torus orbits and $A^*(Y) = \mathrm{Hom}(A_*(Y), \mathbb{Z})$, $c = d$ as Minkowski weights. $\square$

If X is a subvariety of Y defined over $\mathbf{k}$, we may relax the smoothness condition on Y after mandating that X intersects the torus orbits of Y properly.

Definition 9.3. Let Y be a complete toric variety. Let $\widetilde{Y}$ be a smooth toric resolution of Y with fan $\widetilde{\Delta}$, which is a refinement of Δ. Define the *associated cocycle of* X, a Minkowski weight on $\widetilde{\Delta}$ by $c(\tilde{\tau}) = \deg([X] \cdot [V(\widetilde{\tau})])$.

The associated cocycle is well-defined as a Minkowski weight on $\widetilde{\Delta}$. The following proposition shows that it is well-defined on Δ.

Proposition 9.4. *If X is an k-dimensional subvariety of Y, defined over* $\mathbf{k}$, *that intersects the torus orbits properly then the associated cocycle of X is* $-\mathrm{Trop}(X)$.

Proof. Because X intersects the torus orbits properly, by Lemma 8.10, $-\mathrm{Trop}(X)$ is supported on k-dimensional cones in Δ.

We need only show that for every $\tilde{\tau} \in \Delta^{(n-k)}$, the multiplicity $m_{\tilde{\tau}}$ is equal to $c(\tilde{\tau})$. Let $w \in -\tilde{\tau}^{\circ}$. Because intersection product commutes with specialization,

$$\deg([X] \cdot [V(\tilde{\tau})]) = \deg([\mathrm{in}_w(X)] \cdot [V(\tilde{\tau})]).$$

Let $H \subset T$ be the k-dimensional sub-torus corresponding to $\tau \subset T^{\vee}_{\mathbb{R}}$. The underlying cycle of $\mathrm{in}_w(X)$ can be decomposed as

$$[\mathrm{in}_w(X)] = \sum m_i [H \cdot p_i] + D,$$

where $p_i \in (\mathbf{k}^*)^n$ and D is disjoint from the big open torus.

We claim that D is disjoint from $V(\tilde{\tau})$. If it was not, it would have to intersect a proper torus orbit of $V(\tilde{\tau})$. Therefore, it suffices to show that $\mathrm{in}_w(X)$ does not intersect $V(\tilde{\sigma})$ for $\tilde{\sigma} \supset \tilde{\tau}$. If it did, then by Corollary 4.16, there would be $x \in X \cap (\mathbb{K}^*)^n$ so that $\mathrm{in}_w(x) \in V(\tilde{\sigma})$. By Lemma 8.9, $v(x) + w \in \tilde{\sigma}^{\circ}$. Therefore, $v(x) \in -w + \tilde{\sigma}^{\circ} \subset \tilde{\tau}^{\circ} + \tilde{\sigma}^{\circ} \subset \tilde{\sigma}^{\circ}$. But we assumed that $-\mathrm{Trop}(X)$ does not intersect $\tilde{\sigma}^{\circ}$,which is a cone of $\widetilde{\Delta}$ of dimension greater than k.

By a local computation, we see $H \cdot p_i$ meets $V(\tilde{\tau})$ transversely in a single point. Therefore, $c(\tilde{\tau}) = \sum m_i [H \cdot p_i] \cdot [V(\tilde{\tau})] = \sum m_i = m_{\tilde{\tau}}$. $\square$

It follows that the associated cocycle is a pullback by $\pi : Y(\widetilde{\Delta}) \to Y(\Delta)$. Furthermore, the associated cocycle is dual to $[X]$.

Lemma 9.5. *If c is the associated cocycle of $X \subset Y$, then*

$$c \cap [Y] = [X] \in A_k(Y).$$

Proof. Let $\pi : Y(\widetilde{\Delta}) \to Y(\Delta)$ be a smooth toric resolution. By Lemma 9.2, $\pi^* c \cap [Y(\widetilde{\Delta})] = [\pi^{-1}(X)]$. The projection formula tells us

$$c \cap [Y] = c \cap \pi_*([Y(\widetilde{\Delta})] = \pi_*([\pi^{-1}(X)]) = [X]. \quad \square$$

Example 9.6. This gives us the weights for the tropicalization of the hypersurface found in Example 6.6. A top-dimensional cone of $\mathrm{Trop}(V(f))$ corresponds to a one-dimensional face $\Gamma \subset \mathrm{Conv}(-\mathscr{A})$. The multiplicity of that cell is $\deg(V(f)\cdot Y(\Gamma))$. This intersection is defined by

$$\sum_{\omega\in\Gamma} a_\omega x^\omega = 0.$$

This is a polynomial in one variable whose Newton polytope is Γ. Therefore, the number of points in the intersection, hence the multiplicity, is the lattice length of the edge Γ.

9.3. Proof of Bernstein's theorem

For the sake of completeness, we outline a proof of Bernstein's theorem along the lines of the above section. In essence, this proof is a hybrid of the proofs given in [11,37]. We work over $\mathbb{C}$.

Given Laurent polynomials

$$f_1, \ldots, f_n \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}],$$

let Q_i be the Newton polytope of f_i. We summarize the facts we have established in the lemma below.

