Journal of Combinatorial Theory, Series A 117 (2010) 345–353

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Note

Counting labelled trees with given indegree sequence

Rosena R.X. Du [a], Jingbin Yin [b]

[a] Department of Mathematics, East China Normal University, Shanghai 200241, PR China
[b] Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

ARTICLE INFO

Article history:
Received 11 January 2008
Available online 26 June 2009

Keywords:
Labelled tree
Indegree sequence
Partition
Bijection
Lattice

ABSTRACT

For a labelled tree on the vertex set $[n] := \{1, 2, \ldots, n\}$, define the direction of each edge ij to be $i \to j$ if $i < j$. The indegree sequence of T can be considered as a partition $\lambda \vdash n - 1$. The enumeration of trees with a given indegree sequence arises in counting secant planes of curves in projective spaces. Recently Ethan Cotterill conjectured a formula for the number of trees on $[n]$ with indegree sequence corresponding to a partition λ. In this paper we give two proofs of Cotterill's conjecture: one is "semi-combinatorial" based on induction, the other is a bijective proof.

1. Introduction

For a labelled tree on the vertex set $[n] := \{1, 2, \ldots, n\}$, define the direction of each edge ij as $i \to j$ if $i < j$. The indegree sequence of T can be considered as a partition $\lambda \vdash n - 1$. The problem of counting labelled trees with a given indegree sequence was encountered by Ethan Cotterill [2] when counting secant planes of curves in projective spaces. Write $\lambda = \langle 1^{m_1} 2^{m_2} \cdots \rangle$ if λ has m_i parts equal to i. Given $\lambda = \langle 1^{m_1} 2^{m_2} \cdots \rangle \vdash n - 1$, let k be the number of parts of λ, and a_λ be the number of trees on $[n]$ with indegree sequence corresponding to λ. Cotterill [2, p. 29] conjectured the following result:

$$a_\lambda = \frac{(n-1)!^2}{(n-k)!1!^{m_1}2!^{m_2}\cdots m_1!m_2!\cdots}. \tag{1.1}$$

Note that the above formula can also be written as

$$a_\lambda = \frac{(n-1)!}{(n-k)!} \cdot \frac{(n-1)!}{1!^{m_1}m_1!2!^{m_2}m_2!\cdots}, \tag{1.2}$$

in which the second factor on the right-hand side counts the number of partitions π of an $(n-1)$-element set of *type* λ, i.e., the block sizes of π are $\lambda_1, \lambda_2, \ldots$. This suggests that it may help to

E-mail addresses: rxdu@math.ecnu.edu.cn (R.R.X. Du), jbyin@math.mit.edu (J. Yin).

0097-3165/$ – see front matter
doi:10.1016/j.jcta.2009.03.020

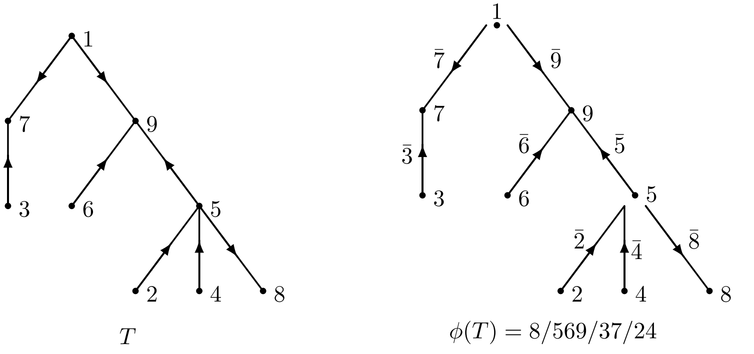

Fig. 1. A tree $T \in \mathcal{T}_{3221}$, and $\phi(T) = 8/569/37/24 \in \Pi_{3221}$.

prove (1.1) if we can find a map $\phi : \mathcal{T}_\lambda \to \Pi_\lambda$ for any $\lambda \vdash n-1$, where $\mathcal{T}_\lambda$ is the set of trees on $[n]$ with indegree sequence λ, and Π_λ is the set of partitions of $[2,n] := \{2,3,\dots,n\}$ of type λ. Richard Stanley (personal communication) suggested that such a map ϕ can be defined as follows.

Given $\lambda \vdash n-1$ and $T \in \mathcal{T}_\lambda$, we can consider T as a rooted tree on $[n]$ with the root 1 "hung up" (see Fig. 1). Now we label the edges of T such that each edge has the same label as the vertex right below it. It is obvious that during the labelling each number in $[2,n]$ is used exactly once. Putting the labels of those edges which point to the same vertex into one block, we get a partition $\pi \in \Pi_\lambda$. Fig. 1 shows a tree $T \in \mathcal{T}_{3221}$, and $\phi(T) = \pi = 8/569/37/24 \in \Pi_{3221}$. We put a bar over the label of each edge to avoid confusion.

