Relative weak compactness of solid hulls in Banach lattices

by Z.L. Chen and A.W. Wickstead

Department of Pure Mathematics, The Queen's University of Belfast, Belfast BT7 INN, Northern Ireland

Communicated by Prof. J. Korevaar at the meeting of April 28, 1997

ABSTRACT

We show that the solid hull of every relatively weakly compact set in a Banach lattice is again relatively weakly compact if and only if the Banach lattice is an order direct sum of a KB-space and an atomic Banach lattice with an order continuous norm. If we assume order continuity of the norm then this is equivalent to requiring that the image of every relatively weakly compact set under the modulus map is again relatively weakly compact. We also show that amongst Banach lattices with an order continuous norm those that have the property that the lattice operations are weakly sequentially continuous are precisely the atomic ones. The final section of the paper is devoted to applications of our earlier results to questions concerning the factorization of compact and weakly compact operators through reflexive Banach lattices.

1. INTRODUCTION

There are a number of results in the Banach lattice literature which depend on knowing about the interaction between the order structure and relatively weakly compact sets. Some results are known about this, for example the solid hulls of positive relatively weakly compact sets are relatively weakly compact precisely in Banach lattices with an order continuous norm. Some desirable results however are not so well understood. It is known that in KB-spaces the solid hull of every relatively weakly compact set is again relatively weakly compact (see [AB2], Theorem 13.8 for proofs of these two results) but we will see in Theorem 2.4 that this does not characterize KB-spaces. In fact we show there that this property is enjoyed precisely by order direct sums of KB-spaces and atomic Banach lattices with an order continuous norm. A related property
that occurs in the literature is that the image of a relatively weakly compact set under the modulus map is again relatively weakly compact. It comes as no surprise that this together with order continuity of the norm is also equivalent.

The proof that atomic Banach lattices with an order continuous norm have the property uses the weak sequential continuity of the lattice operations in such spaces. This has long been known and we are able to show that this continuity characterises such Banach lattices amongst Banach lattices with an order continuous norm.

The final section of this paper is devoted to results on the factorization of compact and weakly compact operators through reflexive Banach lattices. The results in [ABI] rely on the range space having the property that every relatively weakly compact set has a relatively weakly compact solid hull (even though most of the corollaries are stated for the range being a KB-space) or (if attention is restricted to positive operators) having an order continuous norm. We show that these conditions are the weakest possible. This answers several of the questions posed in [ABI].

Prof. Y.A. Abramovich has kindly pointed out to us that results similar to those in section 2 may be found in his paper [A]. We have included our proofs here in the interests of completeness and because of the relatively inaccessible nature of [A].

2. RELATIVE WEAK COMPACTNESS OF SOLID HULLS

Let $(\Omega, \Sigma, \mu)$ be a measure space. $A \in \Sigma$ is called an atom if $\mu(A) > 0$ and for each $B \in \Sigma$ with $B \subseteq A$, then either $\mu(B) = \mu(A)$ or $\mu(B) = 0$. An atom in a vector lattice $E$ is an element $0 \neq a \in E_+$ such that if $b \in E$ with $0 \leq b \leq a$ then $b = \lambda a$ for some $\lambda \in \mathbb{R}$. The atoms in $L^p(\mu)$ are precisely the strictly positive multiples of the characteristic functions of atoms of $\mu$. A vector lattice is said to be purely atomic if the only element that is disjoint from all atoms is the zero element.

**Proposition 2.1.** Suppose that $(\Omega, \Sigma, \mu)$ is a finite measure space and that $\mu$ does not have any atoms, then there exists a sequence of measurable functions $r_n \in L^\infty(\mu)$ such that

1. $|r_n|(t) = 1_{\mu} - \text{almost everywhere}$
2. $\lim_{n \to \infty} \int_\Omega f(t)r_n(t)d\mu = 0 \quad \forall f \in L^1(\mu)$.

