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Small conjugacy classes in the automorphism groups of relatively free groups

Vladimir Tolstykh

Department of Mathematics, Yeditepe University, 34755 Kayışdağı, Istanbul, Turkey

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ABSTRACT

Let F be an infinitely generated free group and let R be a fully invariant subgroup of F such that (a) R is contained in the commutator subgroup F' of F and (b) the quotient group F/R is residually torsion-free nilpotent. Then the automorphism group $\mathrm{Aut}(F/R')$ of the group F/R' is complete. In particular, the automorphism group of any infinitely generated free solvable group of derived length at least two is complete.

This extends a result by Dyer and Formanek (1977) [7] on finitely generated groups F_n/R' where F_n is a free group of finite rank n at least two and R a characteristic subgroup of F_n.

0. Introduction

Recall that a group G is said to be *complete* if all automorphisms of G are inner and the center of G is trivial: $\mathrm{Aut}(G) = \mathrm{Inn}(G) \cong G$. In a series of papers [5–7] Dyer and Formanek justified several conjectures by Baumslag on the automorphism towers of finitely generated relatively free groups. In particular, they proved that the automorphism group of a free group F_n of finite rank $n \geqslant 2$ is complete (and hence the automorphism tower of F_n terminates already after two steps) [5] and that the automorphism group $\mathrm{Aut}(F_n/R')$ of the group F_n/R' is complete where R is a characteristic subgroup of F_n which is contained in the commutator subgroup F_n' of F_n such that the quotient group F_n/R is residually torsion-free nilpotent [7].

In [16–18] the author transferred the main results of [5,6] to the corresponding relatively free groups of infinite rank by proving completeness of the automorphism group of any infinitely generated free group [16] and completeness of the automorphism group of any infinitely generated free nilpotent group of nilpotency class $\geqslant 2$ [17,18]. The aim of the present paper is to extend to infinitely generated relatively free groups the main result of [7].

Let H be a centerless group. According to a theorem by Burnside [2], the automorphism group $\mathrm{Aut}(H)$ of H is complete if and only if the subgroup $\mathrm{Inn}(H)$ of all inner automorphisms of H is a characteristic subgroup of $\mathrm{Aut}(H)$. It is easily seen that the cardinality of the conjugacy class of any inner automorphism in the automorphism group $\mathrm{Aut}(G)$ of an infinitely generated relatively free group G is at most $\mathrm{rank}(G)$. The converse is not true in general, but one can use the subgroup $S(G) \leqslant \mathrm{Aut}(G)$ of all automorphisms of G whose conjugacy class is of cardinality at most $\mathrm{rank}(G)$ as a *naturally defined supergroup* of $\mathrm{Inn}(G)$. Then aiming to apply Burnside's theorem in order to prove completeness of $\mathrm{Aut}(G)$ in the case when G is centerless, one could work to distinguish by group-theoretic means the inner automorphisms of G from other elements of $S(G)$.

In what follows elements of $S(G)$ will be termed elements having *small* conjugacy classes. In Section 1 we obtain some basic results on elements of $S(G)$. Clearly, if $\sigma \in S(G)$, then the index of the centralizer $C(\sigma)$ of σ in $\mathrm{Aut}(G)$ is at most $\mathrm{rank}(G)$. This leads us to start with description of some types of automorphisms of G that can be found in any subgroup Σ of $\mathrm{Aut}(G)$ having index $\leqslant \mathrm{rank}(G)$ in $\mathrm{Aut}(G)$ (Proposition 1.1). We then apply Proposition 1.1 to show that an automorphism

E-mail address: vtolstykh@yeditepe.edu.tr.

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σ of G is in $S(G)$ if and only if there exist a group word $w(*; *_1, \ldots, *_s)$ and a fixed tuple $\vec{u} = u_1, \ldots, u_s$ of elements of G such that

$$\sigma(g) = w(g; u_1, \ldots, u_s) = w(g; \vec{u})$$

for every $g \in G$ (Proposition 1.2). As a corollary of Proposition 1.2 we obtain that $S(F) = \mathrm{Inn}(F)$ for any infinitely generated absolutely free group F and that, in effect, $\mathrm{Inn}(F)$ is the largest (free) normal subgroup of $\mathrm{Aut}(F)$ of cardinality $\mathrm{rank}(F)$. The latter result is an infinite rank analogue of a result by Formanek on the inner automorphisms of nonabelian free groups of finite rank [8]. Note that the results in Section 2 demonstrate that the property $S(G) = \mathrm{Inn}(G)$ is also true for all nonabelian free solvable groups G having infinite rank.

Another corollary of Proposition 1.2 is as follows: if $\mathfrak{V}$ is a variety of groups whose infinitely generated free groups are centerless, then $\mathrm{Aut}(G_1) \cong \mathrm{Aut}(G_2)$ if and only if $\mathrm{rank}(G_1) = \mathrm{rank}(G_2)$ where G_1, G_2 are infinitely generated free groups of $\mathfrak{V}$.

In Section 2 we apply the results obtained in Section 1 to relatively free groups F/R' where F is an infinitely generated free group and R is a fully invariant subgroup of F such that the group ring $\mathbf{Z}[F/R]$ is a domain whose units are trivial. We show that the restriction of any element $\sigma \in S(F/R')$ on the subgroup R/R' of F/R' coincides with the restriction on R/R' of a suitable inner automorphism of the group F/R' (Proposition 2.1). Then by applying the results by Shmel'kin [15] and by Dyer [4], we demonstrate that $S(F/R') = \mathrm{Inn}(F/R')$ and, consequently, that the group $\mathrm{Aut}(F/R')$ is complete in certain important cases (in particular, as we mentioned above, in the case when F/R' is a free solvable group of derived length $\geqslant 2$).

The main result of the paper is proved in Section 3. Assuming additionally that R is contained in the commutator subgroup F' of F and that F/R is residually torsion-free nilpotent, we obtain a group-theoretic characterization of the inner automorphisms of F/R' in the group $\mathrm{Aut}(F/R')$, thereby showing that the automorphism group $\mathrm{Aut}(F/R')$ of the group F/R' is complete.

1. Small conjugacy classes

Let G be an infinitely generated relatively free group, let $\varkappa$ denote the rank of G, and let Γ denote $\mathrm{Aut}(G)$. If $\mathcal{X}$ is a basis of G, we shall denote by $\mathrm{Sym}(\mathcal{X})$ the subgroup of automorphisms of G fixing $\mathcal{X}$ setwise; members of $\mathrm{Sym}(\mathcal{X})$ will be called *permutational* automorphisms of G with respect to $\mathcal{X}$.

Let $\{A_i : i \in I\}$ be a family of subgroups of G. We write $G = \circledast_{i\in I} A_i$, if there is a basis $\mathcal{X}$ of G such that $A_i = \langle \mathcal{A}_i \rangle$ where $\mathcal{A}_i = \mathcal{X} \cap A_i$ for every $i \in I$, the sets $\mathcal{A}_i$ are pairwise disjoint, and their union $\bigcup_{i\in I} \mathcal{A}_i$ is $\mathcal{X}$.

Clearly, if $G = \circledast_{i\in I} A_i$, then given any family $\{\sigma_i \in \mathrm{Aut}(A_i) : i \in I\}$ of automorphisms of subgroups A_i, there is a uniquely determined automorphism σ of G such that $\sigma|_{A_i} = \sigma_i$ for all $i \in I$; we shall denote σ by $\circledast_{i\in I}\sigma_i$.

We shall use the standard notation of the theory of infinite permutation groups. Thus, having a group H acting on a set X, we shall denote by $H_{(Y)}$ and by $H_{\{Y\}}$ the pointwise and the setwise stabilizer of a subset Y of X in H, respectively. Any symbol of the form $H_{*_1,*_2}$ denotes the intersection of subgroups H_{*_1} and H_{*_2}.

In this section we shall study elements of $\mathrm{Aut}(G)$ having 'small' conjugacy classes. We shall say that the conjugacy class σ^Γ of a $\sigma \in \mathrm{Aut}(G)$ is *small* if the cardinality of σ^Γ is at most $\varkappa = \mathrm{rank}(G)$; equivalently, the index $|\Gamma : C(\sigma)| = |\sigma^\Gamma|$ of the centralizer $C(\sigma)$ of σ in Γ is at most $\varkappa$.

We start with a statement on existence of certain stabilizers in the subgroups of $\mathrm{Aut}(G)$ having index $\leqslant \mathrm{rank}(G)$. Recall that a *moiety* of an infinite set I is any subset J of I with $|J| = |I \setminus J|$. The proof of the statement uses the ideas and methods developed by Dixon et al. in [3].