While the map ϕ gives a natural interpretation of the second factor in Eq. (1.2), one can easily check that the preimage of ϕ is not unique: we can get the same partition by applying ϕ to different trees. Let $\mathcal{T}_\pi$ be the set of preimages of $\pi \in \Pi_\lambda$ under the map ϕ, i.e., $\mathcal{T}_\pi = \phi^{-1}(\pi)$, and let $f(\pi) := |\mathcal{T}_\pi|$. Then $\mathcal{T}_\lambda = \bigcup_{\pi \in \Pi_\lambda} \mathcal{T}_\pi$. Our main task is to prove the following theorem.

Theorem 1.1. *Given $\lambda \vdash n-1$ and $\pi \in \Pi_\lambda$, we have*

$$f(\pi) = |\mathcal{T}_\pi| = \frac{(n-1)!}{(n-|\pi|)!},$$

where $|\pi|$ is the number of blocks of π.

In the remainder of this paper we give proofs of this result using two different approaches. In Section 2, we give a "semi-combinatorial" proof based on induction on n. In Sections 3 and 4, we give a bijective proof. Finally in Section 5, some further problems are raised.

2. A semi-combinatorial proof

In this section, we will give an inductive proof of Theorem 1.1.

Lemma 2.2. *The value $f(\pi)$ is independent of $\pi \in \Pi_\lambda$, i.e., for any $\pi_1, \pi_2 \in \Pi_\lambda$, we have $f(\pi_1) = f(\pi_2)$.*

Proof. Since the symmetric group of $[2,n]$ is generated by *adjacent transpositions* $\{s_i\colon 2 \leqslant i \leqslant n-1\}$, where $s_i = (i, i+1)$ is the function that swaps two elements i and $i+1$, it suffices to show that $f(\pi_1) = f(\pi_2)$ for any $\pi_1, \pi_2 \in \Pi_\lambda$ such that by switching i and $i+1$ in π_2 we will get π_1 $(2 \leqslant i \leqslant n-1)$. If i and $i+1$ are in the same block of π_1, then $\pi_1 = \pi_2$. The assertion is trivial in this case. In the following, we will assume that i and $i+1$ are in different blocks of π_1.

In order to prove $f(\pi_1) = f(\pi_2)$, we construct an involution $\varphi_i : \mathcal{T}_{\pi_1} \cup \mathcal{T}_{\pi_2} \to \mathcal{T}_{\pi_1} \cup \mathcal{T}_{\pi_2}$. For any tree $T \in \mathcal{T}_{\pi_1} \cup \mathcal{T}_{\pi_2}$, consider the two vertices labelled i and $i+1$.

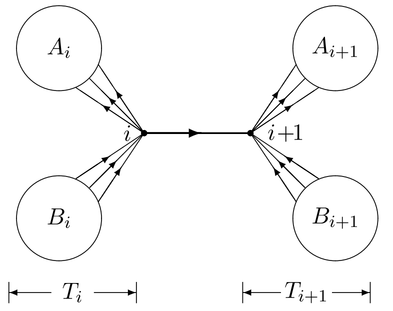

Fig. 2. A partition of the tree T.

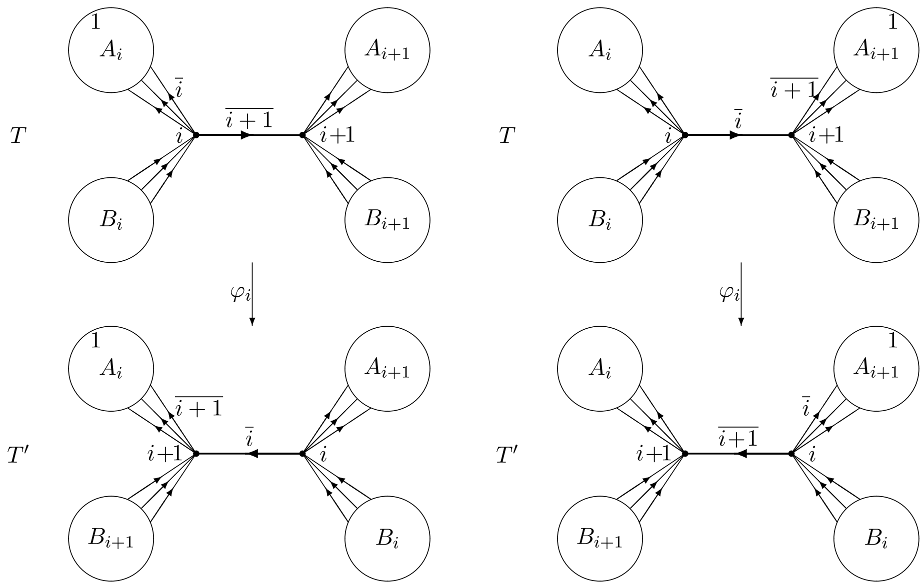

Fig. 3. Map φ_i (left: 1 in A_i, right: 1 in A_{i+1}).

