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Some operator monotone functions

Frank Hansen

Department of Economics, University of Copenhagen, Studiestraede 6, DK-1455 Copenhagen K, Denmark

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Dedicated to Professor Jun Tomiyama on his 77th birthday with respect and affection.

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ABSTRACT

We prove that the functions $t \to (t^q - 1)(t^p - 1)^{-1}$ are operator monotone in the positive half-axis for $0 < p \leqslant q \leqslant 1$, and we calculate the two associated canonical representation formulae. The result is used to find new monotone metrics (quantum Fisher information) on the state space of quantum systems.

1. Introduction

We consider the functions

$$f(t) = \begin{cases} \frac{p}{q} \cdot \frac{t^q - 1}{t^p - 1}, & t > 0,\ t \neq 1, \\ 1, & t = 1 \end{cases} \qquad (1)$$

for positive exponents p and q.

Theorem 1. *The function f in (1) is operator monotone for* $0 < p \leqslant q \leqslant 1$.

Proof. By an elementary calculation we may write

$$f(t) = \int_0^1 (\lambda t^p + 1 - \lambda)^{(q-p)/p} d\lambda.$$

E-mail address: Frank.Hansen@econ.ku.dk

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For $\Im z > 0$ and $0 < \lambda < 1$ we have $0 < \arg(\lambda z^p + 1 - \lambda) < p\pi$. Thus

$$0 < \arg(\lambda z^p + 1 - \lambda)^{(q-p)/p} < (q-p)\pi \leqslant \pi.$$

This shows that the integrand function and hence f is operator monotone [9,5,3]. $\square$

Let $z = re^{i\theta}$ with $r > 0$ and $0 < \theta < \pi$. Then the imaginary part

$$\Im f(z) = \frac{p}{q} \cdot \frac{r^{p+q}\sin(q-p)\theta - r^q \sin q\theta + r^p \sin p\theta}{r^{2p} - 2r^p \cos p\theta + 1}$$

and since f is operator monotone and non-constant, we thus have

$$r^{p+q}\sin(q-p)\theta - r^q \sin q\theta + r^p \sin p\theta > 0 \tag{2}$$

for $r > 0$ and $0 < \theta < \pi$. We need this result later in the paper.

In the first version of the paper we proved (2) directly to obtain operator monotonicity of f. Then Furuta gave an elementary proof using the techniques developed in [6, Proposition 3.1]. Finally, Ando gave the above proof which is the shortest known to the author.

2. Integral representations

Theorem 2. *The function f in* (1) *has the canonical representation*

$$f(t) = \frac{p}{q} + \frac{p}{q}\int_0^\infty \frac{t}{\lambda(t+\lambda)} \cdot \frac{\lambda^{p+q}\sin(q-p)\pi - \lambda^q \sin q\pi + \lambda^p \sin p\pi}{\pi(\lambda^{2p} - 2\lambda^p \cos p\pi + 1)} d\lambda$$

for $0 < p \leqslant q \leqslant 1$.

Proof. The representing measure of the operator monotone function f is calculated by first considering the analytic extension $f(z)$ to the upper complex half-plane, cf. [5]. If $z = re^{i\theta}$ approaches a real $\lambda < 0$ from the upper complex half-plane, then $r \to -\lambda$ and $\theta \to \pi$. Consequently, the imaginary part

$$\Im f(z) \to \frac{p}{q} \cdot \frac{(-\lambda)^{p+q}\sin(q-p)\pi - (-\lambda)^q \sin q\pi + (-\lambda)^p \sin p\pi}{(-\lambda)^{2p} - 2(-\lambda)^p \cos p\pi + 1}.$$

If $z = re^{i\theta}$ approaches zero from the upper complex half-plane, then θ is indeterminate but $r \to 0$ and $\Im f(z) \to 0$. The representing measure [5, Chapter II, Lemma 1] is thus given by

$$d\mu(\lambda) = \frac{p}{q} \cdot \frac{\lambda^{p+q}\sin(q-p)\pi - \lambda^q \sin q\pi + \lambda^p \sin p\pi}{\pi(\lambda^{2p} - 2\lambda^p \cos p\pi + 1)} d\lambda.$$

Since f is both positive and operator monotone it is necessarily of the form

$$f(t) = \alpha t + f(0) + \int_0^\infty \frac{t}{\lambda(t+\lambda)} d\mu(\lambda),$$

where $\alpha \geqslant 0$ and the representing measure μ satisfies

$$\int_0^\infty (\lambda^2+1)^{-1} d\mu(\lambda) < \infty \quad \text{and} \quad \int_0^1 \lambda^{-1} d\mu(\lambda) < \infty.$$

Finally, since the growth of $f(t)$ is smaller than the growth of t in infinity we obtain $\alpha = 0$, and the statement follows.