**Proof.** As $\mu$ has no atoms, we may find disjoint sets $B_{11}, B_{12} \in \Sigma$ with $B_{11} \cup B_{12} = \Omega$ and $\mu(B_{11}) = \mu(B_{12}) = \frac{1}{2} \mu(\Omega)$. Proceeding inductively, we may find sets $B_{ni} \in \Sigma$ such that

(a) $B_{n(2i-1)} \cup B_{n(2i)} = B_{n(n-1)i}$ \hspace{1cm} $i = 1, 2, \ldots, 2^{n-1}$
(b) $\mu(B_{n(2i-1)} \cap B_{n(2i)}) = 0$ \hspace{1cm} $i = 1, 2, \ldots, 2^{n-1}$
(c) $\mu(B_n) = (1/2^n)\mu(\Omega)$ \hspace{1cm} $i = 1, 2, \ldots, 2^n$.

Define also $B_{01} = \Omega$. 

188
Now define
\[ r_n = \sum_{k=1}^{2^n} (-1)^{k-1} \chi_{B_{2^n}}(\mu) \]

It is clear that \(|r_n| = \chi_{\Omega}\), and it is not difficult to verify that
\[ \int_{\Omega} r_n r_m d\mu = 0 \]
for all \(n, m \in \mathbb{N}\) with \(n \neq m\). It follows that \(\{r_n\}_1^\infty\) is an orthogonal sequence in \(L^2(\mu)\), and hence
\[ \lim_{n \to \infty} \int_{\Omega} f r_n d\mu = 0 \]
for all \(f \in L^2(\mu)\) and hence, as \(L^2(\mu)\) is dense in \(L^1(\mu)\), for all \(f \in L^1(\mu)\).

**Theorem 2.2.** Let \(E\) be a Banach lattice with an order continuous norm. If \(E\) does not contain any atoms then for each \(x \in E^+\) there exists a sequence \(x_n \in E\) such that \(|x_n| = x\) and \(x_n \to 0\) weakly as \(n \to \infty\).

**Proof.** For \(x \in E^+\) let \(F\) be the closed ideal of \(E\) generated by \(x\) which has an order continuous norm and a weak unit, so by Theorem 2.7.8 of [MN] and its proof (also see Theorem 120.10 in [Z]) there exists a probability measure space \((\Omega, \mathcal{C}, \mu)\) such that
\[ L^\infty(\mu) \subset F \subset L^1(\mu) \]
\[ L^\infty(\mu) \subset F' \subset L^1(\mu) \]
\(L^\infty(\mu)\) is a dense ideal of \(F\), \(\int_{\Omega} y' y d\mu = \langle y', y \rangle\) for all \(y' \in F', y \in F\) and \(x = \chi_{\Omega}\).

Note that \(F\) does not contain any atoms, as \(E\) does not, so it is clear that \(L^\infty(\mu)\) does not contain any atoms. It follows from Proposition 2.1 above that there exist \(x_n \in L^\infty(\mu) \subset F\) such that \(|x_n| = \chi_{\Omega} = x\) and \(\lim_{n \to \infty} \int_{\Omega} x_n f d\mu = 0\) for all \(f \in L^1(\mu)\). As \(F' \subset L^1(\mu)\) we have \(\lim_{n \to \infty} \langle x', x_n \rangle = \lim_{n \to \infty} \int_{\Omega} x' x_n d\mu = 0\) for all \(x' \in F'\). Thus \(x_n \to 0\) weakly as desired.

**Corollary 2.3.** If \(E\) is a Banach lattice with an order continuous norm then the lattice operations in \(E\) are weakly sequentially continuous if and only if \(E\) is purely atomic.

**Proof.** The ‘if’ part is contained in Theorem 2.5.23 of [MN], whilst the ‘only if’ part follows from applying Theorem 2.2 to the band of elements in \(E\) that disjoint from the atoms.

A Banach lattice \(E\) is said by Groenewegen in [G] to have property \((WL)\) if whenever \(A\) is a relatively weakly compact subset of \(E\) then the set \(|A| = \{ |a| : a \in A \}\) is also relatively weakly compact. Let us remind the reader that the solid hull of a subset \(A\) of a vector lattice \(E\) is the set.
sol(A) = \{ x \in E : \exists a \in A \text{ with } |x| \leq |a| \},

and that a KB-space is a Banach lattice in which every increasing norm-bounded positive sequence is convergent.