If vertices i and $i+1$ are not adjacent, exchanging the labels of these two vertices will give us a new tree T'. Let $\varphi_i(T) = T'$.

If vertices i and $i+1$ are adjacent, let T_i (resp. T_{i+1}) be the largest subtree containing vertex i but not $i+1$ (resp. containing $i+1$ but not i). For $j = i, i+1$, let $T_j = \{j\} \cup A_j \cup B_j$, where A_j (resp. B_j) is the sub-forest such that every edge between itself and vertex j is pointing away from j (resp. pointing to j). (See Fig. 2.)

Considering the position of vertex 1, there are three cases:

Case 1: If vertex 1 is in either A_i or A_{i+1}, make all edges from B_i to vertex i point to vertex $i+1$ instead, make all edges from B_{i+1} to vertex $i+1$ point to vertex i instead, and switch the vertex labels i and $i+1$ (at the same time the direction of the edge between i and $i+1$ will be changed automatically). Then we will get a new tree T'. Let $\varphi_i(T) = T'$. (See Fig. 3.)

Case 2: If vertex 1 is in B_i, let B_i' be the maximum subtree of B_i which contains vertex 1, and let B_i'' be $B_i \backslash B_i'$. Make all edges from B_i' to vertex i point to vertex $i+1$ instead, and switch the vertex labels i and $i+1$ (at the same time the direction of the edge between i and $i+1$ will be changed automatically). Then we will get a new tree T'. Let $\varphi_i(T) = T'$. (See Fig. 4(1).)

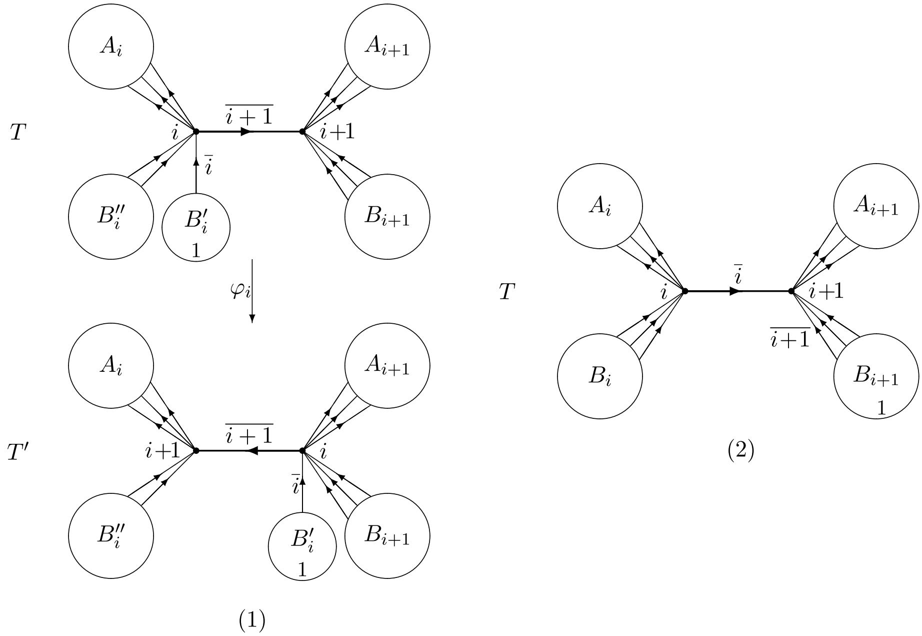

Fig. 4. Map φ_i (left: 1 in B_i, right: 1 in B_{i+1} (impossible)).

Case 3: If vertex 1 is in B_{i+1}, both edges labelled $\bar{i}$ and $\overline{i+1}$ are pointing to vertex $i+1$, i.e., i and $i+1$ are in the same block of π_1 or π_2, then we have a contradiction to the assumption. (See Fig. 4(2).)