Proposition 1.1. *Let G be a relatively free group of infinite rank $\varkappa$, $\mathcal{X}$ a basis of G and Σ a subgroup of the automorphism group $\Gamma = \mathrm{Aut}(G)$ of index at most $\mathrm{rank}(G)$. Then*

(i) *there is a subset $\mathcal{U}$ of $\mathcal{X}$ of cardinality less than $\varkappa$ such that Σ contains the subgroup $\mathrm{Sym}(\mathcal{X})_{(\mathcal{U})}$ of permutational automorphisms with regard to $\mathcal{X}$ which fix $\mathcal{U}$ pointwise;*

(ii) *for every moiety $\mathcal{Z}$ of $\mathcal{X} \setminus \mathcal{U}$, Σ contains the subgroup $\Gamma_{(\mathcal{X}\setminus\mathcal{Z}),\{\langle\mathcal{Z}\rangle\}}$ of automorphisms of G which fix $\mathcal{X} \setminus \mathcal{Z}$ pointwise and preserve the subgroup $\langle\mathcal{Z}\rangle$ generated by $\mathcal{Z}$;*

(iii) *for every moiety $\mathcal{Z} = \{z_i : i \in I\}$ of $\mathcal{X} \setminus \mathcal{U}$, and for every subset $\{v_i : i \in I\}$ of the subgroup $\langle\mathcal{U}\rangle$ generated by $\mathcal{U}$, Σ contains an automorphism σ of G which fixes the set $\mathcal{X} \setminus \mathcal{Z}$ pointwise and takes an element z_i of $\mathcal{Z}$ to $z_i v_i$:*

$$\begin{aligned} \sigma x &= x, \quad x \in \mathcal{X} \setminus \mathcal{Z}, \\ \sigma z_i &= z_i v_i, \quad i \in I. \end{aligned} \tag{1.1}$$

Proof. (0). Before proving (i), we are first going to do some preliminary work for (ii) and (iii).

(a). Let $\mathcal{Y}$ be a subset of $\mathcal{X}$ of cardinality $\varkappa = \mathrm{rank}(G)$. We partition $\mathcal{Y}$ into $\varkappa$ moieties:

$$\mathcal{Y} = \bigsqcup_{i\in I} \mathcal{Y}_i$$

and then partition every $\mathcal{Y}_i$ into countably many moieties:

$$\mathcal{Y}_i = \bigsqcup_{k\in\mathbf{N}} \mathcal{Y}_{i,k}.$$

Fix an index i_0 of I and consider any automorphism α of the subgroup $\langle \mathcal{Y}_{i_0,0}\rangle$ generated by $\mathcal{Y}_{i_0,0}$. We extend α on $\langle \mathcal{Y}_{i_0}\rangle$ as follows:

$$\beta(\alpha) = (\alpha \circledast \alpha^{-1} \circledast \mathrm{id}) \circledast (\alpha \circledast \alpha^{-1} \circledast \mathrm{id}) \circledast \cdots . \tag{1.2}$$

Here, as is quite commonly done for the sake of simplicity, any α (resp. α^{-1}, resp. id) on the right-hand side of (1.2) *indicates* that the action of the restriction of $\beta(\alpha)$ on a subgroup $\langle \mathcal{Y}_{i_0,k}\rangle$ is isomorphic to the action of α (resp. α^{-1}, resp. id) on $\langle \mathcal{Y}_{i_0,0}\rangle$; the corresponding isomorphism of actions is supposed to be induced by a bijection from $\mathcal{Y}_{i_0,0}$ onto $\mathcal{Y}_{i_0,k}$. Any $\circledast$-product of automorphisms below must be treated in a similar fashion.

We then consider the family Λ_α of automorphisms of G such that

$$\lambda = \mathrm{id}_{\langle \mathcal{X}\setminus\mathcal{Y}\rangle} \circledast (\underset{i\in I}{\circledast}\, \beta(\alpha)^{\varepsilon_i}) \tag{1.3}$$

where $\varepsilon_i = 0, 1$ $(i \in I)$.

Observe that $|\Lambda_\alpha| = 2^\varkappa$. Therefore for any subgroup H of Γ having index less than $2^\varkappa$, there are distinct $\lambda_1, \lambda_2 \in \Lambda_\alpha$ with $\lambda_1\lambda_2{}^{-1} \in H$. Clearly, the product $\lambda_1\lambda_2{}^{-1}$ is also of the form (1.3) with ε_i equal either to 0, or to 1, or to -1 $(i \in I)$.

(b). Observe, for future use, that there are permutational automorphisms $\rho_1, \rho_2 \in \mathrm{Aut}(\langle \mathcal{Y}_{i_0}\rangle)$ with regard to the basis $\mathcal{Y}_{i_0}$ such that $\beta(\alpha)^{\rho_1}\beta(\alpha)$ equals the trivial automorphism of $\langle \mathcal{Y}_{i_0}\rangle$ and $\beta(\alpha)^{\rho_2}\beta(\alpha)$ equals the automorphism

$$(\alpha \circledast \mathrm{id} \circledast \mathrm{id}) \circledast (\mathrm{id} \circledast \mathrm{id} \circledast \mathrm{id}) \circledast \cdots , \tag{1.4}$$

that is, coincides with α on $\langle \mathcal{Y}_{i_0,0}\rangle$. The similar argument applies to $\beta(\alpha)^{-1}$: there are permutational automorphisms $\mu_1, \mu_2 \in \mathrm{Aut}(\langle \mathcal{Y}_{i_0}\rangle)$ such that $[\beta(\alpha)^{-1}]^{\mu_1}\beta(\alpha)^{-1}$ is the trivial automorphism of $\langle \mathcal{Y}_{i_0}\rangle$, and $[\beta(\alpha)^{-1}]^{\mu_2}\beta(\alpha)^{-1}$ is the automorphism (1.4).

(i). As

$$|\mathrm{Sym}(\mathcal{X}) : \mathrm{Sym}(\mathcal{X}) \cap \Sigma| \leqslant |\Gamma : \Sigma| \leqslant \mathrm{rank}(G) = |\mathcal{X}|,$$

we get that $\mathrm{Sym}(\mathcal{X}) \cap \Sigma$ is a subgroup of $\Pi = \mathrm{Sym}(\mathcal{X})$ of index at most $|\mathcal{X}|$. Then by Theorem $2^\flat$ of [3], there is a subset $\mathcal{U}$ of $\mathcal{X}$ of cardinality less than $|\mathcal{X}|$ such that

$$\Pi_{(\mathcal{U})} \leqslant \Pi \cap \Sigma = \mathrm{Sym}(\mathcal{X}) \cap \Sigma.$$

(ii). We apply the considerations in (0) to the set $\mathcal{Y} = \mathcal{X} \setminus \mathcal{U}$. We partition $\mathcal{Y}$ as in (0), assuming that $\mathcal{Y}_{i_0,0} = \mathcal{Z}$. Take a $\sigma \in \Gamma_{(\mathcal{X}\setminus\mathcal{Z}),\{\langle\mathcal{Z}\rangle\}}$ and let

$$\alpha = \sigma|_{\langle\mathcal{Z}\rangle} = \sigma|_{\langle\mathcal{Y}_{i_0,0}\rangle}.$$

By the conclusion made in (a) in (0), as the index of Σ in Γ is at most $\varkappa < 2^\varkappa$, Σ contains an element of Λ_α, that is, there are $\varepsilon_i = 0, 1, -1$ $(i \in I)$ some of which are nonzero such that

$$\gamma = \mathrm{id}_{\langle\mathcal{U}\rangle} \circledast (\underset{i\in I}{\circledast}\, \beta(\alpha)^{\varepsilon_i}) \in \Sigma.$$

Using (b) in (0), it is then rather easy to find a permutational automorphism π in $\Pi_{(\mathcal{U})} = \mathrm{Sym}(\mathcal{X})_{(\mathcal{U})}$ which fixes $\mathcal{U}$ pointwise and such that the action of $\gamma^\pi\gamma$ on some $\mathcal{Y}_{j,0}$ is isomorphic to the action of α and the action of $\gamma^\pi\gamma$ on all other sets $\mathcal{Y}_{i,k}$ is trivial. Indeed, let

$$I_0 = \{i \in I : \varepsilon_i = 0\}, \qquad I_1 = \{i \in I : \varepsilon_i = 1\}, \qquad I_{-1} = \{i \in I : \varepsilon_i = -1\}.$$

If I_1 is empty, we switch to γ^{-1}, also a member of Σ. Thus, without loss of generality we may assume that $I_1 \neq \varnothing$. Let us consider the more difficult case, when $i_0 \notin I_1$. Take a j in I_1, and define $\pi \in \Pi_{(\mathcal{U})}$ as follows:

$$\pi = \mathrm{id}_{(\mathcal{U})} \circledast (\underset{i\in I_0}{\circledast}\, \mathrm{id}) \circledast \rho_1 \circledast (\underset{i\in I_1, i\neq j}{\circledast}\, \rho_2) \circledast (\underset{i\in I_{-1}}{\circledast}\, \mu_2), \tag{1.5}$$

where ρ_1, ρ_2, μ_2 were defined in (b) in (0), and ρ_1 in (1.5) describes the action of π on $\langle \mathcal{Y}_j\rangle$.

