On finite simple groups acting on homology 3-spheres

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Abstract

We show that the only nonabelian finite simple group which admits smooth actions on a homology 3-sphere is the dodecahedral or alternating group \( A_5 \).

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1. Introduction

We are interested in finite, and in particular in finite nonsolvable and simple groups acting on homology 3-spheres. Our main result is the following:

**Theorem 1.** The only nonabelian finite simple group which acts faithfully by diffeomorphisms on a homology 3-sphere is the alternating or dodecahedral group \( A_5 \cong \text{PSL}(2, 5) \).

Any action of a nonabelian finite simple group on a homology sphere is nonfree (in arbitrary dimensions: some involution of the group must have nonempty connected fixed point set). The finite groups which admit free actions on homology 3-spheres have periodic cohomology of period four and are discussed in [6]; the only nonsolvable groups which occur are products of the extended dodecahedral group \( A_5^* \cong \text{SL}(2, 5) \) with cyclic groups of relatively prime order. In the papers [9,12], a characterization of the finite groups \( G \) is given which admit arbitrary actions on homology 3-spheres (i.e., possibly with fixed points). The nonsolvable groups \( G \) are either closely related to the extended dodecahedral group \( A_5^* \) (see Theorem 2), or they have a normal subgroup of index at most four of type \( \text{PSL}(2, p) \) (where, for a prime \( p \geq 5 \), we denote by \( \text{PSL}(2, p) \) the projective special linear group...
or linear fractional group in dimension two over the field with \( p \) elements). Now Theorem 1 will follow from:

**Proposition 1.** For a prime \( p \geq 5 \), the group \( G = \text{PSL}(2, p) \) acts as a group of diffeomorphisms of a homology 3-sphere if and only if \( p = 5 \), i.e., \( G \) is isomorphic to the dodecahedral group \( A_5 \).

We are not aware of any general method which gives dimension restrictions for actions of finite groups on homology spheres, in analogy to free actions where the possible dimensions are determined by the cohomological periods of the groups (see [5]).

The proof of Proposition 1 gives also the following result (compare also [4] where it is shown that every finite 3-manifold group satisfies the \( pq \)-conditions).

**Proposition 2.** For prime numbers \( p \) and \( q \), if a group of order \( pq \) acts faithfully on a homology 3-sphere then it is either abelian or dihedral.

Combined with the main results in [9,12], Proposition 1 implies the following:

**Theorem 2.** Let \( G \) be a finite group of orientation preserving diffeomorphisms of a homology 3-sphere. Then one of the following cases occurs.

1. \( G \) is solvable.
2. \( G \) contains a subgroup of index at most two isomorphic to \( A_5 \) or to \( A_5 \times \mathbb{Z}_2 \).
3. (i) \( G \) has a subgroup of index at most two isomorphic to a central product
   \[ A_5 \times \mathbb{Z}_2 \ G_0 \]
   where \( G_0 \) is solvable and both \( A_5 \) and \( G_0 \) act freely.
   
   (ii) \( G \) has a normal subgroup of index at most eight isomorphic to the central product
   \[ A_5 \times \mathbb{Z}_2 \ A_5 \]
   where both factors \( A_5 \) act freely.

By [6], any finite group acting freely on a homology \( n \)-sphere has at most one involution which then belongs to the center of the group. In Theorem 2, we denote by \( \mathbb{Z}_2 \) the subgroup generated by such an involution, and, for two groups \( G_1 \) and \( G_2 \) acting freely, by \( G_1 \times \mathbb{Z}_2 \ G_2 \) the central product of \( G_1 \) and \( G_2 \) with the two central involutions of the groups identified (defined as a factor group of the direct product of the two groups, see [11, p. 137]); thus \( G_1 \) and \( G_2 \) commute elementwise and \( G_1 \cap G_2 = \mathbb{Z}_2 \).

The groups \( G_0 \) in case 3(i) of Theorem 2 act freely on some homology 3-sphere and have periodic cohomology of period four; a list of such groups can be found in [6]. The group \( A_5 \times \mathbb{Z}_2 \ A_5 \), of order 7200, acts orthogonally on the 3-sphere; it is the group of orientation preserving symmetries of the regular 4-dimensional 120-cell (whose boundary is homeomorphic to \( S^3 \)), and also the group of all possible lifts of elements of an isometry group \( A_5 \) of the Poincaré homology 3-sphere to its universal covering \( S^3 \). We refer to [3,
p. 57] for a list of the finite groups acting orthogonally on $S^3$; this list contains various other groups of type 3(i) of Theorem 2.

For actions of finite nonsolvable and simple groups on $\mathbb{Z}_2$-homology spheres and on arbitrary closed 3-manifolds, see [10].

2. Proof of Proposition 1

For a prime $p$, let $G = \text{PSL}(2, p)$ act on a homology 3-sphere $M$. We assume $p > 5$ and will obtain a contradiction. Let $x$ denote a generator of the multiplicative group of the field with $p$ elements. We consider the elements of $\text{PSL}(2, p)$ represented by the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}. $$

Then $A$ generates a subgroup $\mathbb{Z}_p$ of $\text{PSL}(2, p)$, and $B$ generates a subgroup $\mathbb{Z}_q$ which normalizes $\mathbb{Z}_p$, where $q = (p - 1)/2 > 2$ (note that $B^q = -E_2$ which represents the same element of $\text{PSL}(2, p)$ as the unit matrix $E_2$). Thus the subgroup of $\text{PSL}(2, p)$ generated by $A$ and $B$ is a semidirect product $S = \mathbb{Z}_p \ltimes \mathbb{Z}_q$ (a metacyclic group). It is easy to see that no nontrivial element of $\mathbb{Z}_q$ operates trivially on $\mathbb{Z}_p$ (by conjugation), or in other words that the action of $\mathbb{Z}_q$ on $\mathbb{Z}_p$ is faithful.