From the definition of the map, we can easily check that $\phi(T)$ and $\phi(T')$ only differ in the positions of i and $i+1$, i.e., $\phi(T')$ is the same as $\phi(T)$ after switching i and $i+1$. Since $\varphi_i(T) = T'$, we have $\varphi_i : \mathcal{T}_{\pi_1} \cup \mathcal{T}_{\pi_2} \to \mathcal{T}_{\pi_1} \cup \mathcal{T}_{\pi_2}$ is well-defined, and $\varphi_i(\mathcal{T}_{\pi_1}) \in \mathcal{T}_{\pi_2}, \varphi_i(\mathcal{T}_{\pi_2}) \in \mathcal{T}_{\pi_1}$. And by applying φ_i again, we have $\varphi_i(\varphi_i(T)) = T$. Hence, φ_i is an involution with no fixed points. Hence, we have $|\mathcal{T}_{\pi_1}| = |\mathcal{T}_{\pi_2}|$, i.e., $f(\pi_1) = f(\pi_2)$. □

Proof of Theorem 1.1. Now with Lemma 2.2 we can prove Theorem 1.1 by induction on n, the number of vertices.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n-1$, where $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k \geqslant 1$. Then what we need to show is that for any $\pi \in \Pi_\lambda$, we have $f(\pi) = (n-1)!/(n-k)!$.

Base case: If $n = 1$, we have $k = 0$, $\lambda = \emptyset$, $\pi = \emptyset$, and $f(\pi) = 1 = (n-1)!/(n-k)!$.

Inductive step: Assume that the theorem is true for $n-1$ ($\geqslant 1$). Then consider the case for n.

If $\lambda_1 = 1$, then $\lambda = \langle 1^{n-1} \rangle$, $\pi = n/n-1/\cdots/2$ and $k = n-1$. In this case, each $T \in \mathcal{T}_\pi$ is an increasing tree, i.e., the label of any vertex is bigger than the label of its parent, i.e., the directions of edges are pointing away from the root 1. Otherwise, there is at least one vertex with indegree at least 2, contradicting that λ is the indegree sequence. Hence, we can do the bijection as in [3, §1.3] by mapping T to a permutation of $[2, n]$. Or we can use the bijection between labelled trees and Prüfer codes (see, for example, [1, §2.4], or a more generalized forest version [4, §5.3]). But while doing this bijection, what we will get is a subset of all possible Prüfer codes, i.e., a subset of $[n-1] \times [n-2] \times \cdots \times [2] \times [1]$. Both methods show that $f(\pi) = (n-1)! = (n-1)!/(n-k)!$.

Now suppose that $\lambda_1 \geqslant 2$. By Lemma 2.2, we can assume without loss of generality, that both n and $n-1$ are in the same block B_1 of $\pi = \{B_1, B_2, \ldots, B_k\}$. Pick $T \in \mathcal{T}_\pi$. Since n is the largest label, by the definition of π and $\mathcal{T}_\pi$, we know that vertices n and $n-1$ are adjacent. By merging the edge between n and $n-1$ in T, and deleting the label n, we get a new tree $\tilde{T}$ with $n-1$ vertices. There are two possible cases:

Case 1: If the indegree of vertex $n-1$ in T is 0, then $\phi(\tilde{T}) = \{B_1\backslash\{n\}, B_2, \ldots, B_k\} =: \tilde{\pi}_1$.

Case 2: If the indegree of vertex $n-1$ in T is not 0, then there exists $j \in [2,k]$ such that $\phi(\tilde{T}) = \{B_1 \cup B_j\backslash\{n\}, B_2, \ldots, B_{j-1}, B_{j+1}, \ldots, B_k\} =: \tilde{\pi}_j$.

One can easily check that this is a bijection. Thus, $f(\pi) = \sum_{j=1}^{k} f(\tilde{\pi}_j)$. By the induction hypothesis we have

$$f(\pi) = \sum_{j=1}^{k} f(\tilde{\pi}_j) = \frac{((n-1)-1)!}{((n-1)-k)!} + (k-1)\frac{((n-1)-1)!}{((n-1)-(k-1))!} = \frac{(n-1)!}{(n-k)!},$$

which proves the case for n.

Hence it follows by induction that Theorem 1.1 is true for all possible n. □

3. An “almost” bijective proof

The inductive proof in the former section makes Cotterill's conjecture a theorem, but it does not explain combinatorially why there is such a simple factor $(n-1)!/(n-k)!$. In this section, we will try to give a bijective proof to explain this fact.

First we will give some terminology and notation related to posets. Let S be a finite set. We use Π_S to denote the poset (actually a geometric lattice) of all partitions of S ordered by refinement ($\sigma \preccurlyeq \pi$ in Π_S if every block of σ is contained in a block of π). In the following discussion we will consider the case that $S = [2,n]$.

