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Free Products of Completely Positive Maps and Spectral Sets

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1. INTRODUCTION

Let $\mathbb{F}_n$ be the free group on n generators and for each word $w \in \mathbb{F}_n$ let $|w|$ be its length. Haagerup [10] showed that for each $0 < r \leqslant 1$, the function $H_r(w) = r^{|w|}$ is a positive definite function on $\mathbb{F}_n$. Consider the positive definite function $\varphi_r(n) = r^{|n|}$ on $\mathbb{Z}$. Then the free product function $\varphi_r * \varphi_r$ on $\mathbb{F}_2 = \mathbb{Z} * \mathbb{Z}$ coincides with H_r and the result was extended this way by de Michele and Figà-Talamanca [7] and by Bożejko [4, 5]. In [5] it is proved that the free product of the unital positive defined functions $u_i: G_i \to \mathscr{L}(\mathscr{H})$ is still positive defined on the free product group $* G_i$.

The correspondence between the positive defined functions on a discrete group G and the completely positive maps on the full C^*-algebra $C^*(G)$ and the isomorphism $C^*(G_1 * G_2) \simeq C^*(G_1) \check{*} C^*(G_2)$ suggested we consider amalgamated products of unital linear maps on the amalgamated product of a family of unital C^*-algebras over a C^*-subalgebra. In Section 3 we prove that the amalgamated product of a family of unital completely positive B-bimodule maps $\Phi_i: A_i \to C$ is completely positive on the "biggest" amalgamated free product $\check{*}_B A_i$.

As an application of our main result, in Section 2 we obtain some results concerning the dilation of noncommutative families of operators.

2. PRELIMINARIES

Let $(A_i)_{i \in I}$ be a family of unital C^*-algebras containing a common C^*-subalgebra B with $1_{A_i} \in B$ and let $E_i: A_i \to B$ be projections of norm one. Then $A_i = B \oplus A_i^0$, where $A_i^0 = \operatorname{Ker} E_i$, the sum being a sum of

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B-bimodules. Consider the free product $A = *_B A_i$ of A_i with amalgamation over B, which is a unital B-ring. As B-bimodules, one has ([6]):

$$*_B A_i = B \oplus \bigoplus_{n \geqslant 1} \bigoplus_{i_1 \neq \cdots \neq i_n} A^0_{i_1} \otimes_B \cdots \otimes_B A^0_{i_n}.$$

For further reference denote:

$W = B \cup \{c_1 \cdots c_n \mid c_k \in A^0_{i_k},\ i_1 \neq \cdots \neq i_n\}$ the set of words of A;

$W_0 = (W \backslash B) \cup \{1\}$;

$h(c_1 \cdots c_n) = n$ the height of the word $c_1 \cdots c_n$ ($h(b) = 0$ for $b \in B$);

$\tilde{w} = \{1, c_1, c_1 c_2, ..., c_1 c_2 \cdots c_n\}$ for $w = c_1 \cdots c_n \in W$;

$A^0(i_1, ..., i_n) = A^0_{i_1} \oplus A^0_{i_1} \otimes_B A^0_{i_2} \otimes_B A^0_{i_1} \oplus \cdots \oplus A^0_{i_1} \otimes_B \cdots \otimes_B A^0_{i_1}$ for $i_1 \neq \cdots \neq i_n$;

$A_l(i) = B \oplus \bigoplus_{n \geqslant 1} \bigoplus_{i_1 \neq \cdots \neq i_n, i_1 \neq i} A^0_{i_1} \otimes_B \cdots \otimes_B A^0_{i_n}$.

A subset X in W is *complete* if it contains 1 and for every word $w \in X$ it follows that $\tilde{w} \subset X$.

Endowed with the $*$-operation $(c_1 \cdots c_n)^* = c_n^* \cdots c_1^*$ on W which extends on A, $*_B A_i$ is a $*$-complex algebra.

Let C be a unital C^*-algebra with $1_C \in B \subset C$ and let $\Phi_i: A_i \to C$ be unital linear B-bimodule maps, i.e., $\Phi_i(b_1 a b_2) = b_1 \Phi_i(a) b_2$ for all $a \in A_i$, $b_1, b_2 \in B$. It follows that $\Phi_i|_B = \mathrm{id}_B$ and the amalgamated product map $\Phi = *_B \Phi_i: *_B A_i \to C$, defined by $\Phi|_B = \mathrm{id}_B$ and $\Phi(c_1 \cdots c_n) = \Phi_{i_1}(c_1) \cdots \Phi_{i_n}(c_n)$ for $c_k \in A^0_{i_k}$, $i_1 \neq \cdots \neq i_n$ is a well-defined unital linear B-bimodule map. Note that

$$\Phi(ax) = \Phi(a)\,\Phi(x) \qquad \text{for all} \quad a \in A_i,\ x \in A_l(i). \tag{1}$$

A $*$-algebra A is said to satisfy the *Combes axiom* if for each $x \in B$ there is $\lambda(x) > 0$ such that $b^* x^* x b \leqslant \lambda(x) b^* b$ for all $b \in B$. It is easy to check the following analogue of the classical Stinespring dilation:

Let A be a unital $*$-algebra satisfying the Combes axiom and let $\Phi: A \to \mathscr{L}(\mathscr{H})$ be a unital completely positive linear map. Then there exist a Hilbert space K, a $*$-representation $\pi: A \to \mathscr{L}(\mathscr{K})$ and an isometry $V \in \mathscr{L}(\mathscr{H}, \mathscr{K})$ such that

(i) *$\Phi(x) = V^* \pi(x) V$ for all $x \in A$;*

(ii) *$\mathscr{K}$ is the closed linear span of $\pi(A) V \mathscr{H}$.*

Each unital C^*-algebra A_i is spanned by the unitary group $\mathscr{U}(A_i)$, hence the unital $*$-algebra $A = *_B A_i$ is spanned by the products $u_1 \cdots u_m$ with $u_k \in \mathscr{U}(A_{i_k})$, $i_1 \neq \cdots \neq i_m$. Consequently, the unitary group $\mathscr{U}(A)$ spans A and A satisfies the Combes axiom. Actually, the existence of the Stinespring dilation for a unital completely positive linear map Φ on $*_B A_i$ yields the extension of Φ on $\check{*}_B A_i$, the completion of $*_B A_i$ in the greatest C^*-norm

$$\|a\| = \sup\{\|\pi(a)\| \mid \pi \text{ } *\text{-representation of } *_B A_i\}.$$

3. The Main Result

We prove the following theorem.

Theorem 3.1. *Let Φ_i be unital completely positive linear B-bimodule maps. Then their amalgamated product $\Phi = *_B \Phi_i$ is still completely positive on $*_B A_i$.*

Let us outline first the main ideas of the proof.

In Proposition 3.2 we show that $\Phi = *_B \Phi_i$ is completely positive on the $*$-algebra $A = *_B A_i$. We start with the key remark that the positivity of the kernel $K(x, y) = \Phi(y^* x)$ on the reduced words of A yields the complete positivity of Φ. The positivity of K on W_0 is showed combining the action of an amalgamated product of groups on the associated tree [12] used in the group case in [5] with some classical facts concerning the complete positivity. Since the $*$-algebra A satisfies the Combes axiom, the map Φ can be extended to a completely positive map on the C^*-algebra $\check{*}_B A_i$ via a Stinespring construction.

Proposition 3.2. *Let Φ_i be unital completely positive linear B-bimodule maps. Then their amalgamated product $\Phi = *_B \Phi_i$ is still completely positive on the $*$-algebra $A = *_B A_i$.*

Proof. We may assume $I_H \in C \subset \mathscr{L}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$.

For further use, note that each positive element $[y_{ij}]$ in $M_m(\mathscr{L}(\mathscr{H}))$ is a sum of elements of the form $[a_i^* a_j]$, with $a_1, ..., a_m \in \mathscr{L}(\mathscr{H})$. Actually $y_{ij} = \sum_{k=1}^{n_0} a_{ik}^* a_{jk}$ and taking $V_i \in \mathscr{L}(\mathscr{H}, \mathscr{H}^{(n_0)})$, $V_i h = (a_{i1} h, ..., a_{in_0} h)$ we obtain $y_{ij} = V_j^* V_i$ for all $i, j = 1, ..., m$.

Let $x_1, ..., x_n \in A$ and $h_1, ..., h_n \in H$. We must check that

$$\sum_{i,j=1}^{n} (\Phi(x_j^* x_i) h_i \mid h_j) \geqslant 0.$$

Note that each x_i can be written as $x_i = \sum_{r=1}^{N} a_r b_{ir}$, with $a_j \in W_0$, $b_{ir} \in B$ (some b_{ir} could be zero).

Denote

$$\xi_{ir} = b_{ir} h_i, \qquad \text{for} \quad r = 1, ..., N, \ i = 1, ..., n;$$

$$\eta_i = (\xi_{i1}, ..., \xi_{iN}) \in \mathscr{H}^{(N)}, \qquad \text{for} \quad i = 1, ..., n.$$

Since

$$([\Phi(a_s^* a_r)]\eta_i \mid \eta_j) = \sum_{r,s=1}^{N} (\Phi(a_s^* a_r)\xi_{ir} \mid \xi_{js})$$

it follows that

$$\begin{aligned}\sum_{i,j=1}^{n} (\Phi(x_j^* x_i)h_i \mid h_j) &= \sum_{i,j=1}^{n} \sum_{r,s=1}^{N} (\Phi(b_{js}^* a_s^* a_r b_{ir})h_i \mid h_j)\\ &= \sum_{i,j=1}^{n} \sum_{r,s=1}^{N} (\Phi(a_s^* a_r)b_{ir}h_i \mid b_{js}h_j)\\ &= \sum_{i,j=1}^{n} ([\Phi(a_s^* a_r)]\eta_i \mid \eta_j).\end{aligned}$$

Hence it is enough to check that for all finite sets X in W_0 and all maps $f: X \to H$ we have

$$S_X = \sum_{x,y \in X} (\Phi(y^*x) f(x) \mid f(y)) \geqslant 0. \tag{2}$$

Since each finite set $X \subset W_0$ is included in the complete finite set $X = \bigcup_{w \in X} \tilde{w} \subset W_0$, we may assume that X is complete. Note also that Φ is a selfadjoint map.