**Theorem 2.4.** For a Banach lattice E the following assertions are equivalent:

1. Every relatively weakly compact subset of E has a relatively weakly compact solid hull.
2. E has an order continuous norm and has property (W1).
3. E may be written as an order direct sum, $E_1 \oplus E_2$, where $E_1$ is a KB-space and $E_2$ is atomic with an order continuous norm.

**Proof.** (3) $\Rightarrow$ (1). Suppose that (3) holds and $A \subseteq E$ is a relatively weakly compact. Let $P$ be the band projection from E onto $E_1$, then $A_1 = PA$ is relatively weakly compact, so is sol($A_1$) by Theorem 13.8 (b) of [AB2] as $E_1$ is a KB-space. Also $A_2 = (I_E - P)A \subseteq E_2$ is relatively weakly compact. By Proposition 2.5.23 of [MN] the lattice operations on $E_2$ are weakly sequentially continuous and it follows from the Eberlein-Šmulian theorem (Theorem 10.13 of [AB2]) that $|A_2| = \{ |x| : x \in A_2 \}$ is relatively weakly compact. Now Theorem 13.8 (a) of [AB2] implies that sol($A_2$) = sol($|A_2|$) is relatively weakly compact. It follows that sol($A$) is relatively weakly compact as

$$\text{sol}(A) \subseteq \text{sol}(A_1) \oplus \text{sol}(A_2),$$

so that (1) holds.

If (1) holds then certainly every relatively weakly compact subset of $E_+$ has a solid hull which is relatively weakly compact, so that by Theorem 13.8 (a) of [AB2], E is an ideal in its bidual and hence has an order continuous norm. Note also that if $A$ is relatively weakly compact then $|A|$ will be a subset of the relatively weakly compact set sol($A$), so will itself be relatively weakly compact and hence E has property (W1).

(2) $\Rightarrow$ (3). Let $E_2$ be the closed ideal of E generated by all atoms of $E, E_1 = E^{2}_0$, then Corollary 2.4.4 of [MN] implies that $E_2$ is a band of E and $E = E_1 \oplus E_2$.

Clearly $E_2$ is atomic, so we need only show that $E_1$ is a KB-space.

It is clear that $E_1$ does not contain any atoms and that every relatively weakly compact subset of $E_1$ has a relatively weakly compact solid hull. If $E_1$ were not a KB-space then by Theorem 14.12 of [AB2] there exist $k_1 > 0, k_2 > 0$, and a disjoint sequence $\{x_n\} \subseteq (E_1)_+$ such that

$$k_1 \|(\lambda_n)\|_{\infty} \leq \| \sum_{i=1}^{\infty} \lambda_i x_i \| \leq k_2 \|(\lambda_n)\|_{\infty}$$

for all $(\lambda_n) \in c_0$, i.e. $(x_n)$ is equivalent to the natural basis of $c_0$.

According to Theorem 2.3 above we can choose $r_{nm} \in E_1$ such that $|r_{nm}| = x_n$ for all $n, m \in \mathbb{N}$ and, for each $n \in \mathbb{N}, r_{nm} \to 0$ weakly as $m \to \infty$. Let $\mu_n = r_{1n} + r_{2n} + \cdots + r_{nn}$ for each $n \in \mathbb{N}$ then we claim that $\mu_n \to 0$ weakly as
Indeed, it suffices to show that \( \langle x', u_n \rangle \to 0 \) as \( n \to \infty \) for each \( 0 \leq x' \in E'_1 \). Let \( I_n \) be the closed ideal of \( E_1 \) generated by \( x_n \) and let \( x'_n \) denote the restriction of \( x' \) to \( I_n \) so that \( x'_n \geq 0 \). Since \( (I_n, \| \cdot \|_n) \) is an AM-space with strong unit \( x_n \), where \( \| \cdot \|_n \) is defined on \( I_n \) by

\[
\| y \|_n = \inf \{ \lambda \geq 0 : y \in \lambda [-x_n, x_n] \},
\]

\( x'_n \) is bounded for the \( \| \cdot \|_n \)-norm on \( I_n \). I.e. \( x'_n \in (I_n, \| \cdot \|_n)^\prime \).