Lemma 9.7. *Let f_i be a polynomial with Newton polytope Q_i, and $X(\Delta_i)$, the toric variety whose fan is $\Delta_i = N(Q_i)$ The hypersurface $V(f_i)$ intersects torus orbits in $X(\Delta_i)$ properly.*

We know by Example 9.6 that the associated cocycle c_i of $V(f_i)$ is the union of cones of the normal fan of Δ_i of positive codimension where the codimension 1 cones are weighted by the lattice length of the dual edges of Δ_i.

Let Δ be a fan that refines the normal fans of the Δ_i's so that $X(\Delta)$ is smooth. There are birational morphisms from a non-singular variety, $p_i : X(\Delta) \to X(\Delta_i)$. By [37], the mixed volume of $\Delta_1, \ldots, \Delta_n$ is equal to the tropical intersection of the c_i's. By [13], this is equal to $\deg(p_1^* c_1 \cup \ldots \cup p_n^* c_n)$, which is the intersection number of $p_1^{-1}(V(f_1)), \ldots, p_n^{-1}(V(f_n))$ in $X(\Delta)$. This bounds the number of geometric intersections in $(\mathbb{C}^*)^n$.

10. Deformations of subschemes into torus orbits

This section is a generalization of [8, Theorem 2.2]. Let $Y(\Delta)$ be a smooth toric scheme defined over $\mathbf{k}$ and $X \subseteq Y$, a purely k-dimensional closed subscheme. If w is in the relative interior of an m-dimensional cell of the Gröbner complex of X, then $\mathrm{in}_w(X)$ is invariant under an m-dimensional torus. $\mathrm{in}_w(X)$ has components supported in the big open torus of Y and within smaller dimensional torus orbits. In particular if w is in the interior of an open cell of the Gröbner complex, $\mathrm{in}_w(X)$ is invariant under T. Therefore, the maximal components of $\mathrm{in}_w(X)$ are supported on the k-dimensional torus orbits. We can use tropical geometry to determine which torus orbits.

Let σ be a codimension k cone in the fan of Y. Then $V(\sigma)$ is a k-dimensional subscheme.

Theorem 10.1. *Let $w \in T_G^\vee$ be a point in a top dimensional cell of the Gröbner fan. The multiplicity of* $\mathrm{in}_w(X)$ *along $V(\sigma)$ is*

$$\sum_x m_x[T^\wedge : M_x + \sigma^\perp],$$

where the sum is over all x in $-\sigma^\circ \cap (-w + \mathrm{Trop}(X))$ and the intersection multiplicities correspond to the intersection of $-w + \mathrm{Trop}(X)$ and $-\sigma$.

Proof. We may refine Δ so that X intersects torus orbits properly. By the toric version of Chow's lemma, we may further refine Δ by so that Y is smooth and projective. Let W be the complete intersection of k ample hypersurfaces. By applying the Kleiman–Bertini theorem on each torus orbit when choosing hypersurfaces, we may ensure that W intersects torus orbits properly. By ampleness, $W \cap V(\sigma) \neq \emptyset$.

$\mathrm{Trop}(W)$ is a union of cones of Δ of codimension at least k. Let $d = \deg(W \cdot V(\sigma))$. The multiplicity of the cone $-\sigma$ in $\mathrm{Trop}(W)$ is d. By Lemma 8.8, without changing $\mathrm{Trop}(W)$, we may replace W by $\lambda \cdot W$ to ensure that W intersects $t^w \cdot X$ in the interior. If Z is any components of $\mathrm{in}_w(X)$ not supported on $V(\sigma)$, then Z must intersect $V(\sigma)$ in a proper torus orbit. Since W intersects torus orbits properly, W does not intersect Z at any points of $V(\sigma)$.

Now $X \times_Y (t^{-w} \cdot W)$ is a zero-dimensional scheme supported on T. Because specialization commutes with refined intersection product as in Theorem 8.7,

$$\mathrm{in}_w(X \cdot_Y (t^{-w} \cdot W)) = \mathrm{in}_w(X) \cdot_{Y_0} \mathrm{in}_w(t^{-w} \cdot W) = \mathrm{in}_w(X) \cdot_{Y_0} W.$$

We decompose the intersection product of X and $t^{-w} \cdot W$ into contributions with different valuations as in the proof of Theorem 8.7. Some contributions deform to give the intersection product of $\mathrm{in}_w(X)$ and W along the components of $\mathrm{in}_w(X)$ supported on $V(\sigma)$. By Lemma 8.9, $v(t^w x) \in \sigma^\circ$, if and only if $\mathrm{in}_w(x)$ is a point in $\mathcal{O}_\sigma$. Therefore, the components of $X \cap (t^{-w} W)$ that deform to the intersection of W with $V(\sigma)$ are the ones supported on x with

$$v(x) \in (w - \mathrm{Trop}(X)) \cap (w - \mathrm{Trop}(t^{-w} \cdot W)) \cap \sigma^\circ = (w - \mathrm{Trop}(X)) \cap \sigma^\circ.$$

Each point x counts with multiplicity $m_x d[T^\wedge : M_x + \sigma^\perp]$. Since $\deg(W \cdot V(\sigma)) = d$, we divide by d to get the multiplicity of $\mathrm{in}_w(X)$ along $V(\sigma)$. $\square$

Acknowledgments

We would like to thank Bernd Sturmfels for suggesting the connection between tropical cycles and Minkowski weights and Hannah Markwig, David Speyer, Frédéric Bihan, and Sam Payne for helpful comments and corrections.

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