At the next step we conjugate $\gamma^\pi\gamma \in \Sigma$ by an appropriately chosen permutational automorphism $\pi_1 \in \Pi_{(\mathcal{U})} \leqslant \Sigma$, interchanging $\mathcal{Y}_{i_0}$ and $\mathcal{Y}_j$, while fixing pointwise $\mathcal{U}$ and all other $\mathcal{Y}_i$, thereby obtaining σ.

(iii). The proof is basically the same as in (i). Having an automorphism σ of G with (1.1), we find in Σ an automorphism σ^* of G whose action on each of some $\varkappa$ moieties of $\mathcal{X} \setminus \mathcal{U}$, into which $\mathcal{X} \setminus \mathcal{U}$ is partitioned and one of which contains $\mathcal{Z}$, is either trivial, or isomorphic to the action of $\beta(\sigma|_{\langle\mathcal{Z}\rangle})$ (suitably defined) on the moiety containing $\mathcal{Z}$, or to isomorphic the action of $\beta(\sigma|_{\langle\mathcal{Z}\rangle})^{-1}$. Then for a suitable $\pi, \pi_1 \in \Pi_{(\mathcal{U})}$ we have that an element $((\sigma^*)^\pi\sigma^*)^{\pi_1}$ of Σ is equal to σ. □

Proposition 1.2. *Let G be a relatively free group of infinite rank $\varkappa$, $\mathcal{X}$ a basis of G. The conjugacy class of a $\sigma \in \mathrm{Aut}(G)$ is small if and only if there are finitely many elements $u_1, \ldots, u_s$ of $\mathcal{X}$ and a term $w(*; *_1, \ldots, *_s)$ of the language of group theory (a group word in symbols $*, *_1, \ldots, *_s$) such that*

$$\sigma(x) = w(x; u_1, \ldots, u_s) \tag{1.6}$$

for all $x \in \mathcal{X}$ and

$$w(xy; u_1, \ldots, u_s) = w(x; u_1, \ldots, u_s) \cdot w(y; u_1, \ldots, u_s) \tag{1.7}$$

for all $x, y \in \mathcal{X}$ (in effect, $\sigma(g) = w(g; u_1, \ldots, u_s)$ for all $g \in G$).

Proof. Since the conjugacy class of σ is of cardinality $\leqslant \varkappa$, the index of the centralizer $C(\sigma)$ of σ in $\mathrm{Aut}(G)$ is at most $\varkappa$. Hence by Proposition 1.1, for $\Sigma = C(\sigma)$ there is a subset $\mathcal{U}$ of $\mathcal{X}$ having properties (i–iii) listed in this proposition. In particular, $\Pi_{(\mathcal{U})} = \mathrm{Sym}(\mathcal{X})_{(\mathcal{U})} \leqslant C(\sigma)$.

Consider an $x \in \mathcal{X} \setminus \mathcal{U}$. Let $w_x(x; \vec{y}, \vec{u})$ be a term of the language of group theory such that

$$\sigma x = w_x(x; \vec{y}, \vec{u}_x)$$

where $\vec{y}$ is a tuple of elements of $\mathcal{X} \setminus \mathcal{U}$ none of which equals x and $\vec{u}_x$ is a tuple of elements of $\mathcal{U}$.

(a) A permutational automorphism π in $\Pi_{(\mathcal{U})}$ which fixes x and takes $\vec{y}$ to a tuple $\pi\vec{y}$ with $\pi\vec{y} \cap \vec{y} = \varnothing$ must commute with σ. Then $\pi\sigma\pi^{-1}x = \sigma x$ implies that

$$w_x(x; \pi\vec{y}, \vec{u}_x) = w_x(x; \vec{y}, \vec{u}_x).$$

Take an endomorphism ε of G which sends all members of $\pi\vec{y}$ to 1 and fixes all other elements of $\mathcal{X}$. Apply ε to both parts of the last equation:

$$\varepsilon(w_x(x; \pi\vec{y}, \vec{u}_x)) = w_x(\varepsilon(x); \varepsilon(\pi\vec{y}), \varepsilon(\vec{u}_x)) = w_x(x; 1, \ldots, 1, \vec{u}_x) = w_x(x; \vec{y}, \vec{u}_x).$$

It follows that $\sigma x \in \langle x, \vec{u}_x \rangle$, and we can assume that

$$\sigma x = w_x(x; \vec{u}_x).$$

(b) Take another element y in $\mathcal{X} \setminus \mathcal{U}$. Again, σ must commute with a permutational automorphism ρ in $\Pi_{(\mathcal{U})}$ interchanging x and y. Comparing $\rho\sigma\rho^{-1}x$ and σx, we obtain that

$$w_y(x; \vec{u}_y) = w_x(x; \vec{u}_x).$$

But then

$$w_y(y; \vec{u}_y) = w_x(y; \vec{u}_x),$$

after forcing an endomorphism of G fixing $\mathcal{U}$ pointwise and taking x to y to act on both parts of the preceding equation. So the image σy of y can be obtained by replacing occurrences x in $w_x(x; \vec{u})$ by y. We arrive therefore at the conclusion that

$$\sigma z = w(z; \vec{u})$$

where $w(*; *_1, \ldots, *_s)$ is a fixed term and $\vec{u}$ is a fixed tuple of elements of $\mathcal{U}$ for all $z \in \mathcal{X} \setminus \mathcal{U}$.

(c) Let x, y be distinct elements of $\mathcal{X} \setminus \mathcal{U}$. The 'transvection' U which takes x to xy and fixes all other elements of $\mathcal{X}$ belongs to $C(\sigma)$ by part (ii) of Proposition 1.1. The equality $U\sigma U^{-1}x = \sigma x$ then implies that

$$w(xy; \vec{u})w(y; \vec{u})^{-1} = w(x; \vec{u}),$$

or

$$w(xy; \vec{u}) = w(x; \vec{u}) \cdot w(y; \vec{u}). \tag{1.8}$$

As $x, y, \vec{u}$ are all members of some basis of G,

$$w(ab; \vec{u}) = w(a; \vec{u}) \cdot w(b; \vec{u}) \tag{1.9}$$

for every $a, b \in G$ (after acting on both parts of (1.9) by an endomorphism of G fixing $\vec{u}$ pointwise and taking x to a and y to b.)

(d). An argument similar to the one we have used in (a) shows that for every $v \in \mathcal{U}$, the image σv of v is in the subgroup generated by $\mathcal{U}$.

Take an $x \in \mathcal{X} \setminus \mathcal{U}$, an element $v \in \mathcal{U}$ and another 'transvection' U_1 which takes x to xv and fixes $\mathcal{X} \setminus \{x\}$ pointwise. By part (iii) of Proposition 1.1, U_1 commutes with σ. Observe that $U_1(\sigma v) = \sigma v$, since U_1 stabilizes all elements of $\mathcal{U}$. Hence

$$w(x; \vec{u}) = \sigma x = U_1\sigma U_1{}^{-1}x = U_1(\sigma(xv^{-1})) = U_1(\sigma(x)\sigma(v^{-1})) = w(xv; \vec{u})\sigma(v^{-1}).$$

By (1.9), $w(xv; \vec{u}) = w(x; \vec{u})w(v; \vec{u})$, whence $\sigma v = w(v; \vec{u})$, completing the proof of the necessity part.

Conversely, if a term $w(*; *_1, \ldots, *_s)$ and a tuple $\vec{u}$ of $\mathcal{X}$ satisfy (1.6) and (1.7), then

$$\sigma(g) = w(g; \vec{u})$$

for all $g \in G$. Let $\pi \in \mathrm{Aut}(G)$. Hence

$$\pi\sigma\pi^{-1}g = \pi(\sigma(\pi^{-1}g)) = \pi(w(\pi^{-1}g, \vec{u})) = w(g; \pi\vec{u})$$

for all $g \in G$. Therefore there are at most $\varkappa$ conjugates of σ, since there are at most $\varkappa$ elements in the orbit of the tuple $\vec{u}$ under $\mathrm{Aut}(G)$. $\square$

Working with a relatively free group G we shall denote by τ_g the inner automorphism of G determined by a $g \in G$.