Second, we will state the basic definitions. Given $\pi \in \Pi_{[2,n]}$, recall that $\mathcal{T}_\pi$ is the set of labelled trees with preimage π under the map ϕ. Let B, B' be two subsets of $[2,n]$. We say that $B \leqslant B'$ (resp. $B < B'$) if and only if $\min B \leqslant \min B'$ (resp. $\min B < \min B'$). Given $T \in \mathcal{T}_\pi$ and $\pi = \phi(T)$, let $B = \{b_1, b_2, \ldots, b_t\}_<$ be a subset of one of the blocks of π. We define the *Star corresponding to B* to be the subset of T that contains all vertices and edges with labels in the set B, and denote it as Star(B). Induced by the ordering of the subsets of $[2,n]$, we will also get an ordering of the stars. For Star(B), there exists a unique vertex of T with some label, say c, such that the vertex c is attached to one of the edges in Star(B), but $c \notin B$. We call the vertex c the *cut point* of B, and denote it by $c(B)$.

For $T \in \mathcal{T}_\pi$ and $\sigma \preccurlyeq \pi$, we define the *decomposition of T with respect to* $\sigma = \{B_1, B_2, \ldots, B_k\}$ to be $T = (\bigcup_{j=1}^{k} \text{Star}(B_j)) \cup \{\text{vertex } 1\}$, where Star$(B_j)$ are the stars corresponding to B_j in T. In this decomposition, the *leaf-stars* are the stars that do not contain any cut points, i.e., if you remove a leaf-star from T, what's left is still a connected tree.

For example, for the tree T in Fig. 1 we have $\phi(T) = \pi = 8/569/37/24$. Star$(\{3,7\})$, Star$(\{2,4\})$ and Star$(\{8\})$ are all leaf-stars of T, and we have $c(\{3,7\}) = 1$, $c(\{2,4\}) = 5$, $c(\{8\}) = 5$ and $c(\{5,6,9\}) = 1$.

Now we define a variant of the map ϕ, which turns out to be a bijection. For any $\sigma = \{B_1, B_2, \ldots, B_k\} \in \Pi_{[2,n]}$, let $\mathcal{T}_{\succcurlyeq\sigma} = \bigcup_{\pi\succcurlyeq\sigma} \mathcal{T}_\pi$. We define $\phi_\sigma : \mathcal{T}_{\succcurlyeq\sigma} \to [n]^{k-1}$ as follows.

1. Let $T_0 = T$.
2. For $i = 1, 2, \ldots, k$, let Star$(B(i))$ be the largest leaf-star in the decomposition of T_{i-1} with respect to $\sigma\backslash\{B(1), B(2), \ldots, B(i-1)\}$. Then we remove Star$(B(i))$ and keep a record of the vertex it is attached to, i.e., let $\omega_i = c(B(i))$, $T_i = T_{i-1}\backslash\text{Star}(B(i))$.

Let $\phi_\sigma(T) = \omega := \omega_1\omega_2\cdots\omega_{k-1} \in [n]^{k-1}$. (We do not need to include ω_k since it is always 1.)

Theorem 3.3. *For any $\sigma \in \Pi_{[2,n]}$, the map ϕ_σ is a bijection between $\mathcal{T}_{\succcurlyeq\sigma}$ and $[n]^{|\sigma|-1}$.*

Proof. We now define the reverse procedure. Given $\sigma = \{B_1, B_2, \ldots, B_k\} \in \Pi_{[2,n]}$ and $\omega = \omega_1\omega_2\cdots\omega_{k-1} \in [n]^{k-1}$, set $\omega_k = 1$. Define the inverse map $\phi_\sigma^{-1} : [n]^{k-1} \to \mathcal{T}_{\succcurlyeq\sigma}$ as follows. For $i = 1, 2, \ldots, k$:

1. Let $B(i) = \{b_1, b_2, \ldots, b_t\}_<$ be the largest block of $\sigma\backslash\{B(1), B(2), \ldots, B(i-1)\}$ such that $B(i)$ does not contain any number in $\{\omega_i, \omega_{i+1}, \ldots, \omega_{k-1}\}$.

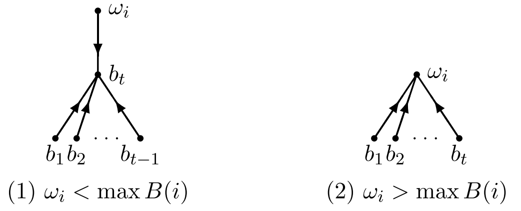

Fig. 5. Two cases when attaching $B(i)$ to ω_i.

