On Some New Inequalities Related to Hardy’s Integral Inequality

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Submitted by J. L. Brenner

Received September 3, 1988

1. INTRODUCTION

One of the many fundamental mathematical discoveries of G. H. Hardy is the following integral inequality [3, Theorem 330]:

If \( p > 1 \), \( m \neq 1 \) \( f(t) \geq 0 \), and \( F \) is defined on \((0, \infty)\) by

\[
F(x) = \int_0^x f(t) \, dt \quad \text{for} \quad m > 1, \\
F(x) = \int_x^\infty f(t) \, dt \quad \text{for} \quad m < 1,
\]

then

\[
\int_0^\infty x^{-m} F(x)^p \, dx < \left( \frac{p}{|m-1|} \right)^p \int_0^\infty x^{-m+p} f(x)^p \, dx \tag{1}
\]

unless \( f(t) \equiv 0 \). The constant on the right is the best possible.

This is known as Hardy’s Inequality, although the name was formerly applied to the case \( m = p \) only [3, Sect. 9.8 and Theorem 327]. There is a vast literature which deals with alternative proofs, generalizations and extensions of (1); see [3–8] and the references therein.

In 1975 Copson [2] proved several interesting generalizations of (1), which are actually integral analogues of some series inequalities that he had found [1] many years earlier. The inequalities in [2] were in turn
generalized by Love [6], who added many allied inequalities, including some in the style of those of Kadlec and Kufner [4]. Still further generalizations are given in the present paper; they were to some extent motivated by the work of Levinson [5], as well as by the papers of Copson [2], Love [6], and Pachpatte [7].

2. PRELIMINARY NOTE

All the integrals involved are Lebesgue integrals; but there is more than one shade of meaning required, as follows.

If $f$ is non-negative measurable it is accepted that $\int f(x) \, dx$ exists with value in $[0, \infty]$; then we write that this integral “exists, finite or infinite.” If the value is in $[0, \infty)$ we write that the integral “exists finite.”

Following more standard usage, we write “$f$ is integrable,” or “$f \in L$,” or “$\int f(x) \, dx$ exists,” whenever $f$ is measurable and $\int |f(x)| \, dx < \infty$.

In this connection we mention the following trivial lemma, which is included in order to clarify steps like (5) which might otherwise appear suspect.

**LEMMA.** If $f, g, h$ are real-valued functions measurable on a measurable set, $f \in L$, $g \in L$, and $\int f(x) \, dx \leq \int g(x) \, dx$, then $\int (f + h)(x) \, dx \leq \int (g + h)(x) \, dx$ if either side of the latter inequality exists.

**Proof.** If the left side exists, $f + h \in L$ as well as $f \in L$; consequently $h \in L$. If instead the right side exists, $h \in L$ similarly. In either case the result now follows from linearity for integrable functions.

3. MAIN THEOREMS

For suitable functions $r_n(t), z(t),$ and $f(t)$ on $(0, \infty),$ and $x \geq 0,$ let $I_n$ and $J_n$ be the operators

$$I_n f(x) = \int_0^x \frac{r_n(t)}{r_n(x)/z(x)} \frac{z'(t)}{f(t)} \, dt, \quad J_n f(x) = \int_x^{\infty} \frac{r_n(t)}{r_n(x)/z(x)} \frac{z'(t)}{f(t)} \, dt.$$

**THEOREM 1.** Let $m > 1,$ $p \geq 1,$ and all hypotheses involving $n$ hold for $n = 1, 2, \ldots, N.$ Let $r_n(x)$ and $w(x)$ be positive and locally absolutely continuous in $(0, \infty).$ Let $z(x)$ be differentiable in $(0, \infty)$ with $z'(x) > 0$ and $z(0+) > 0.$ Let

$$0 < \frac{1}{\lambda_n} \leq \operatorname{ess inf}_{0 < x < \infty} \left\{ 1 + \frac{1}{m-1} \frac{z(x)}{\left( r_n(x) - \frac{w'(x)}{w(x)} \right)} \right\}. \quad (2)$$
If \( f(x) \) is non-negative measurable on \((0, \infty)\), \( F_0(x) = z(x)f(x) \),