Proposition 1.3. *Let G be a centerless relatively free group of infinite rank $\varkappa$.*

(i) *Suppose that the cardinality of the conjugacy class ρ^Γ of a $\rho \in \Gamma$ is less than $\varkappa$. Then ρ is the identity;*

(ii) *the cardinal*

$$\min\{|\pi^\Gamma| : \pi \in \Gamma, \pi \neq \mathrm{id}\}$$

is equal to $\varkappa$;

(iii) *the conjugacy class of a nonidentity $\sigma \in \Gamma$ is small if and only if $|\sigma^\Gamma| \leqslant |\pi^\Gamma|$ for every nonidentity $\pi \in \Gamma$;*

(iv) *the subgroup S of all elements of Γ whose conjugacy class is small is a characteristic subgroup of Γ.*

Proof. (i). Let $\mathcal{X}$ be a basis of G. By Proposition 1.2, there is a term $w(*; *_1, \ldots, *_s)$ and elements $\vec{u} = u_1, \ldots, u_s \in \mathcal{X}$ such that

$$\rho(g) = w(g; \vec{u})$$

for all $g \in G$. As

$$|\{\tau_x \rho {\tau_x}^{-1} : x \in \mathcal{X} \setminus \vec{u}\}| \leqslant |\rho^\Gamma| < \varkappa,$$

there are distinct $x_1, x_2 \in \mathcal{X} \setminus \vec{u}$ such that

$$\tau_{x_1} \rho {\tau_{x_1}}^{-1} = \tau_{x_2} \rho {\tau_{x_2}}^{-1},$$

or

$$\rho \tau_{{x_1}^{-1} x_2} \rho^{-1} = \tau_{{x_1}^{-1} x_2}$$

or $\rho({x_1}^{-1} x_2) = {x_1}^{-1} x_2$ because G is centerless. Therefore

$$w({x_1}^{-1} x_2; \vec{u}) = {x_1}^{-1} x_2,$$

whence $w(g; \vec{u}) = g = \rho(g)$ for all $g \in G$, since the element ${x_1}^{-1} x_2$ and the elements of the tuple $\vec{u}$ all occur in a suitable basis of G.

(ii). Write λ for the cardinal

$$\min\{|\pi^\Gamma| : \pi \in \Gamma, \pi \neq \mathrm{id}\}.$$

By (i), $\varkappa \leqslant \lambda$. On the other hand, for any inner automorphism τ_g determined by a nonidentity element $g \in G$

$$\lambda \leqslant |\tau_g^\Gamma| = \varkappa.$$

(iii). By (ii).

(iv). By (iii). □

Corollary 1.4. *Let $\mathfrak{V}$ be a variety of groups whose free groups are centerless. Then for any infinitely generated free groups $G_1, G_2 \in \mathfrak{V}$*

$$\mathrm{Aut}(G_1) \cong \mathrm{Aut}(G_2) \iff \mathrm{rank}(G_1) = \mathrm{rank}(G_2).$$

Proof. By Proposition 1.3(ii). □

In [8] Formanek proved that the subgroup $\mathrm{Inn}(F_n)$ of the automorphism group $\mathrm{Aut}(F_n)$ of a free group F_n of finite rank $n \geqslant 2$ is the only free normal subgroup of $\mathrm{Aut}(F_n)$ of rank n. Our next corollary extends this result to free groups of infinite rank.

Corollary 1.5. *Let $F = F_\varkappa$ be a free group of infinite rank $\varkappa$. Then $\sigma \in \mathrm{Aut}(F_\varkappa)$ has small conjugacy class if and only if σ is an inner automorphism of $F_\varkappa$. Consequently, $\mathrm{Inn}(F_\varkappa)$ is the largest (free) normal subgroup of $\mathrm{Aut}(F_\varkappa)$ of cardinality $\varkappa$.*

Proof. Let the conjugacy class of a $\sigma \in \mathrm{Aut}(F)$ be small. Take a basis $\mathcal{X}$ of F and choose a subset $\mathcal{U}$ of $\mathcal{X}$ of cardinality $< \varkappa$ as in the proof of Proposition 1.2. Take an $x \in \mathcal{X} \setminus \mathcal{U}$, and partition $\mathcal{X} \setminus (x \cup \mathcal{U})$ into two moieties:

$$\mathcal{X} \setminus (x \cup \mathcal{U}) = \mathcal{Y}_0 \sqcup \mathcal{Y}_1.$$

Then by Proposition 1.1 (ii) the following automorphisms ρ_1, ρ_2, ρ_3 that act identically on $\mathcal{U}$ belong to the centralizer $C(\sigma)$ of σ:

$$\begin{array}{llllll} \rho_1: & x \to x, & \rho_2: & x \to x, & \rho_3: & x \to x^{-1}, \\ & y \to x^{-1} y x, & & y \to y, & & y \to y, \quad (y \in \mathcal{Y}_1), \\ & y \to y, & & y \to x^{-1} y x, & & y \to y, \quad (y \in \mathcal{Y}_2). \end{array}$$

The product $\rho = \rho_3\rho_2\rho_1$ is an automorphism of F which fixes $\mathcal{U}$ pointwise, inverts x and takes every $y \in \mathcal{Y}_1 \cup \mathcal{Y}_2$ to its conjugate by x. It is proved in [16, Lemma 4.2] that any automorphism of F commuting with ρ takes x either to vxv^{-1}, or $vx^{-1}v^{-1}$, where v is in the fixed-point subgroup of ρ, that is, in $\langle \mathcal{U} \rangle$. It follows that

$$\sigma(z) = w_0(z; v) = vzv^{-1} \quad (z \in \mathcal{X}),$$

or

$$\sigma(z) = w_1(z; v) = vz^{-1}v^{-1} \quad (z \in \mathcal{X}).$$

But in the second case it is not true for the term w_1 that

$$w_1(xy; v) = v(xy)^{-1}v^{-1} = w_1(x; v) \cdot w_1(y; v) = vx^{-1}y^{-1}v^{-1}$$

for every $x, y \in \mathcal{X}$. Hence σ is an inner automorphism of F, as claimed. □

Remark 1.6. A theorem by Burnside [2] states that given a centerless group G such that the group $\mathrm{Inn}(G)$ is a characteristic subgroup of $\mathrm{Aut}(G)$, we have that $\mathrm{Aut}(G)$ is complete. It then follows from Corollary 1.6 that the automorphism group of any infinitely generated free group is complete (a result proven in [16] by a different method).

2. Relatively free groups F/R'

Recall that a *derivation* of a given group G in a G-module (a module over the group ring $\mathbf{Z}[G]$) M is any map $D : G \to M$ such that

$$D(ab) = D(a) + aD(b)$$

for every $a, b \in G$ (here $aD(b)$ is the result of the action of a scalar $a \in G \subseteq \mathbf{Z}[G]$ on a vector $D(b) \in M$.)

As it has been proved by Fox [9] if F is a free group with a basis $(X_i : i \in I)$ then for any prescribed elements $Y_i \in \mathbf{Z}[F]$ there is a unique derivation D of F in $\mathbf{Z}[F]$ such that

$$D(X_i) = Y_i \quad (i \in I).$$

In particular, for every $i \in I$ there is a derivation D_i of F such that

$$D_i(X_j) = \delta_{ij} \quad (i, j \in I).$$

Now let R be a normal subgroup of F and let R' denote the commutator subgroup of R; the quotient group R/R' will be denoted by $\widehat{R}$.

We shall write $^-$ for the homomorphism $\mathbf{Z}[F] \to \mathbf{Z}[F/R]$ of group rings induced by the natural group homomorphism $F \to F/R$; it is convenient to use the same symbol $^-$ to denote the homomorphism $\mathbf{Z}[F/R'] \to \mathbf{Z}[F/R]$ induced by the natural homomorphism $F/R' \to F/R$.

Clearly, any F/R-module can be in a natural way viewed as an F- and as an F/R'-module. Consider a free F/R-module M with free generators $(t_i : i \in I)$. Then it is easy to see that the map

$$\partial(aR') = \sum \overline{D_i(a)}t_i \tag{2.1}$$

where a runs over F is a well-defined derivation of F/R' in M, since $\overline{D_i(b)} = 0$ for every $b \in R'$ and for every $i \in I$.

A famous result by Magnus from [12] is tantamount to the fact that $\partial : F/R' \to M$ is injective (see, for instance, [13]). Moreover, the following properties

$$\partial(r_1r_2) = \partial(r_1) + \bar{r}_1\partial(r_2) = \partial(r_1) + \partial(r_2), \tag{2.2}$$
$$\partial(g\widehat{R} * r) = \partial(grg^{-1}) = \partial(g) + \bar{g}\partial(r) - \overline{grg^{-1}}\partial(g) = \bar{g}\partial(r)$$

are true for all $r_1, r_2 \in \widehat{R} = R/R'$ and for all $g \in F/R'$. One can therefore state that R/R' and $\partial(R/R')$, viewed as F/R-modules, are isomorphic via ∂.