If X contains an element, then $X = \{1\}$, hence

$$S_X = (\Phi(1) f(1) \mid f(1)) = \|f(1)\|^2 \geqslant 0.$$

If X contains two elements, then $X = \{1, a\}$, with $a \in A_i^0$, hence the 2-positivity of Φ_i yields

$$S_X = \sum_{x,y \in X} (\Phi_i(y^*x) f(x) \mid f(y)) \geqslant 0.$$

By induction, let $X \subset W_0$ be a complete finite set, $f: X \to \mathscr{H}$ be a map, choose a word $w_0 = a_1 \cdots a_m$ $(a_k \in A_{i_k}^0,\ i_1 \neq \cdots \neq i_m)$ such that $m = h(w_0) = \max_{w \in X} h(w) \geqslant 1$ and define

$$X_2 = X \cap \{b_1 \cdots b_m \mid b_k \in A_{i_k}^0,\ k = \overline{1, m}\};$$

$$X_1 = X \setminus X_2.$$

The complete finite set X_1 does not contain w_0, hence Card $X_1 <$ Card X and $S_{X_1} \geqslant 0$. Consequently, there exist the operators $V_x \in \mathscr{L}(\mathscr{H}, \mathscr{H}^{(\infty)})$ such that

$$\Phi(y^*x) = V_y^* V_x \qquad \text{for all} \quad x, y \in X_1. \tag{3}$$

For further use, we remark that for any $x_1 \in X_1$, $x_2 = x_0 a \in X_2$ with $x_0 = c_1 \cdots c_{m-1}$, $c_k \in A^0_{i_l}$, $k = \overline{1, m-1}$, $a \in A^0_{i_m}$ we have $x_0^* x_1 \in A_l(i_m)$, hence

$$\Phi(x_2^* x_1) = \Phi(a^* x_0^* x_1) = \Phi(a^*)\, \Phi(x_0^* x_1). \tag{4}$$

Let $x_1 = d_1 \cdots d_n \in X_1$ ($d_k \in A^0_{j_k}$, $j_1 \neq \cdots \neq j_n$, $n \leqslant m$) and let n_0 be the greatest index with $i_r = j_r$ for any $r \leqslant n_0$. Then, it is easy to verify that

$$c^*_{n_0} \cdots c^*_1 d_1 \cdots d_{n_0} \in B \oplus A^0_{(i_{n_0}, \ldots, i_1)}. \tag{5}$$

The following two cases may appear:

(i) For $n_0 < m-1$, (5) yields ($i_{n_0+1} \neq i_{n_0}$, $j_{n_0+1} \neq j_{n_0} = i_{n_0}$, $i_{n_0+1} \neq j_{n_0+1}$)

$$\begin{aligned} x_0^* x_1 = c^*_{m-1} \cdots c_1^* d_1 \cdots d_n \in A^0_{i_{m-1}} \otimes \cdots \otimes A^0_{i_{n_0+1}} \\ \otimes (B \oplus A^0_{(i_{n_0}, \ldots, i_1)}) \otimes A^0_{j_{n_0+1}} \otimes \cdots \otimes A^0_{j_n} \subset A_l(i_m). \end{aligned}$$

(ii) For $n_0 = m-1$, the maximality of $h(w_0)$ in X yields $m = n$. Since $a^*_{m-1} \cdots a_1^* c_1 \cdots c_{m-1} \in B \oplus A^0_{(i_{m-1}, \ldots, i_1)}$ and $j_m \neq i_m$, $i_{m-1} \neq i_m$, we obtain

$$x_0^* x_1 \in A^0_{j_m} \oplus A^0_{(i_{m-1}, \ldots, i_1)} \otimes A^0_{j_m} \subset A_l(i_m).$$

Let us remark also that for any $x_0 = c_1 \cdots c_{m-1}$, $x_0' = c_1' \cdots c'_{m-1}$, $c_k, c_k' \in A^0_{i_k}$, $i_1 \neq \cdots \neq i_{m-1}$ we get

$$x_0'^* x_0 = b(x_0, x_0') + d(x_0, x_0'), \tag{6}$$

where $d(x_0, x_0') \in A^0_{(i_{m-1}, \ldots, i_1)}$ and the kernel $b: X_0 \times X_0 \to B$ with $X_0 = \{x_0 = c_1 \cdots c_{m-1} \mid c_k \in A^0_{i_k}$ and there exists an $a \in A^0_{i_m}$ with $x_0 a \in X_2\}$ is positive definite ($i_1, \ldots, i_m$ being fixed by the choice of w_0 in X). Indeed, an inductive argument shows us that

$$x_0'^* x_0 = b_{m-1}(x_0, x_0') + d(x_0, x_0'),$$

where $d(x_0, x_0') \in A^0_{(i_{m-1}, \ldots, i_1)}$, $b_k(x_0, x_0') = E_{i_k}\ (c_k'(x_0')^*\, b_{k-1}(x_0, x_0')\, c_k(x_0))$ for all $k = 2, \ldots, m-1$ and $b_0(x_0, x_0') = 1$. Since the conditional expectations E_{i_k} are completely positive and the matrix with all entries equal to 1 is positive, the positivity of b on X_0 follows inductively. Consequently, we get a family $\{\beta_w(x_0)\}_{w \in X_0}$ in B such that

$$b(x_0, x_0') = \sum_{w \in X_0} \beta_w(x_0')^* \beta_w(x_0) \qquad \text{for all} \quad x_0, x_0' \in X_0.$$