Write \( \| x'_n \|_o = \| x'_n \| (I_n, \| \cdot \|_n)^\prime \) then

\[
\sum_{n=1}^{\infty} \| x'_n \|_o \leq k_2 \| x' \| < \infty
\]
as

\[
\sum_{i=1}^{\infty} \| x'_i \|_o = \sum_{i=1}^{n} x'_i(x_i)
\]

\[= \sum_{i=1}^{n} x'(x_i)\]

\[= x' \left( \sum_{i=1}^{n} x_i \right)\]

\[\leq \| x'\| \sum_{i=1}^{n} x_i\]

\[\leq k_2 \| x' \| < \infty\]

for all \( n \in \mathbb{N} \).

Now for each \( \varepsilon > 0 \) there exists \( N \) such that \( \sum_{i=N+1}^{\infty} \| x'_i \|_o < \varepsilon \) so that for each \( n > N \),

\[
| x'(u_n) | = \left| \sum_{i=1}^{n} x'(r_{in}) \right| \leq \left| \sum_{i=1}^{N} x'(r_{in}) \right| + \left| \sum_{i=N+1}^{n} x'(r_{in}) \right| \leq \left| \sum_{i=1}^{N} x'(r_{in}) \right| + \left| \sum_{i=N+1}^{n} \| x'_i \|_o \| r_{in} \| (I_n, \| \cdot \|_n) \right| \leq \left| \sum_{i=1}^{N} x'(r_{in}) \right| + \varepsilon \cdot \varepsilon.
\]

Thus

\[
\limsup_{n \to \infty} | x'(u_n) | \leq \lim_{n \to \infty} \left| \sum_{i=1}^{N} x'(r_{in}) \right| + \varepsilon = \varepsilon
\]
as \( r_{in} \to 0 \) weakly as \( n \to \infty \) for each \( 1 \leq i \leq N \). Note that \( \varepsilon \) is arbitrary so it follows that \( x'(u_n) \to 0 \) as \( n \to \infty \) for each \( 0 \leq x' \in E'_1 \). I.e. \( u_n \to 0 \) weakly as \( n \to \infty \).

If \( A = \{ u_n \}_{n=1}^{\infty} \) then \( A \) is relatively weakly compact and \( |A| = \{|u_n| : n \in \mathbb{N} \} \).
Since \( \{x_n\} \) are disjoint and \(|r_{ni}| = x_n\) for all \( i \in \mathbb{N} \), we have \( r_{ni} \perp r_{mj} \) for all \( n, m, i, j \in \mathbb{N} \) with \( n \neq m \) and hence
\[
|u_n| = x_1 + x_2 + \ldots + x_n \uparrow .
\]

Also
\[
||u_n| - |u_m|| \geq k_1 > 0
\]
for all \( n \neq m \). Therefore \(|u_n|\) does not contain any weakly convergent subsequence as any such subsequence would converge in norm to its limit by Dini's theorem. Thus \(|A|\) is not relatively weakly compact, which contradicts (2). This completes the proof of the theorem.  

3. FACTORIZATION OF WEAKLY COMPACT OPERATORS

The factorization theorem of Aliprantis and Burkinshaw for weakly compact operators ([AB]), Theorem 2.2 and part 2 of Theorem 1.7, states that if \( X \) is a Banach space, \( F \) a Banach lattice, \( T : X \to F \) is weakly compact and the solid hull of the image under \( T \) of the unit ball in \( X \) is relatively weakly compact then \( T \) factors through a reflexive Banach lattice \( E \) in such a way that \( T = QP \) where \( P \) is a bounded linear operator and \( Q \) is an interval preserving lattice homomorphism. It follows immediately that if the solid hull of every relatively weakly compact subset of \( F \) is again relatively weakly compact then such a factorization will be possible for all weakly compact operators \( T : X \to E \). The following result shows that it is only for such \( F \) that this factorization is possible.