Theorem 3. *Let* $0 < p \leqslant q \leqslant 1$. *The function f in* (1) *has the canonical exponential representation*

$$f(t) = \frac{p}{q}\left(\frac{1-\cos q\frac{\pi}{2}}{1-\cos p\frac{\pi}{2}}\right)^{1/2} \exp\int_0^\infty \left(\frac{\lambda}{\lambda^2+1} - \frac{1}{\lambda+t}\right) h(\lambda) d\lambda,$$

where

$$h(\lambda) = \frac{1}{\pi} \arctan \frac{\lambda^{p+q} \sin(q-p)\pi - \lambda^q \sin q\pi + \lambda^p \sin p\pi}{\lambda^{p+q} \cos(q-p)\pi - \lambda^q \cos q\pi - \lambda^p \cos p\pi + 1} \quad \lambda > 0. \tag{3}$$

(*Notice that* $0 \leqslant h(\lambda) \leqslant 1/2$ *for every* $\lambda > 0$).

Proof. The exponential representation of f is obtained by considering the operator monotone function $\log f(t)$. The analytic continuation $\log f(z)$ to the upper complex half-plane has positive imaginary part bounded by π, cf. [1,5,7]. The representing measure of $\log f(t)$ is therefore absolutely continuous with respect to Lebesgue measure with Radon–Nikodym derivative bounded by one. We calculate and obtain the expression

$$\begin{aligned}\Im \log f(z) &= \frac{1}{2i} \log \frac{(r^q e^{iq\theta} - 1)(r^p e^{-ip\theta} - 1)}{(r^p e^{ip\theta} - 1)(r^q e^{-iq\theta} - 1)} \\ &= \frac{1}{2i} \log \left[\frac{a + ib}{(r^{2p} - 2r^p \cos p\theta + 1)^{1/2} (r^{2q} - 2r^q \cos q\theta + 1)^{1/2}} \right]^2,\end{aligned}$$

where

$$\begin{aligned} a &= r^{p+q} \cos(q-p)\theta - r^q \cos q\theta - r^p \cos p\theta + 1, \\ b &= r^{p+q} \sin(q-p)\theta - r^q \sin q\theta + r^p \sin p\theta. \end{aligned}$$

We recognize from (2) that $b > 0$, and since the imaginary part of $(a+ib)^2$ is positive, we obtain that also $a > 0$. The expression inside the square is thus a complex number of modulus one in the first quadrant of the complex plane. It is of the form $\exp i\phi(z)$ for some $\phi(z)$ with $0 < \phi(z) < \pi/2$, where

$$\phi(z) = \arctan \frac{r^{p+q} \sin(q-p)\theta - r^q \sin q\theta + r^p \sin p\theta}{r^{p+q} \cos(q-p)\theta - r^q \cos q\theta - r^p \cos p\theta + 1}$$

and $\Im \log f(z) = \phi(z)$. We let $z = re^{i\theta}$ approach a real $\lambda < 0$ from the upper complex half-plane and obtain

$$\Im \log f(z) \to \arctan \frac{(-\lambda)^{p+q} \sin(q-p)\pi - (-\lambda)^q \sin q\pi + (-\lambda)^p \sin p\pi}{(-\lambda)^{p+q} \cos(q-p)\pi - (-\lambda)^q \cos q\pi - (-\lambda)^p \cos p\pi + 1}.$$

The representing measure is thus given by the weight function

$$h(\lambda) = \frac{1}{\pi} \arctan \frac{\lambda^{p+q} \sin(q-p)\pi - \lambda^q \sin q\pi + \lambda^p \sin p\pi}{\lambda^{p+q} \cos(q-p)\pi - \lambda^q \cos q\pi - \lambda^p \cos p\pi + 1} \quad \lambda > 0,$$

and we obtain the exponential representation

$$f(t) = \exp \left[\beta + \int_0^\infty \left(\frac{\lambda}{\lambda^2 + 1} - \frac{1}{\lambda + t} \right) h(\lambda) d\lambda \right],$$

where $\beta = \Re \log f(i)$. By a tedious calculation we obtain

$$\beta = \log p - \log q + \frac{1}{2} \log \frac{1 - \cos q\frac{\pi}{2}}{1 - \cos p\frac{\pi}{2}}$$

and the statement is proved. □

3. Applications to quantum information theory

Definition 4. We denote by $\mathscr{F}_{\mathrm{op}}$ the set of functions $f\colon \mathbf{R}_+ \to \mathbf{R}_+$ satisfying

(i) f is operator monotone,
(ii) $f(t) = tf(t^{-1})$ for all $t > 0$,
(iii) $f(1) = 1$.

The following result was proved in [2, Theorem 2.1].