Moreover, for $x=x_0a,\ x'=x_0'a'\in X_2$ as above we get

$$\begin{aligned}\Phi(x'^*x)&=\Phi(a'^*x_0'^*x_0a)=\Phi(a'^*(b(x_0,x_0')+d(x_0,x_0'))a)\\&=\Phi(a'^*b(x_0,x_0')a)+\Phi(a'^*d(x_0,x_0')a)\\&=\Phi(a'^*b(x_0,x_0')a)+\Phi(a'^*)\,\Phi(d(x_0,x_0'))\,\Phi(a)\\&=\Phi(a'^*b(x_0,x_0')a)+\Phi(a'^*(x_0'^*x_0-b(x_0,x_0'))a)\\&=\Phi(a'^*)\,\Phi(x_0'^*x_0)\,\Phi(a)+\Phi_{i_m}(a'^*b(x_0,x_0')a)\\&\quad-\Phi_{i_m}(a'^*)\,b(x_0,x_0')\Phi_{i_m(a)}.\end{aligned}\tag{7}$$

Applying the Kadison inequality for the unital completely positive map $\Phi'\colon M_N(A_{i_m})\to\mathscr{L}(\mathscr{H}^{(N)})$, $\Phi'([a_{ij}])=[\Phi_{i_m}(a_{ij})]$ for $[a_{ij}]\in M_N(A_{i_m})$ we obtain

$$\Phi'(x^*)\,\Phi'(x)\leqslant\Phi'(x^*x)\qquad\text{for all}\quad x\in M_N(A_{i_m}).$$

Letting

$$x=\begin{bmatrix}w_1&\cdots&w_N\\0&\cdots&0\\\cdots&\cdots&\cdots\\0&\cdots&0\end{bmatrix},\ w_1,\dots,w_N\in A_{i_m}$$

one obtains

$$\sum_{i,j=1}^{N}(\Phi_{i_m}(w_j^*w_i)\xi_i\mid\xi_j)\geqslant\sum_{i,j=1}^{N}(\Phi_{i_m}(w_j^*)\,\Phi_{i_m}(w_i)\xi_i\mid\xi_j)$$
$$\text{for all}\quad w_1,\dots,w_N\in A_{i_m},\ \xi_1,\dots,\xi_N\in\mathscr{H}.$$

In particular, for $N=\operatorname{card}X_2$ we obtain

$$\begin{aligned}&\sum_{x_0a,x_0'a'\in X_2}(\Phi_{i_m}(a'^*\beta_w(x_0')^*\,\beta_w(x_0)a)\,f(x_0a)\mid f(x_0'a'))\\&\quad\geqslant\sum_{x_0a,x_0'a'\in X_2}(\Phi_{i_m}(a'^*\beta_w(x_0')^*)\,\Phi_{i_m}(\beta_w(x_0)a)\,f(x_0a)\mid f(x_0'a'))\\&\quad=\sum_{x_0a,x_0'a'\in X_2}(\Phi_{i_m}(a'^*)\,\beta_w(x_0')^*\,\beta_w(x_0)\,\Phi_{i_m}(a)\,f(x_0a)\mid f(x_0'a')),\end{aligned}$$

hence, summing over $w\in X_0$,

$$\begin{aligned}&\sum_{x_0a,x_0'a'\in X_2}(\Phi_{i_m}(a'^*b(x_0,x_0')a)\,f(x_0a)\mid f(x_0'a'))\\&\quad\geqslant\sum_{x_0a,x_0'a'\in X_2}(\Phi_{i_m}(a'^*)\,b(x_0,x_0')\,\Phi_{i_m}(a)\,f(x_0a)\mid f(x_0'a')).\end{aligned}\tag{8}$$

Taking into account (7), (8), and (3) we obtain $(x_0 \in X_1)$

$$\begin{aligned}\sum_{x_0a,\, x_0'a' \in X_2} &(\Phi(a'^* x_0'^* x_0 a) f(x_0 a) \mid f(x_0' a')) \\ &\geqslant \sum_{x_0a,\, x_0'a' \in X_2} (\Phi(a'^*)\, \Phi(x_0'^* x_0)\, \Phi(a) f(x_0 a) \mid f(x_0' a')) \\ &= \left\| \sum_{x_0 a \in X_2} V_{x_0} \Phi(a) f(x_0 a) \right\|^2. \end{aligned} \tag{9}$$