\[
F_n(x) = z(x) I_n I_{n-1} \cdots I_1 f(x)
\]
each exist for some positive \( x \) and hence for all, and

\[
\frac{w(x)}{z(x)^{m-1}} F_n(x)^p \to 0 \quad \text{as} \quad x \to 0+,
\]

then

\[
\left( \int_0^\infty \frac{z'(x)}{z(x)^m} w(x) F_n(x)^p \, dx \right)^{1/p} \leq A \left( \int_0^\infty \frac{z'(x)}{z(x)^m} w(x) F_0(x)^p \, dx \right)^{1/p}
\]

where

\[
A = \left( \frac{p}{m-1} \right)^N \prod_{n=1}^N \alpha_n.
\]

**Theorem 2.** This has the same hypotheses and conclusion as Theorem 1 except for replacement of

\[
m > 1 \quad \text{by} \quad m < 1,
\]

\[
z(0+) > 0 \quad \text{by} \quad z(+\infty) < \infty,
\]

\[
F_n(x) = z(x) I_n I_{n-1} \cdots I_1 f(x) \quad \text{by} \quad F_n(x) = z(x) J_n J_{n-1} \cdots J_1 f(x),
\]

\[
\frac{w(x)}{z(x)^{m-1}} F_n(x)^p \to 0 \quad \text{as} \quad x \to 0+ \quad \text{by}
\]

\[
\frac{w(x)}{z(x)^{m-1}} F_n(x)^p \to 0 \quad \text{as} \quad x \to +\infty.
\]

**4. Proof of Theorem 1**

(i) Let \( 0 < X < \infty \). Integrating by parts formally, we obtain

\[
(m-1) \left[ \int_0^X \frac{z'(x)}{z(x)^m} w(x) F_n(x)^p \, dx \right] + \int_0^X \left\{ \frac{w'(x)}{z(x)^{m-1}} F_n(x)^p + \frac{w(x)}{z(x)^{m-1}} pF_n(x)^{p-1} \right. \\
\left. \times \left( \frac{z'(x)}{z(x)} F_{n-1}(x) - \frac{r_n'(x)}{r_n(x)} F_n(x) \right) \right\} \, dx.
\]
The integral on the left of (4) exists because

\[ \frac{z'(x)}{z(x)} \quad \text{and} \quad \frac{w(x)}{z(x)^{m-1}} F_n(x)^p \]

are respectively integrable and bounded (because continuous) on \([0, X]\).

The integrated terms also exist; hence so does the integral on the right of (4). Since the integrated term at \(X\) is non-positive, the whole right side is

\[ \leq \int_0^X \left\{ \frac{z'(x)}{z(x)^m} w(x) F_n(x)^p \frac{z(x)}{z'(x)} \left( \frac{w'(x)}{w(x)} - \frac{p r_n(x)}{r_n(x)} \right) \right\} \, dx + p F_n(x)^{p-1} F_{n-1}(x) \frac{z(x)}{z(x)^m} w(x) \]

By the lemma,

\[ \int_0^X \frac{z'(x)}{z(x)^m} w(x) F_n(x)^p \left\{ m - 1 - \frac{z(x)}{z'(x)} \left( \frac{w'(x)}{w(x)} - \frac{p r_n(x)}{r_n(x)} \right) \right\} \, dx \leq p \int_0^X \frac{z'(x)}{z(x)^m} w(x) F_n(x)^{p-1} F_{n-1}(x) \, dx \quad (5) \]

if either side exists; and we show in (ii) that the right side does.