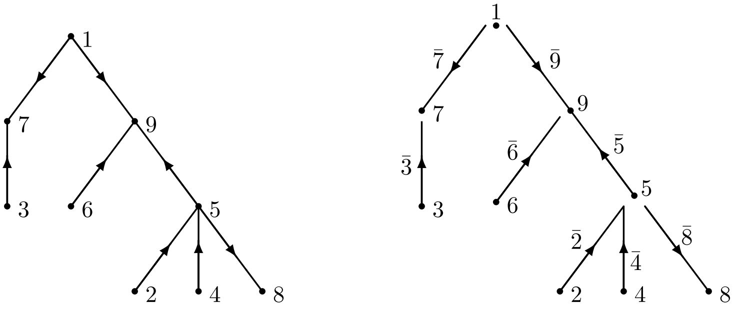

Fig. 6. A tree $T \in \mathcal{T}_{3221}$, with $\phi(T) = 8/569/37/24 \in \Pi_{3221}$, $\sigma = 8/7/6/59/3/24 \prec \pi$, and $\phi_\sigma(T) = 59715$.

2. Attach the vertices in $B(i)$ to ω_i according to the following two cases:

   Case 1: If $b_t > \omega_i$, we connect vertices $b_1, b_2, \ldots, b_t$ and ω_i such that the edges between $b_1, b_2, \ldots, b_{t-1}, \omega_i$ and b_t are all pointing to b_t (see Fig. 5(1));

   Case 2: If $b_t < \omega_i$ we simply connect $b_1, b_2, \ldots, b_t$ and ω_i such that all edges between $b_1, b_2, \ldots, b_t$ and ω_i are all pointing to ω_i (see Fig. 5(2)).

It is easy to see that after all k steps, we get a tree $T := \phi_\sigma^{-1}(\omega) \in \mathcal{T}_{\succcurlyeq\sigma}$. One can easily check that ϕ_σ is a bijection. □

Example 3.4. For the tree T in Fig. 6, let $\sigma = 8/7/6/59/3/24$. We then have $B(1) = \{8\}$, $\omega_1 = c(B(1)) = 5$; $B(2) = \{6\}$, $\omega_2 = c(B(2)) = 9$; $B(3) = \{3\}$, $\omega_3 = c(B(3)) = 7$; $B(4) = \{7\}$, $\omega_4 = c(B(4)) = 1$; $B(5) = \{2, 4\}$, $\omega_5 = c(B(5)) = 5$; $B(6) = \{5, 9\}$, $\omega_6 = c(B(6)) = 1$ (which we do not write). Thus we have $\phi_{8/7/6/59/3/24}(T) = 59715 \in [9]^5$.

Proof of Theorem 1.1. Let $g(\sigma) = |\mathcal{T}_{\succcurlyeq\sigma}|$. From the bijection $\phi_\sigma : \mathcal{T}_{\succcurlyeq\sigma} \to [n]^{|\sigma|-1}$ we know that $g(\sigma) = n^{|\sigma|-1}$. Recall from Section 2 that $f(\pi) = |\mathcal{T}_\pi|$. Since $\mathcal{T}_{\succcurlyeq\sigma} = \bigcup_{\pi \succcurlyeq \sigma} \mathcal{T}_\pi$ is a disjoint union, we have

$$\sum_{\pi \succcurlyeq \sigma} f(\pi) = n^{k-1}, \quad \text{for any } \sigma \in \Pi_{[2,n]}. \tag{3.1}$$

It is now sufficient to prove that the unique solution of the above equations is $f(\pi) = (n-1)!/(n-|\pi|)!$, for any $\pi \in \Pi_{[2,n]}$.

First, since Eq. (3.1) holds for any $\pi, \sigma \in \Pi_{[2,n]}$ such that $\pi \succcurlyeq \sigma$, we have, by the poset structure of $\Pi_{[2,n]}$, that the solution f to Eq. (3.1) (valid for all $\sigma \in \Pi_{[2,n]}$) is unique.

Second, let $\sigma = \{B_1, B_2, \ldots, B_k\}$. Then the interval $[\sigma, \hat{1}_{[2,n]}]$ is isomorphic in an obvious way to the lattice of partitions of the set $\{B_1, B_2, \ldots, B_k\}$. Hence $[\pi, \hat{1}_{[2,n]}] \cong \Pi_{[k]}$, where $\hat{1}_{[2,n]}$ is the maximum element of $\Pi_{[2,n]}$. Thus we have

$$\sum_{\pi \succcurlyeq \sigma} \frac{(n-1)!}{(n-|\pi|)!} = \sum_{\tau \in \Pi_{[k]}} \frac{(n-1)!}{(n-|\tau|)!}$$
$$= \sum_{j=1}^{k} S(k,j) \frac{(n-1)!}{(n-j)!}$$
$$= \frac{1}{n} \sum_{j=1}^{k} S(k,j) n(n-1)\cdots(n-j+1)$$
$$= n^{k-1},$$

where $S(k,j)$ is the *Stirling number of the second kind*, i.e., the number of partitions of a k-set into j blocks. The last equation follows from a standard Stirling number identity, see e.g., identity (24d) in [3, §1.4]. Thus, $(n-1)!/(n-|\pi|)!$ is a possible solution to Eqs. (3.1).