Finally, (3), (9), and (4) yield

$$\begin{aligned} S_X = &\sum_{x,\, y \in X_1} (\Phi(y^* x) f(x) \mid f(y)) \\ &+ \sum_{x_0a,\, x_0'a' \in X_2} (\Phi(a'^* x_0'^* x_0 a) f(x_0 a) \mid f(x_0' a')) \\ &+ 2 \operatorname{Re} \sum_{\substack{x \in X_1 \\ x_0 a \in X_2}} (\Phi(a^* x_0^* x) f(x) \mid f(x_0 a)) \\ \geqslant &\left\| \sum_{x \in X_1} V_x f(x) \right\|^2 + \left\| \sum_{x_0 a \in X_2} V_{x_0} \Phi(a) f(x_0 a) \right\|^2 \\ &+ 2 \operatorname{Re} \sum_{\substack{x \in X_1 \\ x_0 a \in X_2}} (V_x f(x) \mid V_{x_0} \Phi(a) f(x_0 a)) \\ = &\left\| \sum_{x \in X_1} V_x f(x) + \sum_{x_0 a \in X_2} V_{x_0} \Phi(a) f(x_0 a) \right\|^2 \geqslant 0. \quad \blacksquare \end{aligned}$$

COROLLARY 3.3. *Let A_i be unital C^*-algebras, $\mathscr{S}_i \subset A_i$ be unital linear subspaces and $L_i: \mathscr{S}_i \to \mathscr{L}(\mathscr{H})$ be unital completely contractive maps. Then the L_i extend to a completely contractive map on the free product C^*-algebra $\check{*} A_i$.*

Proof. By the Arveson extension theorem, each L_i extends to a unital completely positive map $\Phi_i: A_i \to \mathscr{L}(\mathscr{H})$ and by the previous theorem, taking $B = \mathbb{C}$ and each E_i a state on A_i, the map $* \Phi_i$ is unital and completely positive on $* A_i$. $\blacksquare$

Remark 1. Actually, there is a canonical way to describe the GNS representation of the conditional expectation $E = *_B E_i$, via the construction from [14, Section 5], that we shall recall briefly.

For each i, denote by $\mathscr{H}_i$ the separation and completion of A_i with respect to $\|a\|_{E_i} = \|E_i(a^* a)\|^{1/2}$. The B-valued inner product $\langle a, b \rangle = E_i(b^* a)$ on A_i yields an inner product on $\mathscr{H}_i$ which is a Hilbert

B-module. Each A_i can be expressed as a direct sum of B-bimodules $A_i = B \oplus \operatorname{Ker} E_i$ and this decomposition gives rise to the orthogonal direct sum of Hilbert B-modules $\mathscr{H}_i = B \oplus \mathscr{H}_i^0$. The left multiplication on A_i yields the GNS $*$-homomorphism $\pi_i\colon A_i \to \mathscr{L}(\mathscr{H}_i)$ with $E_i(a) = \langle \pi_i(a)\xi_i, \xi_i\rangle$. Denoting $\chi_i = \pi_i|_B$ one obtains $\chi_i(b')(b \oplus h) = b'b \oplus \chi_i^0(b')h$. One denotes $\xi_i = 1_B \oplus 0 \in B \oplus \mathscr{H}_i^0$ and one defines $(\mathscr{H}, \xi) = *(\mathscr{H}_i, \xi_i)$ by

$$\mathscr{H} = B \oplus \bigoplus_{n \geqslant 1} \bigoplus_{i_1 \neq \cdots \neq i_n} \mathscr{H}_{i_1}^0 \otimes_B \cdots \otimes_B \mathscr{H}_{i_n} = B \oplus \mathscr{H}^0$$

$$\xi = 1_B \oplus 0 \in \mathscr{H}.$$

One considers

$$\mathscr{H}_l(i) = B \oplus \bigoplus_{n \geqslant 1} \bigoplus_{\substack{i_1 \neq \cdots \neq i_n \\ i_1 \neq i}} \mathscr{H}_{i_1}^0 \otimes_B \cdots \otimes_B \mathscr{H}_{i_n}^0$$

and the isomorphisms $V_i\colon \mathscr{H} \to \mathscr{H}_i \otimes_B \mathscr{H}_l\,(\mathrm{i})$ defined by

$$V_i \xi = \xi_i \otimes \xi$$

$$V_i(h_1 \otimes \cdots \otimes h_n) = \begin{cases} h_1 \otimes (h_2 \otimes \cdots \otimes h_n) & \text{if } i_1 = i,\ n \geqslant 2 \\ h_1 \otimes \xi & \text{if } i_1 = i,\ n = 1 \\ \xi_i \otimes (h_1 \otimes \cdots \otimes h_n) & \text{if } i_1 \neq i, \end{cases}$$

where $h_k \in \mathscr{H}_{i_k}^0$, $i_1 \neq \cdots \neq i_n$.

Define the $*$-homomorphisms $\sigma_i\colon A_i \to \mathscr{L}(\mathscr{H})$ by $\sigma_i = \lambda_i \circ \pi_i$, where $\lambda_i\colon \mathscr{L}(\mathscr{H}_i) \to \mathscr{L}(\mathscr{H})$, $\lambda_i(T) = V_i^{-1}(T \otimes I) V_i$, and let $\sigma = *_B \sigma_i$.