**Theorem 3.1.** For a Banach lattice \( F \) the following assertions are equivalent:

1. Every relatively weakly compact subset of \( F \) has a relatively weakly compact solid hull.

2. For each Banach space \( X \), every weakly compact operator \( T : X \to F \) factors through a reflexive Banach lattice \( E \) with \( T = QP \) where \( P : X \to E \) is a bounded linear operator and \( Q : E \to F \) is an interval preserving lattice homomorphism. Moreover, if \( X \) is a Banach lattice and \( T \geq 0 \) then we may take \( P \geq 0 \).

3. Every weakly compact operator \( T : \ell_1 \to F \) factors through a Banach lattice \( E \) such that \( T = QP \) where \( P : \ell_1 \to E \) is bounded linear and \( Q : E \to F \) is weakly compact interval preserving.

4. The collection of all weakly compact operators from \( \ell_1 \) into \( F \) forms an ideal in \( \mathcal{L}'(\ell_1, F) = \mathcal{L}(\ell_1, F) \).

**Proof.** It is clear that (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3). It is well-known that every linear bounded operator from \( \ell_1 \) into any Banach lattice has a modulus, so that \( \mathcal{L}'(\ell_1, F) = \mathcal{L}(\ell_1, F) \). Assume (3), so that each weakly compact operator \( T : \ell_1 \to F \) factors through a Banach lattice \( E \) such that \( T = QP \) where \( P : \ell_1 \to E \) and \( Q : E \to F \) is weakly compact interval preserving. Since the modulus of \( P \) exists it is easy to verify that \(|T| - |QP| \leq Q|P|\) and hence
Also, if \( S, T \in L(\ell_1, F) \) are such that \( 0 \leq S \leq T \) and \( T \) is weakly compact, then (still assuming (3)) \( T \) can be factored through a Banach lattice \( E \) such that \( T = QP \) where \( Q \) is weakly compact and interval preserving. So \( 0 \leq S \leq T = QP \leq Q|P| \) and as above we have \( S \) weakly compact. It follows that the collection of all weakly compact operators from \( \ell_1 \) into \( F \) forms indeed an ideal in \( L(\ell_1, F) \). Thus (3) \( \Rightarrow \) (4) holds.

Finally, let us assume (4) and prove (1). We do this using Theorem 2.4 and showing that \( F \) has an order continuous norm and property (W1). Theorem 17.10 of [AB2] shows that \( F \) must have an order continuous norm (since the dual of \( \ell_1 \) certainly does not have), so we need to establish property (W1). Suppose that this fails then, using the Eberlein–Šmulian theorem, there is a sequence \( (x_n) \) in \( F \) such that the set \( \{x_n : n \in \mathbb{N}\} \) is relatively weakly compact but the set \( \{|x_n| : n \in \mathbb{N}\} \) is not. Define \( T : \ell_1 \to F \) by \( T(\lambda_n) = \sum_{n=1}^{\infty} \lambda_n x_n \) then Corollary 10.16 of [AB2] implies that \( T \) is weakly compact. From (4) \( |T| \) exists (and it is easy to verify that \( |T|(\lambda_n) = \sum_{n=1}^{\infty} |\lambda_n| |x_n| \)) and is weakly compact. Taking \( e_n \) for the usual basis vector in \( \ell_1 \), the weak compactness of \( |T| \) shows that the sequence \( (|T|e_n) = (|x_n|) \) is relatively weakly compact. This contradiction establishes (1). \( \Box \)

More results on the order structure of spaces of weakly compact operators are contained in [CW]. In the similar way to the last result we have:

**Theorem 3.2.** For Banach lattice \( F \) the following assertions are equivalent:

1. \( F \) has an order continuous norm.
2. For each Banach lattice \( E \), every positive weakly compact operator \( T : E \to F \) factors through a reflexive Banach lattice \( G \) with \( T = QP \) where \( P : E \to G \) is positive linear and \( Q : G \to F \) is interval preserving.
3. Every positive compact operator \( T : \ell_1 \to F \) factors through a Banach lattice \( G \) with \( T = QP \) where \( P : \ell_1 \to G \) is bounded and \( Q : G \to F \) is weakly compact and interval preserving.