According to a result by Auslander and Lyndon [1], if the quotient group F/R is infinite, the group F/R' is centerless; we shall use this fact in Corollary 2.2 and Theorem 3.3 below.

Lemma 2.1. *Let F be an infinitely generated free group, R a fully invariant subgroup of F such that the group ring $\mathbf{Z}[F/R]$ has no zero divisors and all its units are trivial:*

$$U(\mathbf{Z}[F/R]) = \pm F/R.$$

Suppose that $\sigma \in \mathrm{Aut}(F/R')$ has small conjugacy class in $\mathrm{Aut}(F/R')$. Then the restriction of σ on the group $\widehat{R} = R/R'$ coincides with the restriction on $\widehat{R}$ of a suitable inner automorphism of F/R', that is, there is a $v \in F/R'$ such that

$$\sigma r = vrv^{-1}.$$

for every r in R/R'.

Proof. Fix a basis $\mathcal{B}$ of the free group F and let $\mathcal{X}$ be the image of $\mathcal{B}$ under the natural homomorphism $F \to F/R'$. By Proposition 1.2, there are elements $u_1, \ldots, u_s$ of $\mathcal{X}$ and a term $w(*; *_1, \ldots, *_s)$ of the language of group theory such that w satisfies (1.6) and (1.7) and

$$\sigma(z) = w(z; \vec{u})$$

for all $z \in F/R'$. Suppose that

$$w(x; \vec{u}) = v_1 x^{k_1} \ldots v_m x^{k_m}$$

where elements $v_2, \ldots, v_m$ from the subgroup $\langle \vec{u} \rangle$ generated by the elements $\vec{u} = u_1, \ldots, u_s$ are nontrivial, $k_1, \ldots, k_{m-1}$ are nonzero integers, while $v_1 \in \langle \vec{u} \rangle$ and x^{k_m} could be equal to identity.

We show that the sum $l = k_1 + \cdots + k_m$ of exponents of x is 1. Indeed, by (1.7)

$$w(xy; \vec{u}) = w(x; \vec{u}) w(y; \vec{u}), \tag{2.3}$$

for all $x, y \in \mathcal{X}$. Assume that x, y are distinct members of $\mathcal{X}$. Take an endomorphism of F/R' sending all u_i to 1, while preserving x and y, and apply it to both parts of (2.3):

$$(xy)^l = x^l y^l.$$

Let X be the element of $\mathcal{B}$ whose image is x. Consider the derivation D_X of F which takes X to 1 and takes to 0 all other elements of $\mathcal{B}$. Let then ∂_x be the derivation of F/R' in $\mathbf{Z}[F/R]$ induced by D_X:

$$\partial_x(aR') = \overline{D_X(a)} \quad (a \in F).$$

We have that

$$\partial_x((xy)^l) = \partial_x(x^l).$$

Let, for instance, $l > 0$. Then

$$\begin{aligned} \partial_x((xy)^l) &= 1 + \overline{xy} + \cdots + \overline{xy}^{l-1} = \partial_x(x^l) \\ &= 1 + \bar{x} + \cdots + \bar{x}^{l-1}. \end{aligned}$$

We apply an endomorphism of the group ring $\mathbf{Z}[F/R]$ induced by the endomorphism of F/R fixing all elements of $\overline{\mathcal{X}} \setminus \{\bar{y}\}$ and taking $\overline{xy} \to 1$ to both parts of the last equation:

$$l = \partial_x(x^l).$$

The same is true when $l < 0$. Thus

$$\partial_x(x^l) = l$$

which means that $l = 0$, or $l = 1$. The former is clearly impossible, since σ is an automorphism of F/R'. Hence $l = k_1 + \cdots + k_m = 1$, as claimed.

Observe also that after applying to both parts of (2.3) an endomorphism of F/R taking both x, y to 1 and fixing all u_i, we get that

$$w(1; \vec{u}) = 1.$$

In particular,

$$v_1 v_2 \ldots v_m = 1,$$

and then

$$w(x; \vec{u}) = \prod_{i=1}^{m} c_i x^{k_i} c_i^{-1},$$

where

$$c_i = v_1 \ldots v_i \quad (i = 1, \ldots, m).$$

Let $\partial : F/R' \to \mathbf{Z}[F/R]$ be a derivation (2.1) of F/R' associated with the basis $\mathcal{B}$ of F we have chosen above. By (2.2), for every $r \in R/R'$ we have that

$$\begin{aligned} \partial(\sigma(r)) &= \partial\left(\prod_{i=1}^{m} c_i r^{k_i} c_i^{-1}\right) = \sum_{i=1}^{m} k_i \bar{c}_i \, \partial(r) \\ &= \left(\sum k_i \bar{c}_i\right) \partial(r). \end{aligned}$$

Let us denote the element $\sum k_i\bar{c}_i \in \mathbf{Z}[F/R]$ by f_σ. As the conjugacy class of the inverse σ^{-1} of σ is also small, the same argument applies to σ^{-1}: there is an element $f_{\sigma^{-1}}$ of $\mathbf{Z}[F/R]$ with

$$\partial(\sigma^{-1}(r)) = f_{\sigma^{-1}}\partial(r) \quad [r \in R/R'].$$

The proof of Corollary 1 in [4] demonstrates that R/R' is a fully invariant subgroup of F/R', provided that the group ring $\mathbf{Z}[F/R]$ has no zero divisors. Then $\sigma^{-1}(r) \in R/R'$ for every $r \in R/R'$ and hence

$$\partial(r) = \partial(\sigma(\sigma^{-1}(r))) = f_\sigma\partial(\sigma^{-1}(r)) = f_\sigma f_{\sigma^{-1}}\partial(r),$$

or

$$(1 - f_\sigma f_{\sigma^{-1}})\partial(r) = 0.$$

As $\mathbf{Z}[F/R]$ has no divisors of zero,

$$1 = f_\sigma f_{\sigma^{-1}} = f_{\sigma^{-1}} f_\sigma$$

and as $\mathbf{Z}[F/R]$ has only trivial units,

$$f_\sigma = \bar{v}, \quad \text{or} \quad f_\sigma = -\bar{v}$$

for some $v \in F/R'$, whence

$$\partial(\sigma(r)) = \bar{v}\,\partial(r), \quad \text{or} \quad \partial(\sigma(r)) = -\bar{v}\,\partial(r)$$

for all $r \in R/R'$. In the first case we are done:

$$\partial(\sigma(r)) = \partial(vrv^{-1})$$

and $\sigma(r) = vrv^{-1}$, since ∂ is injective. In the second case

$$k_1\bar{c}_1 + \cdots + k_m\bar{c}_m = -\bar{v},$$

which is impossible, since the vector on the left-hand side has augmentation $k_1 + \cdots + k_m = 1$, whereas the vector on the right-hand side has augmentation -1. □

Generalizing an earlier result by Shmel'kin [15] on free solvable groups, Dyer [4] proved the following result: if F is a free group and a normal subgroup R is such that the quotient group F/R is torsion-free and either is solvable, or has nontrivial center and is not cyclic-by-periodic, then any automorphism of the group F/R' which fixes R/R' pointwise is an inner automorphism of F/R' determined by an element of R/R'. We have therefore the following corollary of Lemma 2.1.

Corollary 2.2. *Let F be an infinitely generated free group and R a fully invariant subgroup of F such that the quotient group F/R satisfies the conditions of Dyer's theorem and all units of the group ring $\mathbf{Z}[F/R]$ are trivial. Then the group $\mathrm{Aut}(F/R')$ is complete. In particular, the automorphism group of any infinitely generated free solvable group of derived length $\geqslant 2$ is complete.*

Proof. First, observe that as F/R must be torsion-free by the conditions, triviality of units of $\mathbf{Z}[F/R]$ implies that $\mathbf{Z}[F/R]$ has no zero divisors (see, for instance, [11, Section 6].)

By Proposition 1.3(iv), the subgroup S of elements of $\mathrm{Aut}(F/R')$ whose conjugacy class is small is a characteristic subgroup of $\mathrm{Aut}(F/R')$. By Lemma 2.1 and by the quoted result by Dyer from [4], S equals $\mathrm{Inn}(F/R')$. For any automorphism σ of F/R' whose conjugacy class is small is inner: the restriction of σ on R/R' coincides with the restriction on R/R' of a suitable inner automorphism τ_v of F/R'; then $\tau_{v^{-1}}\sigma$ fixes R/R' pointwise, and $\tau_{v^{-1}}\sigma = \tau_r$ for some $r \in R/R'$. Hence $\mathrm{Inn}(F/R') = S$ is a characteristic subgroup of $\mathrm{Aut}(F/R')$, and then the group $\mathrm{Aut}(F/R')$ is complete (by Burnside's theorem quoted in Remark 1.6).