A simple calculus shows that for each word $c = c_1 \cdots c_m (c_k \in A_{i_k}^0$, $i_1 \neq \cdots \neq i_m)$ one obtains

$$\sigma_{i_1}(c_1) \cdots \sigma_{i_m}(c_m)\xi = E_{i_1}(c_1) \cdots E_{i_m}(c_m)\xi + h_0, \qquad \text{with} \quad h_0 \in \mathscr{H}^0.$$

Consequently

$$\langle \sigma(c)\xi, \xi\rangle = \langle E_{i_1}(c_1) \cdots E_{i_m}(c_m)\xi, \xi\rangle = E_{i_1}(c_1) \cdots E_{i_m}(c_m) = E(c).$$

Remark 2. Let G be a discrete group and let $C^*(G)$ be the associated C^*-algebra, endowed with the canonical trace. Denote by ε_g, $g \in G$ the canonical unitaries of $C^*(G)$. There is a 1–1 correspondence $\varphi \leftrightarrow \Phi$ between the positive defined functions on G and the completely positive linear maps on $C^*(G)$ given by $\Phi(\varepsilon_g) = \varepsilon(g)$, $g \in G$.

Let G_i be discrete groups and let $\varphi_i\colon G_i \to \mathscr{L}(\mathscr{H})$ be unital positive defined functions. Then the corresponding $\Phi_i\colon C^*(Gi) \to \mathscr{L}(\mathscr{H})$ are completely positive and their free product $*\Phi_i\colon \check{*} C^*(G_i) = C^*(*G_i) \to \mathscr{L}(\mathscr{H})$

related to the canonical traces on $C^*(G_i)$ is still completely positive. Since $\Phi = *\Phi_i$ corresponds to $\varphi = *\varphi_i$ it follows that $\varphi = *\varphi_i$ is positive defined on $*G_i$. Hence our result implies Theorem 7.1 in [5].

4. Dilations for Noncommutative Families of Operators

Let Ω be a compact set in the complex plane and let $R(\Omega)$ denote the algebra of rational functions whose poles lie off Ω, endowed with the norm $\|f\| = \sup_{z\in\Omega} |f(z)| = \sup_{z\in\partial\Omega} |f(z)|$, where $\partial\Omega$ denotes the boundary of Ω.

If T is a bounded linear operator on the Hilbert space $\mathscr{H}$ whose spectrum $\sigma(T)$ is contained in Ω, there is a unital homomorphism $\rho_T\colon R(\Omega)\to\mathscr{L}(\mathscr{H})$, $\rho_T(f)=f(T)$. When ρ_T is a (completely) contractive map, Ω is called a *(complete) spectral* set for T.

The commutative C^*-algebra $C(\partial\Omega)$ contains the selfadjoint subspace $\mathscr{S} = R(\Omega)+R(\Omega)^*$. When $\mathscr{S}$ is dense in $C(\partial\Omega)$, $R(\Omega)$ is called a Dirichlet algebra on $\partial\Omega$ and this is the case for simply connected sets.

By the classical von Neumann inequality, the closed unit disk $\bar{\mathbb{D}} = \{z\in\mathbb{C} \mid |z|\leqslant 1\}$ with the boundary $\mathbb{T}$ is a spectral set for every contraction T; hence the homomorphism ρ_T extends to a positive linear map on $C(\mathbb{T})$, which is completely positive by the commutativity of $C(\mathbb{T})$. The Stinespring dilation of this map yields the Szökefalvi-Nagy unitary dilation of T.

The same technique shows that whether the operator T has a spectral set Ω with $R(\Omega)$ Dirichlet algebra on $\partial\Omega$, then it has a $\partial\Omega$-normal rational dilation [9]. This means that there are a Hilbert space $\mathscr{K}$ containing $\mathscr{H}$ and a normal operator $N\in\mathscr{L}(\mathscr{K})$ with $\sigma(N)\subset\partial\Omega$ such that

$$f(T) = P_{\mathscr{H}} f(N)|_{\mathscr{H}} \qquad \text{for all} \quad f\in R(\Omega).$$

When Ω is a symmetric annulus, this result is still true; this is a profound result of Agler [1]. These results show that in these cases the complete contractivity of ρ_T is implied by his contractivity.

Pick for each compact Ω a probability measure μ on $\partial\Omega$ and denote $C(\partial\Omega)^0 = \{f\in C(\partial\Omega) \mid \int_{\partial\Omega} f(z)\,d\mu(z)=0\}$. When $\partial\Omega$ is a finite system of rectifiable Jordan curves and $z_0\in\mathring{\Omega}$, we can take

$$\mu_{z_0}(f) = \frac{1}{2\pi i}\int_{\partial\Omega}\frac{f(z)}{z-z_0}\,dz \qquad \text{for all} \quad f\in R(\Omega).$$

In this case all analytic polynomials that vanish in z_0 are contained in $C(\partial\Omega)^0$. When $\Omega=\bar{\mathbb{D}}$ the measure μ_0 on $\mathbb{T}$ is exactly the canonical trace on $C(\mathbb{T}) = C^*(\mathbb{Z})$.