Suppose that the action of $\mathbb{Z}_p$ on the homology sphere $M$ is not free. By classical Smith fixed point theory, the fixed point set of $\mathbb{Z}_p$ on $M$ is connected, that is a simple closed curve $K$ in $M$ (see, e.g., [1, Theorem 4.9, p. 144]). As $\mathbb{Z}_q$ normalizes $\mathbb{Z}_p$, it maps $K$ to itself acting as a group of rotations along or around $K$ (no element can act as a reflection on $K$ because $q > 2$). The actions of $\mathbb{Z}_p$ and $\mathbb{Z}_q$ are standard in a small invariant regular neighbourhood of $K$, therefore all elements of $S$ commute in such a regular neighbourhood. But by a result of Newman ([8]; see also [2]), a periodic diffeomorphism of a manifold which is the identity on a nonempty open subset is the identity. It follows that $S$ is an abelian group which is a contradiction.

Thus the action of $\mathbb{Z}_p$ on the homology sphere $M$ is not free. By classical Smith fixed point theory, the fixed point set of $\mathbb{Z}_p$ on $M$ is connected, that is a simple closed curve $K$ in $M$ (see, e.g., [1, Theorem 4.9, p. 144]). As $\mathbb{Z}_q$ normalizes $\mathbb{Z}_p$, it maps $K$ to itself acting as a group of rotations along or around $K$ (no element can act as a reflection on $K$ because $q > 2$). The actions of $\mathbb{Z}_p$ and $\mathbb{Z}_q$ are standard in a small invariant regular neighbourhood of $K$, therefore all elements of $S$ commute in such a regular neighbourhood. But by a result of Newman ([8]; see also [2]), a periodic diffeomorphism of a manifold which is the identity on a nonempty open subset is the identity. It follows that $S$ is an abelian group which is a contradiction.

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Thus the action of $\mathbb{Z}_p$ on $M$ is free. Let $N = M/\mathbb{Z}_p$ be the quotient manifold, so $M$ is a cyclic regular covering of $N$. As $M$ is a homology 3-sphere, it follows easily that $H_1(N) \cong \pi_1(N)_{\text{ab}} \cong \mathbb{Z}_p$ (homology with coefficients in $\mathbb{Z}$). Then $N$ is a homology lens space, i.e., has the homology and cohomology of a lens space. The action of $\mathbb{Z}_q$ on $M$ projects to an action on $N$, and the induced $\mathbb{Z}_q$-action on $H_1(N) \cong \mathbb{Z}_p$ is given by the action of $\mathbb{Z}_q$ on $\mathbb{Z}_p$ determined by $S$.

Let $f$ be a generator of $\mathbb{Z}_q$ acting as a diffeomorphism on $N$. We will obtain a contradiction by applying a classical argument used in the homotopy classification of lens spaces which shows that the action of the induced map $f_*$ on $H_1(N) \cong \mathbb{Z}_p$ is multiplication by $\pm 1$ (see [7, Theorem 69.3, p. 411], or [4]).

The induced map $f_*$ acts on $H_1(N) \cong \mathbb{Z}_p$ by multiplication with an integer $k$. Then also the induced map $f^*$ on cohomology $H^1(M; \mathbb{Z}/p) \cong \mathbb{Z}_p$, with coefficients in $\mathbb{Z}/p$, is multiplication by $k$, in particular for a generator $u$ of $H^1(N; \mathbb{Z}/p)$ we have $f^*(u) = ku$. Let

$$\beta^* : H^1(N; \mathbb{Z}/p) \to H^2(N; \mathbb{Z}/p)$$

be the boundary map.
be the Bockstein homomorphism associated to the exact coefficient sequence

\[ 0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0. \]

It is easy to see that \( \beta^* \) is an isomorphism in our situation, so \( \beta^*(u) \) is a generator of \( H^2(N; \mathbb{Z}/p) \). As the cup-product for a manifold is a dual pairing, the element \( u \cup \beta^*(u) \) is a generator of \( H^3(N; \mathbb{Z}/p) \cong \mathbb{Z}_p \). The induced map \( f^* \) is the identity on \( H^3(N; \mathbb{Z}/p) \) (because \( f \) is orientation preserving). It follows

\[
u \cup \beta^*(u) = f^*(u \cup \beta^*(u)) = f^*(u) \cup f^*(\beta^*(u)) = f^*(u) \cup \beta^*(f^*(u)) = k^2(u \cup \beta^*(u))
\]

which implies \( k^2 \equiv 1 \mod p \). Thus the induced map \( f_* \) on \( H_1(N) \cong \mathbb{Z}_p \) is multiplication by \( \pm 1 \).

It follows that also the action in \( S = \mathbb{Z}_p \rtimes \mathbb{Z}_q \) of a generator of \( \mathbb{Z}_q \) on \( \mathbb{Z}_p \) is multiplication by \( \pm 1 \). In any case the square of the generator operates trivially which is a contradiction because the action of \( \mathbb{Z}_q \) on \( \mathbb{Z}_p \) is faithful and \( q > 2 \) (in the case \( p = 5 \) we have \( q = 2 \) and \( S \) is a dihedral subgroup of order ten of the alternating group \( A_5 \)).

This finishes the proof of Theorem 1.

References