**Proof.** It is clear that (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3). To show that (3) \( \Rightarrow \) (1) let \( 0 \leq y_n \uparrow \leq y \). Define \( S, T : \ell_1 \to F \) by \( S(\lambda_n) = \sum_{n=1}^{\infty} \lambda_n y_n \) and \( T(\lambda_n) = (\sum_{n=1}^{\infty} \lambda_n) y \) for all \( (\lambda_n) \in \ell_1 \). Then \( 0 \leq S \leq T \) and \( T \) is compact so it follows from (3) that \( T \) admits a factorization through a Banach lattice \( G \) with \( T = QP \) where \( P : \ell_1 \to G \) is linear bounded and \( Q : G \to F \) is weakly compact and interval preserving. Thus \( 0 \leq S \leq T = QP \leq Q|P| \) and in the same way as in the proof of Theorem 3.1 above we have that \( S \) is weakly compact, and hence \( (Se_n) = (y_n) \) contains a weakly convergent subsequence. It follows from the monotoneness of \( (y_n) \) and
Dini's theorem that \((y_n)\) is norm convergent. Therefore \(F\) has an order continuous norm. □

The following result is a dual version of the last two results.

**Theorem 3.3.** For Banach lattice \(E\) the following assertions are equivalent:

1. \(E'\) has an order continuous norm. (i.e. \(E'\) is a KB-space)
2. For each Banach space \(Y\), every weakly compact operator \(T : E \to Y\) factors through a reflexive Banach lattice \(G\) such that \(T = PJ\) where \(J : E \to G\) is a lattice homomorphism and \(P : G \to Y\) is linear bounded. Moreover if \(Y\) is a Banach lattice and \(T \geq 0\) then we may take \(P \geq 0\).
3. Every positive compact operator \(T : E \to \ell_\infty\) factors through a Banach lattice \(G\) such that \(T = PJ\) where \(J : E \to G\) is a weakly compact lattice homomorphism and \(P : G \to \ell_\infty\) is linear and bounded.
4. Every positive weakly compact operator \(T : E \to C([0,1])\) factors through a reflexive Banach lattice.

**Proof.** It is clear that (1) \(\Rightarrow\) (2) \(\Rightarrow\) (3) and that (2) \(\Rightarrow\) (4). Suppose that (3) holds and suppose that we have, in \(E', 0 \leq x_n' \uparrow x'.\) Define \(S, T : E \to \ell_\infty\) by \(S(x) = (x_n(x))_{n=1}^\infty\) and \(T(x) = x'(x)e_0,\) where \(e_0 = (1, 1, \ldots) \in \ell_\infty.\) Then \(0 \leq S \leq T\) and \(T,\) being rank 1, is compact so by (3) \(T\) admits a factorization through a Banach lattice \(G\) with \(T = PJ\) where \(J : E \to G\) is a weakly compact lattice homomorphism and \(P : G \to \ell_\infty\) is linear and bounded. Noting that each operator from a Banach lattice into \(\ell_\infty\) has a modulus we have \(0 \leq S \leq T = PJ \leq |P|J,\) and by Theorem 1.1 of [Ar] \(S = QJ\) for some \(0 \leq Q \leq |P|\). It follows that \(S\) is weakly compact and so is \(S'\) by Gantmacher's theorem. It is routine to check that \(S'\) contains a weakly convergent subsequence. Thus the monotonicity of \(\{x_n'\}\) implies that \(x_n'\) is norm convergent. Therefore \(E'\) has an order continuous norm and have established that (3) \(\Rightarrow\) (1).