Let $(\Omega_i)_i$ be spectral sets for $T_i \in \mathscr{L}(\mathscr{H})$ and let μ_i be probability measures on $\partial\Omega_i$. If $f = f_1 \cdots f_m \in *C(\partial\Omega_i)$ with $f_k \in C(\partial\Omega_{i_k})^0$, $f_k \in R(\Omega_{i_k}) + R(\Omega_{i_k})^*$, $i_1 \neq \cdots \neq i_m$; denote

$$f((T_i)_i) = f_1(T_{i_1}) \cdots f_m(T_{i_m}).$$

The function $z \to z$ on Ω_i viewed in $*C(\partial\Omega_i)$ is denoted by ι_i. Denote also

$$\mathscr{S}(*\Omega_i) = \{f = f_1 \cdots f_m \mid f_k \in C(\partial\Omega_{i_k})^0, f_k \in R(\Omega_{i_k}) + R(\Omega_{i_k})^*, i_1 \neq \cdots \neq i_m\}.$$

Let $(T_i)_i$ be bounded linear operators on the Hilbert space $\mathscr{H}$, let Ω_i be compact sets in $\mathbb{C}$ with $\sigma(T_i) \subset \Omega_i$ for every i and let μ_i be probability measures on $\partial\Omega_i$. Suppose that there exist a Hilbert space $\mathscr{K}$ which contains $\mathscr{H}$ and $N_i \in \mathscr{L}(\mathscr{K})$ normal operators with $\sigma(N_i) \subset \partial\Omega_i$ for every i such that for each $f \in \mathscr{S}(*\Omega_i)$ it follows that

$$f((T_i)_i) = P_{\mathscr{H}} f((N_i)_i) \mid_{\mathscr{H}}.$$

Then $(N_i)_i$ is called a *$(\partial\Omega_i)_i$-normal rational dilation* for $(T_i)_i$. When $\mathscr{K}$ is the closed linear span of $\{f((T_i)_i)\mathscr{H} \mid f \in \mathscr{S}(*\Omega_i)\}$ the dilation $(N_i)_i$ is *minimal.*

PROPOSITION 4.1. *Let $(\Omega_i)_i$ be complete spectral sets for $T_i \in \mathscr{L}(\mathscr{H})$, each $\partial\Omega_i$ being endowed with a probability measure μ_i.*
Then $(T_i)_i$ has a minimal $(\partial\Omega_i)_i$-normal rational dilation and

$$\|f((T_i)_i)\| \leqslant \|f\|_{*C(\partial\Omega_i)} \quad \textit{for every} \quad f \in \mathscr{S}(*\Omega_i).$$

Proof. The unital completely contractive homomorphisms ρ_{T_i} extend to the completely positive linear maps $\Phi_i \colon C(\partial\Omega_i) \to \mathscr{L}(\mathscr{H})$. By Theorem 3.1, the free product $\Phi = *\Phi_i \colon *C(\partial\Omega_i) \to \mathscr{L}(\mathscr{H})$ related to the probability measures μ_i is still completely positive. Let $(\pi, \mathscr{K})$ be the Stinespring dilation of Φ and let $N_i = \pi(\iota_i)$. It is straightforward that $\sigma(N_i) \subset \partial\Omega_i$ and for each $f = f_1 \cdots f_m \in \mathscr{S}(*\Omega_i)$ one obtains:

$$\begin{aligned} f((T_i)_i) &= f_1(T_{i_1}) \cdots f_m(T_{i_m}) = \rho_{T_{i_1}}(f_1) \cdots \rho_{T_{i_m}}(f_m) \\ &= \Phi_{i_1}(f_1) \cdots \Phi_{i_m}(f_m) = \Phi(f) = P_{\mathscr{H}} \pi(f) \mid_{\mathscr{H}} \\ &= P_{\mathscr{H}} \pi(f_1) \cdots \pi(f_m) \mid_{\mathscr{H}} = P_{\mathscr{H}} f((N_i)_i) \mid_{\mathscr{H}} \end{aligned}$$

and

$$\|f((T_i)_i)\| = \|\Phi(f)\| \leqslant \|f\|_{*C(\partial\Omega_i)}. \quad \blacksquare$$

Whether T_i are contractions, $\Omega_i = \bar{\mathbb{D}}$ for every i and f is a free polynomial, this inequality was obtained in [5].

Remark. Let $(U_i)_i$ be a minimal unitary dilation for $(T_i)_i$ and let $\mathscr{K}_-$ be the closed linear span of $\{U_{i_1}^{n_1} \cdots U_{i_m}^{n_m} \mathscr{H} \mid n_k \in \mathbb{Z}_-, i_1 \neq \cdots \neq i_m\}$. Then $\mathscr{K}_-$ is invariant for each U_i^*; hence $V_i = U_i^*|_{\mathscr{K}_-}$ are isometries, $W_i = V_i^*$ are coisometries, and

$$\begin{aligned}(W_i h \mid U_{i_1}^{n_1} \cdots U_{i_m}^{n_m} h') &= (V_i^* h \mid U_{i_1}^{n_1} \cdots U_{i_m}^{n_m} h') = (h \mid U_i U_{i_1}^{n_1} \cdots U_{i_m}^{n_m} h') \\ &= (h \mid T_i^* T_{i_1}^{*\,-n_1} \cdots T_{i_m}^{*\,-n_m} h') \\ &= (T_i h \mid T_{i_1}^{*\,-n_1} \cdots T_{i_m}^{*\,-n_m} h') = (T_i h \mid U_{i_1}^{n_1} \cdots U_{i_m}^{n_m} h'),\end{aligned}$$

for every $h, h' \in H$, $n_k \in \mathbb{Z}_-$, $i_1 \neq \cdots \neq i_m$. Consequently W_i are coisometries on $\mathscr{K}_-$ with $W_i|_{\mathscr{H}} = T_i$. This is a well-known remark of Durszt and Szökefalvi-Nagy [8].