Recall that free polynilpotent groups (in particular, free solvable groups) are orderable [14]. Also, the group ring $\mathbf{Z}[G]$ of an orderable group G has only trivial units [11, Section 6]. Thus the conditions of the corollary are met by any infinitely generated free solvable group $F/F^{(k)}$ of derived length $k \geqslant 2$, and hence the automorphism group $\mathrm{Aut}(F/F^{(k)})$ of $F/F^{(k)}$ is complete. □

3. Residually torsion-free nilpotent relatively free groups F/R'

Till the end of this section F will denote an infinitely generated free group, R a fully invariant subgroup of R, and G the quotient group F/R'. We shall assume throughout the section that the quotient group F/R is residually torsion-free nilpotent.

Recall that if $\mathcal{P}$ is a property of groups, a group H is said to be *residually* $\mathcal{P}$, if for every nonidentity element h of H, there is a surjective homomorphism from H onto a group with $\mathcal{P}$ such that the image of h under this homomorphism is not trivial.

By a quite standard argument, every residually orderable group is orderable. As any torsion-free nilpotent group is orderable, we obtain that the group F/R is orderable, and according to the remarks we have made at the end of the previous section the group ring $\mathbf{Z}[F/R]$ is a domain whose units are trivial. So Lemma 2.1 applies to $G = F/R'$.

As usual $\gamma_k(G)$, where $k \in \mathbf{N}$, denotes the k-th term of the lower central series of G ($\gamma_1(G) = G$ and $\gamma_{k+1}(G) = [G, \gamma_k(G)]$ for all natural numbers $k \geqslant 1$). As in [7] we define the series $(\overline{\gamma}_k(G) : k \geqslant 1)$ where

$$\overline{\gamma}_k(G) = \{g \in G : g^m \in \gamma_k(G) \text{ for some integer } m \neq 0\}.$$

Clearly, $G/\overline{\gamma}_k(G)$ is a torsion-free nilpotent group of class at most $k - 1$. According to a result by Hartley [10, Theorem D2], if F/R is residually torsion-free nilpotent, so is F/R'. Thus G is residually torsion-free nilpotent, and hence

$$\bigcap_{k\geqslant 1} \overline{\gamma}_k(G) = \{1\}.$$

For the sake of formality, we shall say that a relation X on a group H is *definable* in H, if X admits a description in H in terms of group operation. For instance, X is definable in H if X is the set of realizations in H of a suitable formula of some logic. Any definable relation on a given group H is invariant under all automorphisms of H.

Working with a subgroup H of G, we shall denote by I_H the group $\{\tau_h : h \in H\}$ of all inner automorphisms of G determined by members of H.

Lemma 3.1. *Let S be the subgroup of all automorphisms of G whose conjugacy class is small. Then*

(i) $S = \mathrm{Inn}(G) \cdot S_{(\widehat{R})}$ *where $S_{(\widehat{R})}$ is the subgroup of all elements of S fixing $\widehat{R}$ pointwise;*

(ii) *the subgroup $S_{(\widehat{R})}$ is the Hirsch–Plotkin radical (the maximal locally nilpotent subgroup) of the group S;*

(iii) *if $R \leqslant F'$, then S' coincides with the subgroup $I_{G'}$ of all inner automorphisms of G determined by elements of G'. In particular,*

$$I_{\widehat{R}} = S_{(\widehat{R})} \cap S';$$

(iv) *elements of the form $\tau_x\gamma$ where x is an element of G whose image under the natural homomorphism $G = F/R' \to F/R$ is a primitive element of the group F/R and $\gamma \in S_{(\widehat{R})}$ form a definable family of the group* $\mathrm{Aut}(G)$.

Proof. (i) Observe that by Lemma 2.1, every element σ of S can be written in the form

$$\tau_v({\tau_v}^{-1}\sigma) = \tau_v\gamma$$

where the automorphism $\gamma = {\tau_v}^{-1}\sigma$ fixes $\widehat{R}$ pointwise and its conjugacy class is small.

(ii) We base our argument on the fact that $\widehat{R} = R/R'$ is the Hirsch–Plotkin radical of the group $G = F/R'$ [7].

According to Corollary 2 in [4], if an automorphism π belongs to the subgroup $\mathrm{Aut}(G)_{(\widehat{R})}$, that is, if it fixes $\widehat{R}$ pointwise, then

$$x^{-1}\pi x \in \widehat{R}$$

for all $x \in G$. It follows that

$$\pi x = x r_x \tag{3.1}$$

where $r_x \in \widehat{R}$ for all $x \in G$, and hence the group $\mathrm{Aut}(G)_{(\widehat{R})}$ is abelian. As $\widehat{R}$ is a characteristic subgroup of G, the group $\mathrm{Aut}(G)_{(\widehat{R})}$ is a normal subgroup of $\mathrm{Aut}(G)$. It follows that $S_{(\widehat{R})}$ is a normal abelian subgroup of $\mathrm{Aut}(G)$.

Let $\tau_g\gamma$ where $\gamma \in S_{(\widehat{R})}$ be an element of S which is not in $S_{(\widehat{R})}$. In particular, $g \in G \setminus \widehat{R}$. Let r be a nonidentity element of $\widehat{R}$; clearly, $\tau_r \in S_{(\widehat{R})}$. As it is shown in [7],

$$[g, g, \ldots, g, r] \neq 1 \tag{3.2}$$

(see the proof of Theorem 3.5 in [7]). But then

$$[\tau_g\gamma, \tau_g\gamma, \ldots, \tau_g\gamma, \tau_r] = \tau_{[g,g,\ldots,g,r]} \neq \mathrm{id}$$

in $\mathrm{Aut}(G)$. Hence there is no locally nilpotent subgroup of S properly containing $S_{(\widehat{R})}$.

(ii) Recall that $I_{G'}$ denotes the group of inner automorphisms of G determined by elements of G'. Clearly, $I_{G'} \leqslant S'$, since $\mathrm{Inn}(G) \leqslant S$. Consider a commutator of elements of S:

$$\rho = \tau_a\gamma\tau_b\delta\gamma^{-1}{\tau_a}^{-1}\delta^{-1}{\tau_b}^{-1}$$

where $\gamma, \delta \in S_{(\widehat{R})}$. Then

$$\rho = \tau_a\tau_{\gamma(b)}\tau_{\delta(a^{-1})}\tau_{b^{-1}}.$$

By (3.1) there exist $r_b, s_a \in \widehat{R}$ such that $\gamma(b) = br_b$ and $\delta(a) = as_a$. Then ρ is the inner automorphism determined by the element

$$abr_b{s_a}^{-1}a^{-1}b^{-1}$$

of $G'\widehat{R} = G'$, since $R \leqslant F'$.

(iv) By (i), $S = \mathrm{Inn}(G) \cdot S_{(\widehat{R})}$ and then

$$\begin{aligned} S/S_{(\widehat{R})} &= \mathrm{Inn}(G) \cdot S_{(\widehat{R})}/S_{(\widehat{R})} \cong \mathrm{Inn}(G)/(\mathrm{Inn}(G) \cap S_{(\widehat{R})}) \\ &\cong I_G/I_{\widehat{R}} \cong G/\widehat{R} \cong F/R. \end{aligned}$$

Thus $S/S_{(\widehat{R})}$ is a relatively free group isomorphic to the group F/R. As the family of all primitive elements a given relatively free group H is definable in H, the result follows. To explain in terms of group operation that an element z of H is primitive, one explains that z can be included into some basis of H; a basis X of H being a subset of H such that any map from X into H can be extended to a homomorphism from H into H. □

Proposition 3.2. *Let $R \leqslant F'$. Suppose that the following conditions are true for an automorphism $\sigma \in \mathrm{Aut}(G)$:*
(a) *the conjugacy class of σ is small;*
(b) *the image of σ under the natural homomorphism $S \to S/S_{(\widehat{R})}$ is a primitive element of the relatively free group $S/S_{(\widehat{R})} \cong F/R$;*
(c) *the group $L(\sigma) = \mathrm{NC}(\sigma)I_{G'}$ contains no element of $S_{(\widehat{R})} \setminus I_{\widehat{R}}$.*
It follows that σ is an inner automorphism of G, that $\mathrm{NC}(\sigma)I_{G'} = \mathrm{Inn}(G)$, and that $\mathrm{Inn}(G)$ is a characteristic subgroup of $\mathrm{Aut}(G)$.