Hence, by uniqueness, we have $f(\pi) = (n-1)!/(n-|\pi|)!$. □

Remark. In fact, given Eq. (3.1), we can solve for f by using the dual form of the Möbius inversion formula:

$$f(\pi) = \sum_{\sigma \geqslant \pi} \mu(\pi,\sigma) g(\sigma),$$

where the coefficient $\mu(\pi,\sigma)$ is the Möbius function of $\Pi_{[2,n]}$, which can be calculated explicitly, [3, Example 3.10.4].

4. The real bijective map

Although we gave a bijection ϕ_σ in Section 3, we needed to prove Theorem 1.1 by solving equations, and we still do not have a very good bijection that maps $\mathcal{T}_\lambda$ to a set of cardinality a_λ for any $\lambda \vdash n-1$.

Let $\phi' : \mathcal{T}_\lambda \to \Pi_\lambda \times [n]^{k-1}$, $T \mapsto (\pi, \phi_\pi(T))$, where $\pi = \phi(T)$. Since ϕ_π is a bijection, we have that ϕ' is an injection. Let $\Omega_\pi := \phi_\pi(\mathcal{T}_\pi)$. Then $\phi'(\mathcal{T}_\lambda) = \{\{\pi\} \times \Omega_\pi\colon\ \pi \in \Pi_\lambda\} =: (\Pi \times \Omega)_\lambda$. Thus, $\phi' : \mathcal{T}_\lambda \to (\Pi \times \Omega)_\lambda$ is the bijection we are looking for.

Example 4.5. Assume $\pi = \{B_1, B_2\}$. For any $T \in \mathcal{T}_\pi$, we have $\phi'(T) = (\pi, \max\{c(B_1), c(B_2)\})$, and $\Omega_\pi = [n-1]$, $f(\pi) = n-1$.

Example 4.6. When $\lambda = \langle 1^{n-1} \rangle$, Π_λ contains only the partition $\hat{0}_{[2,n]} = n/n-1/\cdots/2$. As also pointed out in the proof in Section 2, $\mathcal{T}_{\hat{0}_{[2,n]}}$ is the set of all increasing trees on $[n]$, in this case we have $\Omega_{\hat{0}_{[2,n]}} = [n-1] \times [n-2] \times \cdots \times [1]$, and for each $T \in \mathcal{T}_{\hat{0}_{[2,n]}}$ and $\phi'(T) = (\pi, \omega)$, ω is the Prüfer code of T.

Though it seems quite hard to find what Ω_π's are, there still exists a very good relation among them.

Theorem 4.7. *For any $\pi_1, \pi_2 \in \Pi_{[2,n]}$, if $\pi_2 \succ \pi_1$, we have that for any $T \in \mathcal{T}_{\succcurlyeq \pi_2}$, $\phi_{\pi_2}(T)$ is a subsequence of $\phi_{\pi_1}(T)$. In particular, if $\pi_2 = \phi(T)$, we have $\phi'(T) = (\pi_2, \omega)$ and ω is a subsequence of $\phi_{\pi_1}(T)$.*

Proof. It suffices to prove the assertion for all covering pairs. Assume that $\pi_2 \succ\!\!\!\cdot\ \pi_1$. Thus there exist two blocks B and B' of π_1 which become one block in π_2.

Assume $\phi_{\pi_1}(T) = \omega = \omega_1\omega_2\cdots\omega_{k-1}$, $\phi_{\pi_2}(T) = \omega' = \omega'_1\omega'_2\cdots\omega'_{k-2}$ and $\omega_k = \omega'_{k-1} = 1$. Then there exist $1 \leqslant r, s \leqslant k$ such that B and B' are removed from T at steps r and s, respectively, in process ϕ_{π_1}. Assume, without loss of generality, that $r < s$. Then it is easy to see that $\omega'_l = \omega_l$ for $1 \leqslant l < r$.

For step r in process ϕ_{π_2}, there are two cases:

Case 1: If $B < B'$ and $\mathrm{Star}(B \cup B')$ is a leaf-star, then it must be that $s = r + 1$. At this step, we remove $\mathrm{Star}(B \cup B')$. Then $\omega'_r = \omega_s$, $\omega'_l = \omega_l$ for $r < l \leqslant k - 2$.

Case 2: If $B > B'$, or $\mathrm{Star}(B \cup B')$ is not a leaf-star, then we will remove $\mathrm{Star}(B \cup B')$ at the step that we remove $\mathrm{Star}(B')$ in the process ϕ_{π_1}, i.e., $\omega'_l = \omega_{l+1}$ for $r \leqslant l \leqslant k - 2$.