Such a minimal unitary dilation for $(T_i)_i$ is unique. Given two minimal unitary dilations $(U_i)_i$ and $(U_i')_i$ on $\mathscr{K}$ and respectively $\mathscr{K}'$, there is a unitary $\Theta: \mathscr{K} \to \mathscr{K}'$ such that $\Theta U_i = U_i' \Theta$.

Actually, the previous proposition and the results of Foias and Agler imply

COROLLARY 4.2. *Let $(\Omega_i)_i$ be spectral sets for $T_i \in \mathscr{L}(\mathscr{H})$, with $R(\Omega_i)$ Dirichlet algebra on $\partial\Omega_i$ or Ω_i symmetric annulus for every i. Then $(T_i)_i$ has a $(\partial\Omega_i)_i$-normal rational dilation.*

An operator $T \in \mathscr{L}(\mathscr{H})$ has a *unitary ρ-dilation* $(\rho > 1)$ whether there exist a Hilbert space $\mathscr{K}$ containing $\mathscr{H}$ and a unitary $U \in \mathscr{L}(\mathscr{K})$ such that

$$T^n = \rho P_{\mathscr{H}} U^n|_{\mathscr{H}}, \qquad \text{for every} \quad n \geq 1.$$

For example, by a result of Berger, each operator T having the numerical radius $w(T) \leq 1$ has a 2-unitary dilation ([11, Proposition 3.13]).

Let $T \in \mathscr{L}(\mathscr{H})$ have a ρ-unitary dilation. Denoting by V the isometry $\mathscr{H} \subsetneq \mathscr{K}$, one obtains:

$$\begin{bmatrix} \rho T^* \cdots T^{*n} \\ T\rho \cdots T^{*n-1} \\ \cdots\cdots\cdots\cdots \\ T^n T^{n-1} \cdots \rho \end{bmatrix}$$

$$= \rho \begin{bmatrix} V & & & \\ & V & & \\ & & \ddots & \\ & & & V \end{bmatrix}^* \cdot \begin{bmatrix} I & U^* \cdots U^{*n} \\ U & I \cdots U^{*n-1} \\ & \cdots\cdots\cdots\cdots \\ U^n & U^{n-1} \cdots I \end{bmatrix} \cdot \begin{bmatrix} V & & & \\ & V & & \\ & & \ddots & \\ & & & V \end{bmatrix} \geq 0.$$

This inequality yields the positivity of the unital linear map $\Phi : \mathscr{P}_+ + \bar{\mathscr{P}}_+ \to \mathscr{L}(\mathscr{H})$, $\Phi(p + \bar{q}) = (1/\rho)[(p(T) + q(T)^*) + (\rho - 1)(p(0) + \overline{q(0)}) \cdot I]$, where $\mathscr{P}_+$ is the algebra of analytic polynomials (for details see [11; 2.6, 3.13]).

COROLLARY 4.3. *Let $T_i \in \mathscr{L}(\mathscr{H})$ be operators having unitary ρ-dilations. Then there exist a Hilbert space $\mathscr{K}$ containing $\mathscr{H}$ and unitaries $U_i \in \mathscr{L}(\mathscr{K})$ such that for each $p = p_1 \cdots p_m$; p_k polynomial, $p_k(0) = 0$, $i_1 \neq \cdots \neq i_m$ we have*

$$p((T_i)_i) = \rho^m P_{\mathscr{H}}\, p((U_i)_i|_{\mathscr{H}}.$$

Proof. Each positive linear map $\Phi_i \colon C(\mathbb{T}) \to \mathscr{L}(\mathscr{H})$, where $\Phi_i(p + \bar{q}) = (1/\rho)[(p(T_i) + q(T_i)^*) + (\rho - 1)(p(0) + \overline{q(0)}) \cdot I]$ if $p, q \in \mathscr{P}_+$ is completely positive and unital; hence $\Phi = * \Phi_i$ is completely positive on $* C(\mathbb{T})_i$. Consequently the Stinespring dilation $(\pi, \mathscr{K})$ of Φ yields the unitaries $U_i = \pi(\iota_i)$ and for each free polynomial $p = p_1 \cdots p_m$ we have

$$p((T_i)_i) = p_1(T_{i_1}) \cdots p_m(T_{i_m}) = \rho^m \Phi(p) = \rho^m P_{\mathscr{H}}\, p((U_i)_i)|_{\mathscr{H}}. \quad \blacksquare$$

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