Proof. Let

$$\sigma = \tau_x \gamma \tag{3.3}$$

where $\gamma \in S_{(\widehat{R})}$. Suppose, towards a contradiction, that σ is not an inner automorphism of G. This implies that $\gamma \in S_{(\widehat{R})} \setminus I_{\widehat{R}}$. As under the natural homomorphisms $F/R' \to F/R$ and $F/R \to F/F'$ the element x goes to a primitive element of F/F', there is a $c \in G'$ such that cx is a primitive element of G.

The group $L(\sigma)$ then contains the element

$$\tau_c \tau_x \gamma = \tau_{cx} \gamma.$$

This enables us to assume without loss of generality that x in (3.3) is already a primitive element of G.

Since (c) is satisfied by σ, whenever elements of the form $\tau_a \gamma_1$ and $\tau_a \gamma_2$, where $a \in G$ and $\gamma_1, \gamma_2 \in S_{(\widehat{R})}$, both belong to $L(\sigma)$, the elements γ_1, γ_2 must be congruent modulo $I_{\widehat{R}}$:

$$\tau_a \gamma_1, \tau_a \gamma_2 \in L(\sigma) \Rightarrow \gamma_1 \equiv \gamma_2 \pmod{I_{\widehat{R}}}; \tag{3.4}$$

otherwise $\gamma_2{}^{-1} \tau_a{}^{-1} \cdot \tau_a \gamma_1 \in S_{(\widehat{R})} \setminus I_{\widehat{R}}$, contradicting (c).

In particular, for any automorphism π of G stabilizing our primitive element x, we have by (3.4) that

$$\gamma^\pi \equiv \gamma \pmod{I_{\widehat{R}}}.$$

where $\gamma^\pi = \pi \gamma \pi^{-1}$.

Fix a basis $\mathcal{X}$ of G containing x. By Proposition 1.2 and by (3.1)

$$\gamma(t) = tv(t; u_1, \ldots, u_k) \quad (t \in \mathcal{X})$$

where $\vec{u} = u_1, \ldots, u_k$ are some (fixed) members of $\mathcal{X}$, $v(*; *_1, \ldots, *_k)$ is a term/word of the language of group theory such that

$$ztv(zt; \vec{u}) = zv(z; \vec{u}) \cdot tv(t; \vec{u})$$

for all $z, t \in \mathcal{X}$ and

$$v(t; \vec{u}) \in \widehat{R}$$

for all $t \in \mathcal{X}$.

(1) *Suppose first that x does not belong to the tuple $\vec{u}$*. Consider then an automorphism π of G acting on $\mathcal{X}$ as a permutation which fixes x and takes $\vec{u}$ to a tuple $\pi\vec{u}$ having no common element with $\vec{u}$: $\pi\vec{u} \cap \vec{u} = \varnothing$. By (3.4), there is an $s \in \widehat{R}$ such that $\gamma^\pi = \tau_s \gamma$. This implies that

$$zv(z; \pi\vec{u}) = szs^{-1}v(z; \vec{u})$$

for all $z \in \mathcal{X}$. Since $\mathcal{X}$ is infinite, there is a $t \in \mathcal{X}$ such that the letter t does not appear in the word s, nor $t \in \vec{u}$, nor $t \in \pi\vec{u}$. We then apply an endomorphism of G taking all elements $\pi\vec{u}$ to 1 and fixing all other elements of $\mathcal{X}$ to both parts of the equation

$$tv(t; \pi\vec{u}) = sts^{-1}v(t; \vec{u}),$$

thereby getting that

$$t = s_0 t s_0{}^{-1} v(t; \vec{u}),$$

or

$$s_0{}^{-1}ts_0 = tv(t; \vec{u}).$$

It follows that

$$s_0{}^{-1}zs_0 = zv(z; \vec{u})$$

for every $z \in \mathcal{X}$, and then γ is an inner automorphism determined by an element $s_0{}^{-1} \in \widehat{R}$, a contradiction.

(2) *Suppose now that x is a member of $\vec{u}$* and $\vec{u} = x, u_2, \ldots, u_k$. Write $\vec{u}_0$ for the tuple $u_2, \ldots, u_k$. As above, we consider an automorphism π of G acting on $\mathcal{X}$ as a permutation, fixing x and such that tuples $\vec{u}_0$ and $\pi\vec{u}_0$ are disjoint. Then

$$zv(z; x, \pi\vec{u}_0) = szs^{-1}v(z; x, \vec{u}_0)$$

for all $z \in \mathcal{X}$. Working with the endomorphism of G taking all elements $\pi\vec{u}_0$ to 1 and fixing pointwise $\mathcal{X} \setminus \{\pi\vec{u}_0\}$, we see that

$$tw(t; x) = s_0ts_0{}^{-1}v(t; x, \vec{u}_0),$$

or

$$s_0{}^{-1}ts_0w(t; x) = tv(t; x, \vec{u}_0), \tag{3.5}$$

where $w(*; *_1)$ is a fixed group word/term of the language of groups, for all $t \in \mathcal{X}$ (at first for $t \in \mathcal{X}$ that are not members of $\vec{u}_0$, πu_0, and the set of letters of $\mathcal{X}$ forming s, then for all $t \in \mathcal{X}$.)

Eq. (3.5) means that

$$\gamma = \tau_{s_0^{-1}}\delta$$

where

$$\delta t = tw(t; x) \quad (t \in \mathcal{X}).$$

Clearly, δ is in $S_{(\widehat{R})}$, is not an inner automorphism of G, and the element $\tau_x\delta$ is a member of $L(\sigma)$.

Our goal is to show that δ is the identity automorphism; this will imply, as in (1) above, that γ is an inner automorphism, which is impossible.

Claim 1. *For every $t \in \mathcal{X}$ and for every natural number k*

$$w(t; x^k) = w(t; x)^k.$$

Let y be a member of $\mathcal{X}$ which is not equal to x. We start with some two elements of $L(\sigma)$ of the form $\tau_y\eta$ where $\eta \in S_{(\widehat{R})}$ to gain more information about $w(*; *_1)$. First, we see that $\tau_y\delta^\pi$ belongs to $L(\sigma)$ where $\pi \in \mathrm{Aut}(G)$ interchanges x and y, while fixing all other elements of $\mathcal{X}$. Second, let ρ be the automorphism of G which takes x to xy and fixes $\mathcal{X} \setminus \{x\}$ pointwise. Then

$$(\tau_x\delta)^\rho\tau_x{}^{-1}\delta^{-1} = \tau_{xyr^{-1}x^{-1}}\delta^\rho\delta^{-1} \in L(\sigma),$$

where $\delta^\rho(x) = xr$ and $r \in \widehat{R}$. As for a suitable element c of G' we have that $cxyr^{-1}x^{-1} = y$, it follows that $\tau_y\delta^\rho\delta^{-1}$ is also in $L(\sigma)$. Hence by (3.4), there exists an $s \in \widehat{R}$ with

$$\delta^\rho\delta^{-1} = \tau_s\delta^\pi.$$

Comparing the images of a $t \in \mathcal{X} \setminus \{x\}$ under the automorphisms participating in both parts of the last equation, one obtains that

$$tw(t; xy)w(t; x)^{-1} = sts^{-1}w(t; y). \tag{3.6}$$

In particular,

$$yw(y; xy)w(y; x)^{-1} = sys^{-1}$$

for $t = y$.

Let $k > 1$. Consider the endomorphism ε of G taking y to x^k and fixing $\mathcal{X} \setminus \{y\}$ pointwise. Then

$$x^kw(x^k; x^{k+1})w(x^k; x)^{-1} = \varepsilon(s)x^k\varepsilon(s)^{-1},$$

whence $x^k = \varepsilon(s)x^k\varepsilon(s)^{-1}$. Clearly, $x^k \notin \widehat{R}$, since F/R is torsion-free and $\varepsilon(s) \in \widehat{R}$. Hence $\varepsilon(s) = 1$. We then apply ε to both parts of (3.6), assuming that t is an arbitrary element of $\mathcal{X} \setminus \{x\}$:

$$tw(t; x^{k+1})w(t; x)^{-1} = tw(t; x^k).$$

By the induction hypothesis $w(t; x^k) = w(t; x)^k$ and the result follows.