In both cases, we have that ω' is a subsequence of ω, i.e., $\phi_{\pi_2}(T)$ is a subsequence of $\phi_{\pi_1}(T)$. $\square$

Example 4.8. Let $\pi = 8/569/37/24$ and $\sigma = 8/7/6/59/3/24$, so $\pi \succ \sigma$. For the tree T in Fig. 6, we have $T \in \mathcal{T}_{\succcurlyeq\sigma}$, and $\phi_\pi(T) = 515$, which is a subsequence of $\phi_\sigma(T) = 59715$.

By the proof of Theorem 4.7, for any $\sigma \in \Pi_{[2,n]}$ we can define a bijection from $\bigcup_{\pi \succ \sigma} \Omega_\pi$ to $[n]^{k-1} \backslash \Omega_\sigma$ such that each sequence will be a subsequence of its image. Inductively using this bijection, we can find out all Ω_σ's. But when $|\sigma|$ gets larger and larger, it will become more and more difficult to find out what this bijection is explicitly.

5. Remarks

We want to remark that the bijection we defined in Section 3 can be considered as a generalization of the Prüfer codes for labelled trees: instead of deleting (attaching) vertices one by one, we are dealing with groups of vertices with respect to a partition of $[2, n]$. Moreover, the bijection ϕ' together with Theorem 4.7 suggests a structure on the set of labelled trees $\{\mathcal{T}_\pi \colon \pi \in \Pi_{[2,n]}\}$ as a lattice isomorphic to $\Pi_{[2,n]}$ under the map $\mathcal{T}_\pi \mapsto \pi$.

The following problems are still interesting to consider.

1. Given $\pi \in \Pi_{[2,n]}$, Theorem 4.7 shows how to find Ω_π explicitly, i.e., Ω_π is the subset of $[n]^{|\pi|-1}$ with sequences corresponding to its subsequences from Ω_σ deleted, for any $\sigma \succ \pi$. For example, let $\pi = 45/3/2$, we have

$$\Omega_\pi = \begin{Bmatrix} 11 & 12 & \cancel{13} & 14 & \cancel{15} \\ 21 & 22 & \cancel{23} & 24 & \cancel{25} \\ 31 & 32 & \cancel{33} & 34 & \cancel{35} \\ 41 & 42 & 43 & \cancel{44} & \cancel{45} \\ \cancel{51} & \cancel{52} & \cancel{53} & \cancel{54} & \cancel{55} \end{Bmatrix},$$

where $13, 23, 33, 44$ correspond to its subsequences $1, 2, 3, 4$ in $\Omega_{45/23}$, $15, 25, 53, 45$ correspond to its subsequences $1, 2, 3, 4$ in $\Omega_{3/245}$, $51, 52, 35, 54$ correspond to its subsequences $1, 2, 3, 4$ in $\Omega_{345/2}$, and 55 correspond to its subsequence $\emptyset$ in Ω_{2345}.

However, the "corresponding relationship", between sequences and its subsequences described inductively in the proof of Theorem 4.7, depends highly on the set $\{\sigma \in \Pi_{[2,n]} \colon \sigma \succ \pi\}$, and it is not easy to describe in general. Hence, it would be nice if one can give a simple description of this relationship, and use it to characterize Ω_π.

2. In the proof of Theorem 1.1 in Section 2, we mentioned that when $\lambda = \langle 1^{n-1} \rangle$, we can map an increasing tree to a permutation of $[2, n]$ [3, §1.3]. Is it possible to generalize this bijection to any λ by mapping a tree in $\mathcal{T}_\lambda$ to $(\phi(T), w)$, where w is a length $k - 1$ permutation of an $(n - 1)$-element set?

Acknowledgments

The authors want to thank Prof. Richard P. Stanley for his inspiring suggestions and helpful discussions, and also thank the referee for helpful comments on an earlier version of this paper. The first author is supported by the National Science Foundation of China under Grant No. 10726048 and No. 10801053, Shanghai Educational Development Foundation under the Chenguang Project No. 2007CG29, and Shanghai Leading Academic Discipline Project No. B407.

References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, North-Holland, New York, 1980, c1976.
[2] E. Cotterill, Geometry of curves with exceptional secant planes, arXiv:math.AG/0706.2049.
[3] R.P. Stanley, Enumerative Combinatorics, vol. 1, Wadsworth and Brooks/Cole, Pacific Grove, CA, 1986; second printing: Cambridge University Press, Cambridge, 1996.
[4] R.P. Stanley, Enumerative Combinatorics, vol. 2, Cambridge University Press, Cambridge, 1999.