Theorem 5. *A function $f \in \mathscr{F}_{\rm op}$ admits a canonical representation*

$$f(t) = \frac{1+t}{2} \exp \int_0^1 \frac{(\lambda^2-1)(1-t)^2}{(\lambda+t)(1+\lambda t)(1+\lambda)^2} h(\lambda) d\lambda, \tag{4}$$

where the weight function $h : [0,1] \to [0,1]$ is measurable. The equivalence class containing h is uniquely determined by f. Any function on the given form is in $\mathscr{F}_{\rm op}$.

In addition, for $z = re^{i\theta}$ with $r > 0$ and $0 < \theta < \pi$, the weight function h in the above theorem appears as

$$h(\lambda) = \frac{1}{\pi} \lim_{z \to -\lambda} \Im \log f(z)$$

for almost all $\lambda \in (0,1]$. Notice that $r \to \lambda$ and $\theta \to \pi$ when $z \to -\lambda$.

A monotone metric is a map $\rho \to K_\rho(A,B)$ from the set $\mathscr{M}_n$ of positive definite $n \times n$ density matrices to sesquilinear forms $K_\rho(A,B)$ defined on $M_n(\mathbf{C})$ satisfying:

1. $K_\rho(A,A) \geqslant 0$, and equality holds if and only if $A = 0$.
2. $K_\rho(A,B) = K_\rho(B^*,A^*)$ for all $\rho \in \mathscr{M}_n$ and all $A,B \in M_n(\mathbf{C})$.
3. $\rho \to K_\rho(A,A)$ is continuous on $\mathscr{M}_n$ for every $A \in M_n(\mathbf{C})$.
4. $K_{T(\rho)}(T(A),T(A)) \leqslant K_\rho(A,A)$ for every $\rho \in \mathscr{M}_n$, every $A \in M_n(\mathbf{C})$ and every stochastic mapping $T : M_n(\mathbf{C}) \to M_m(\mathbf{C})$.

A mapping $T : M_n(\mathbf{C}) \to M_m(\mathbf{C})$ is said to be stochastic if it is completely positive and trace preserving. A monotone metric [4,10] is given on the form

$$K_\rho(A,B) = \operatorname{Tr} A^* c(L_\rho, R_\rho) B, \tag{5}$$

where c is a so called Morozova–Chentsov function and $c(L_\rho, R_\rho)$ is the function taken in the pair of commuting left and right multiplication operators (denoted L_ρ and R_ρ, respectively) by ρ. The Morozova–Chentsov functions are of the form

$$c(x,y) = \frac{1}{y g(xy^{-1})}, \quad x,y > 0, \tag{6}$$

where $g \in \mathscr{F}_{\rm op}$.

There is an involution $f \to f^\#$ on the set of positive operator monotone functions f defined in the positive half-axis given by

$$f^\#(t) = t f(t^{-1}), \quad t > 0,$$

cf. [8]. It plays a role in the following result.

Theorem 6. *The function*

$$c(x,y) = \frac{q}{p} \cdot \frac{x^p - y^p}{x^q - y^q} (xy)^{-(1-q+p)/2}, \quad x,y > 0$$

is a Morozova–Chentsov function for $0 < p \leqslant q \leqslant 1$. The generating operator monotone function g in (6) has the exponential representation

$$g(t) = \frac{1+t}{2} \exp \int_0^1 \frac{(\lambda^2-1)(1-t)^2}{(\lambda+t)(1+\lambda t)(1+\lambda)^2} \left(\frac{1-q+p}{2} + h(\lambda) \right) d\lambda,$$

where the weight function h is the restriction to the unit interval of the function given in (3).

Proof. We first notice that the function

$$g(t) = \frac{p}{q} \cdot \frac{t^q - 1}{t^p - 1} t^{(1-q+p)/2}, \quad t > 0 \tag{7}$$

generates $c(x, y)$ according to (6). We thus have to prove that $g \in \mathscr{F}_{\rm op}$ for $0 < p \leqslant q \leqslant 1$. Choosing the function f as defined in (1), we obtain

$$f^{\#}(t) = t^{1-q+p} f(t), \quad t > 0.$$

Since obviously the function

$$\sqrt{f(t) f^{\#}(t)} = t^{(1-q+p)/2} f(t), \quad t > 0$$

is a fix-point under the involution # and the geometric mean is operator monotone, we obtain $g \in \mathscr{F}_{\rm op}$ as desired. We have thus proved the first part of the theorem. Setting $z = re^{i\theta}$ for $r > 0$ and $0 < \theta < \pi$ we calculate that

$$\Im \log g(z) = \frac{1-q+p}{2}\theta + \Im \log f(z).$$

The result now follows by Theorem 5 and the remarks below. □

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