Claim 2. *For every $t \in \mathcal{X}$ the element $w(t;x)$ is in $\gamma_3(G)$, the third term of the lower central series of G.*

As we observed above, δ fixes $\widehat{R}$ pointwise, and adding that to the fact that δ has small conjugacy class, we get that

$$\delta(r) = rw(r;x) = r,$$

for all $r \in \widehat{R}$, whence $w(r;x) = 1$. Also $w(1,x) = 1$ and then we can write $w(t;x)$ where $t \in \mathcal{X}$ as a product of conjugates of powers of t:

$$w(t;x) = \prod_{i=1}^{n} x^{k_i} t^{s_i} x^{-k_i}.$$

Since $w(t,x) \in \widehat{R} \leqslant G'$, the sum of exponents s_i is zero. Substitute an arbitrary $r \in \widehat{R}$ for t in the last equality and take the standard derivative, keeping in mind that $w(r;x) = 1$:

$$0 = \partial(w(r;x)) = \sum_{i=1}^{n} s_i \bar{x}^{k_i} \partial(r) = \left(\sum_{i=1}^{n} s_i \bar{x}^{k_i}\right) \partial(r).$$

Therefore

$$\sum_{i=1}^{n} s_i \bar{x}^{k_i} = 0. \tag{3.7}$$

Suppose that there are exactly l pairwise distinct exponents k_i participating in (3.7), say, $m_1, \ldots, m_l$. Due to linear independence of powers of $\bar{x}$ over $\mathbf{Z}$, there must be a partition of $\{1, 2, \ldots, n\}$ into l pairwise disjoint sets

$$\{1, 2, \ldots, n\} = A_1 \sqcup \cdots \sqcup A_l$$

such that for a particular j, for every $i_1, i_2 \in A_j$, integers s_{i_1}, s_{i_2} are coefficients of x^{m_j} in (3.7) and

$$\sum_{i \in A_j} s_i = 0.$$

Observe that in a nilpotent group H of class two all conjugates of a given element of H are commuting. Then we have for every $t \in \mathcal{X}$:

$$\begin{aligned} w(t;x) &= \prod_{i=1}^{n} x^{k_i} t^{s_i} x^{-k_i} \equiv \prod_{j=1}^{l} \prod_{i \in A_j} (x^{m_j} t^{s_i} x^{-m_j}) \pmod{\gamma_3(G)} \\ &\equiv \prod_{j=1}^{l} (x^{m_j} t^{\sum_{i \in A_j} s_i} x^{-m_j}) \equiv \prod_{j=1}^{l} (x^{m_j} t^0 x^{-m_j}) \equiv 1 \pmod{\gamma_3(G)}. \end{aligned}$$

Claim 3. *For every $k \geqslant 3$ and for every $t \in \mathcal{X} \setminus \{x\}$*

$$w(t;x) \equiv 1 \pmod{\overline{\gamma}_k(G)}.$$

Hence $w(t;x) = 1$ and δ is the identity automorphism.

Claim 2 takes care of the induction base. Assume that $w(t;x) \in \overline{\gamma}_k(G)$, that is,

$$w(t;x)^m = w(t;x^m) \in \gamma_k(G) \tag{3.8}$$

for some natural number $m > 1$ (the equality is justified by Claim 1).

Observe that the subgroup $\overline{\gamma}_k(G)$ is invariant under all endomorphisms of G. As δ has small conjugacy class, $\delta(z) = zw(z;x)$ for every $z \in G$ by Proposition 1.2. Hence for every $t \in \mathcal{X}$ and for every $q \in \mathbf{Z}$

$$\delta(t^q) = (tw(t;x))^q = t^q w(t^q;x).$$

Let ε be an endomorphism of G taking x to x^{ml} where $l \in \mathbf{Z}$ and stabilizing every element of $\mathcal{X} \setminus \{x\}$. After application of ε to the last equality, we see that

$$t^q w(t^q, x^{ml}) = (tw(t;x^{ml}))^q.$$

Since elements of $\gamma_k(G)$ commute modulo $\gamma_{k+1}(G)$ with all elements of G and since $w(t;x^{ml}) = w(t;x^m)^l \in \gamma_k(G)$ by Claim 1,

$$t^q w(t^q, x^{ml}) = (tw(t;x^{ml}))^q \equiv t^q w(t;x^{ml})^q \pmod{\gamma_{k+1}(G)}.$$

Then, again by Claim 1,

$$w(t^q, x^{ml}) \equiv w(t;x)^{qml} \pmod{\gamma_{k+1}(G)}. \tag{3.9}$$

Due to invariance of $\gamma_k(G)$ under endomorphisms of G, we obtain from (3.8) that $w(t^m; x^m) \in \gamma_k(G)$. Therefore $w(t^m; x^m)$ can be written as a product of basis commutators of weight k modulo $\gamma_{k+1}(G)$:

$$w(t^m; x^m) \equiv \prod_i b_i(t, x) \pmod{\gamma_{k+1}(G)}.$$

Consider an endomorphism of G taking both x and t to their squares and apply it to the last congruence:

$$w(t^{2m}; x^{2m}) \equiv \prod_i b_i(t^2, x^2) \pmod{\gamma_{k+1}(G)}. \tag{3.10}$$

It is easy to see that $b_i(t^2, x^2) \equiv b_i(t, x)^{2^k} \pmod{\gamma_{k+1}(G)}$; for instance,

$$[t^2, x^2, t^2] \equiv [t, x^2, t^2]^2 \equiv [t, x, t^2]^{2^2} \equiv [t, x, t]^{2^3} \pmod{\gamma_4(G)}.$$

This implies that the element on the right-hand side of (3.10) is congruent to $w(t^m; x^m)^{2^k}$, and, further, to $w(t, x)^{m^2 2^k}$ by (3.9). By the same Eq. (3.9), the element on the left-hand side of (3.10) is congruent to $w(t; x)^{4m^2}$. Therefore

$$w(t; x)^{4m^2} \equiv w(t; x)^{2^k m^2} \pmod{\gamma_{k+1}(G)},$$

or

$$w(t; x)^{(2^k-4)m^2} \equiv 1 \pmod{\gamma_{k+1}(G)}.$$

As $k \geqslant 3$, $2^k > 4$, and we are done. □

Theorem 3.3. *Let F be an infinitely generated free group, R a fully invariant subgroup of F which is contained in the commutator subgroup of F. Suppose that the quotient group F/R is residually torsion-free nilpotent. Then the automorphism group* $\mathrm{Aut}(F/R')$ *of the group F/R' is complete.*

Proof. By Burnside's theorem and by Proposition 3.2. □

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References

[1] M. Auslander, R.C. Lyndon, Commutator subgroups of free groups, Amer. J. Math. 77 (1955) 929–931.
[2] W. Burnside, Theory of Groups of Finite Order, Dover Publications, Inc., New York, 1955.
[3] J. Dixon, P.M. Neumann, S. Thomas, Subgroups of small index in infinite symmetric groups, Bull. Lond. Math. Soc. 18 (1986) 580–586.
[4] J.L. Dyer, A criterion for automorphisms of certain groups to be inner, J. Aust. Math. Soc. Ser. A 21 (1976) 179–184.
[5] J.L. Dyer, E. Formanek, The automorphism group of a free group is complete, J. Lond. Math. Soc. 11 (1975) 181–190.
[6] J.L. Dyer, E. Formanek, Automorphism sequences of free nilpotent group of class two, Math. Proc. Cambridge. Philos. Soc. 79 (1976) 271–279.
[7] J.L. Dyer, E. Formanek, Characteristic subgroups and complete automorphism groups, Amer. J. Math. 99 (1977) 713–753.
[8] E. Formanek, Characterizing a free group in its automorphism group, J. Algebra. 133 (1990) 424–432.
[9] R. Fox, Free differential calculus I, Ann. of Math. 37, 547–560.
[10] B. Hartley, The residual nilpotence of wreath products, Proc. Lond. Math. Soc. 20 (1970) 365–392.
[11] T.Y. Lam, A first course in noncommutative rings, in: Graduate Texts in Mathematics, vol. 131, Springer-Verlag, New York, 2001.
[12] W. Magnus, On a theorem of Marshall Hall, Ann. of Math. (2) 40 (1939) 764–768.
[13] V.N. Remeslennikov, V.G. Sokolov, Certain properties of the Magnus embedding, Algebra Logika 9 (1970) 566–578.
[14] A.L. Shmel'kin, Free polynilpotent groups, Dokl. Akad. Nauk SSSR 151 (1963) 73–75.
[15] A.L. Shmel'kin, Two remarks on free solvable groups, Algebra Logika 6 (1967) 95–109.
[16] V. Tolstykh, The automorphism tower of a free group, J. Lond. Math. Soc. (2) 61 (2000) 423–440.
[17] V. Tolstykh, Free two-step nilpotent groups whose automorphism group is complete, Math. Proc. Cambridge Philos. Soc. 131 (2001) 73–90.
[18] V. Tolstykh, Infinitely generated free nilpotent groups: completeness of the automorphism groups, Math. Proc. Cambridge Philos. Soc. 147 (2009) 541–566.