Now we suppose that (4) is true but (1) is false. According to Theorem 2.4.2 of [MN], \(E\) contains a closed sublattice \(H\) which is lattice isomorphic to \(\ell_1.\) Let \(J : H \to \ell_1\) be the corresponding lattice isomorphism. From Proposition 2.3.11 of [MN] there is a positive projection \(P\) from \(E\) onto \(H.\) Also by the theorem \(A\) of [T] there exists a weakly compact subset \(W\) of \(C([0,1])\) such that for each reflexive Banach lattice \(G\) and each bounded operator \(S : G \to C([0,1]),\) \(S(U) \not\subseteq W,\) where \(U\) is the closed unit ball of \(G.\)

Choose \(x_n \in W\) such that \(\{x_n : n \in \mathbb{N}\}\) is dense in \(W.\) Define \(T : \ell_1 \to C([0,1])\) by \(T(\lambda_n) = \sum_{n=1}^\infty \lambda_n x_n\) then Theorem 10.15 of [AB2] implies that \(T\) is weakly compact and hence so is \(TJP : E \to C([0,1]).\) By (4) we can find a reflexive Banach lattice \(G\) such that \(TJP = SR\) where \(R : E \to G\) and \(S : G \to C([0,1])\) are linear bounded.

Let \(y_n = J^{-1}e_n \in H \subseteq E,\) then \(\|y_n\| \leq \|J^{-1}\|\) for all \(n \in \mathbb{N}.$ Replacing \(R\) by \(R/(\|R\| \cdot \|J^{-1}\|)\) and \(S\) by \(\|R\|\|J^{-1}\|S\) we may assume that \(\|R\|\|J^{-1}\| = 1\) and
thus we have $\|Ry_n\| \leq \|R\|\|y_n\| \leq \|R\|J^{-1}\| = 1$ for all $n \in \mathbb{N}$. I.e. each $Ry_n \in U$. Moreover

$$x_n = Te_n = TJu_n = TJPy_n = SRy_n = S(Ry_n) \in S(U)$$

for all $n \in \mathbb{N}$, and it follows that $W \subset S(U)$ which contradicts the definition of $W$. So (4) ⇒ (1) holds. □

4. FACTORIZATION BY COMPACT LATTICE HOMOMORPHISM

The following lemma has a routine proof which we omit.

**Lemma 4.1.** Suppose that Banach lattice $F$ is atomic with an order continuous norm, $X$ is a Banach space, $T : X \to F$ is a compact operator and that $F_1$ denotes the closed ideal of $F$ generated by $TX$. Then $F_1$ is a band in $F$ and the set $\Gamma$ of all atoms of $(F_1)_+$ with a norm 1 is countable.

**Theorem 4.2.** For Banach lattice $F$ the following assertions are equivalent:

(1) $F$ is atomic with an order continuous norm.

(2) For each Banach space $X$, every compact operator $T : X \to F$ factors through a reflexive Banach lattice $G$ such that $T = QP$ where $P : X \to G$ is linear and bounded and $Q : G \to F$ is a compact interval preserving lattice homomorphism. Moreover if $X$ is a Banach lattice and $T \geq 0$ then we may take $P \geq 0$.

(3) Every positive compact operator $T : \ell_1 \to F$ factors through a Banach lattice $G$ with $T = QP$ where $P : \ell_1 \to G$ is bounded and $Q : G \to F$ is compact and interval preserving.

**Proof.** We first show that (1) ⇒ (2). According to Lemma 4.1 above we may assume that the set $\Gamma$ of all atoms of $F_+$ with norm 1 is countable, say $\Gamma = \{e_n\}_{n=1}^{\infty}$. It is easy to verify that $\{e_n\}$ is a Schauder basis of $F$ and that for each $x \in F$, we have $x = \sum_{n=1}^{\infty} \lambda_n e_n$ where $\lambda_n = \max\{\lambda \in \mathbb{R}_+ : \lambda e_n \leq x\}$. Moreover, if $x = \sum_{n=1}^{\infty} \lambda_n e_n \in F$ then $x \geq 0$ if and only if $\lambda_n \geq 0$ for all $n \in \mathbb{N}$.

Define $P_n : F \to F$ by $P_n(x) = \sum_{k=1}^{n} \lambda_k e_k$ if $x = \sum_{k=1}^{\infty} \lambda_k e_k$. Then $0 \leq P_n \leq P_{n+m} \leq I_F$ for all $n, m \in \mathbb{N}$. Since $T$ is compact it is routine to check that $\|T - P_n T\| \to 0$ as $n \to \infty$. Therefore we can choose $1 \leq n_1 \leq n_2 \leq n_3 \leq \ldots$ such that $\|P_{n_k} - P_{n_{k+1}}\| < 1/4^k$ for each $k \in \mathbb{N}$. Write $Q_1 = P_{n_1}$ and $Q_k = P_{n_k} - P_{n_{k-1}}$ for $k \geq 2$ and let $F_k = Q_k F$ for each $k \in \mathbb{N}$ then

(a) $0 \leq Q_k \leq I_F$ for all $k$.

(b) $\|Q_k T\| \leq 1/4^k$ for all $k \geq 2$.

(c) $\dim F_k = n_k - n_{k-1}$ for $k \in \mathbb{N}$ (defining $n_0 = 0$)

(d) $F_k \perp F_m$ for all $k, m \in \mathbb{N}$ with $k \neq m$.

For $p \in (1, \infty)$ let

$$E_p = \ell_p(F_k) = \{(x_k) : x_k \in F_k, \sum_{k=1}^{\infty} \|x_k\|^p < \infty\}$$
with norm \( \| (x_k) \| = \left( \sum_{k=1}^{\infty} |x_k|^p \right)^{1/p} \) and pointwise ordering, so that \( E_p \) is a reflexive Banach lattice.

Define \( P : X \to E_p \) and \( Q : E_p \to F \) by \( Px = (2^k Q_k T x_k) \sum_{k=1}^{\infty} \) and \( Q(x_k) = \sum_{k=1}^{\infty} 1/(2^k)x_k \). It is clear that \( P \) is compact and that \( P \geq 0 \) if \( T \geq 0 \). Also it is routine to verify that \( Q \) is a compact interval preserving lattice homomorphism and \( T = QP \) as desired.

That (2) \( \Rightarrow \) (3) is clear, so we conclude the proof by showing that (3) \( \Rightarrow \) (1). From Theorem 2.1 of [W] it suffices to show that \((\ell_1, F)\) has the compact domination property. If \( S, T \in L(\ell_1, F) \) with \( 0 \leq S \leq T \) and \( T \) is compact, then by (3) \( T \) factors through a Banach lattice \( G \) with \( T - QP \) where \( P : \ell_1 \to G \) is bounded and \( Q : G \to F \) is compact and interval preserving. As \( 0 \leq S \leq T = QP \leq Q |P| \) we have \( 0 \leq S' \leq |P'| Q' \) and it follows from Theorem 1.1 of [Ar] that \( S' = RQ' \) for some \( 0 \leq R \leq |P'| \). Thus \( S' \), and hence \( S \), is compact as desired. \( \square \)

Again we also have a dual version.

**Theorem 4.3.** For a Banach lattice \( E \) the following assertions are equivalent:

1. \( E' \) is atomic with an order continuous norm.
2. For each Banach space \( Y \), every compact operator \( T : E \to Y \) factors through a reflexive Banach lattice \( G \) with \( T = PJ \) where \( J : E \to G \) is a compact lattice homomorphism and \( P : G \to Y \) is linear and bounded. Moreover if \( Y \) is a Banach lattice and \( T \geq 0 \) then we may take \( P \geq 0 \).
3. Every positive compact operator \( T : E \to \ell_\infty \) factors through a Banach lattice \( G \) with \( T = PJ \) where \( J : E \to G \) is a compact lattice homomorphism and \( P : G \to \ell_\infty \) is linear and bounded.

**REFERENCES**


(Received February 1997)