(ii) Both sides of (5) are integrals of non-negative measurable functions, since by hypothesis the braced expression on the left is almost everywhere not less than the positive number \((m - 1)/\alpha_n\). To prove the finiteness of the right side of (5), the integral is

\[ \int_0^X \left( \frac{z'(x)}{z(x)^m} w(x) F_n(x)^p \right)^{(p-1)/p} \left( \frac{z'(x)}{z(x)^m} w(x) F_{n-1}(x)^p \right)^{1/p} \, dx \]

\[ \leq \left( \int_0^X \frac{z'(x)}{z(x)^m} w(x) F_n(x)^p \, dx \right)^{(p-1)/p} \]

\[ \times \left( \int_0^X \frac{z'(x)}{z(x)^m} w(x) F_{n-1}(x)^p \, dx \right)^{1/p} \quad (6) \]

by Hölder's inequality if \(p > 1\), and trivially if \(p = 1\). Now the integral on the left of (4) exists for \(n = 1, 2, \ldots, N\), as proved in (i). Consequently both factors on the right of (6) are finite for these \(n\), with the possible exception of the last factor when \(n = 1\). If that factor were infinite, the right side of (3) would be infinite and there would be nothing to prove; so that possibility can be dismissed. Thus the right side of (5) is finite and (5) is established.
(iii) By (2), (5), and (6),
\[
\frac{m - 1}{m} \int_0^x \frac{z'(x)}{z(x)^m} w(x) F_n(x)^p \, dx \\
\leq \int_0^x \frac{z'(x)}{z(x)^m} w(x) F_n(x)^{p - 1} F_n(x)^{-1} \, dx \\
\leq \left( \int_0^x \frac{z'(x)}{z(x)^m} w(x) F_n(x)^p \, dx \right)^{1/p - 1/p} \\
\times \left( \int_0^x \frac{z'(x)}{z(x)^m} w(x) F_{n-1}(x)^p \, dx \right)^{1/p}.
\]

Since the integral on the left of (4) exists, we can divide the extreme members of (7) by the first factor on its right, if this is not zero. This gives
\[
\left( \int_0^x \frac{z'(x)}{z(x)^m} w(x) F_n(x)^p \, dx \right)^{1/p} \\
\leq \frac{m^2}{m - 1} \left( \int_0^x \frac{z'(x)}{z(x)^m} w(x) F_{n-1}(x)^p \, dx \right)^{1/p}.
\]

If the integral on the left of (8) were zero, the hypotheses on \( z \) and \( w \) would make \( F_n(x) \) zero for almost all \( x \) in \( (0, X) \), and consequently for all since \( F_n \) is continuous. Then by definition of \( I_n \), \( F_{n-1}(x) \) would be zero for almost all \( x \) in \( (0, X) \), so that (8) would hold trivially, both sides being zero.

Thus (8) is established, for \( n = 1, 2, \ldots, N \). It follows that
\[
\left( \int_0^x \frac{z'(x)}{z(x)^m} w(x) F_n(x)^p \, dx \right)^{1/p} \leq A \left( \int_0^x \frac{z'(x)}{z(x)^m} w(x) F_0(x)^p \, dx \right)^{1/p};
\]

and the desired inequality (3) follows from this if we make \( X \to \infty \).

5. PROOF OF THEOREM 2

This proceeds almost word for word as in the preceding proof, with the changed meaning of \( F_n(x) \). The other main changes are the replacement of (4) and (5), respectively, by
\[
(1 - m) \int_x^\infty \frac{z'(x)}{z(x)^m} w(x) F_n(x)^p \, dx = \left[ \frac{w(x)}{z(x)^{m-1}} F_n(x)^p \right]_x \\
- \int_x^\infty \left\{ \frac{w'(x)}{z(x)^{m-1}} F_n(x)^p - \frac{w(x)}{z(x)^{m-1}} p F_n(x)^{p - 1} \right\} dx \\
\times \left( \frac{z'(x)}{z(x)} F_{n-1}(x) + \frac{r_n(x)}{r_n(x)} F_n(x) \right) \right) \, dx
\]
and
\[
\int_{x}^{\infty} \frac{z'(x)}{z(x)^m} w(x) F_\alpha(x)^p \left\{ 1 - m + \frac{z(x)}{z'(x)} \left( \frac{w'(x)}{w(x)} - p \frac{r_n'(x)}{r_n(x)} \right) \right\} dx 
\leq p \int_{X}^{\infty} \frac{z'(x)}{z(x)^m} w(x) F_\alpha(x)^p \mathcal{J}_0 F_{\alpha - 1}(x) dx. \tag{11}
\]

Inequalities (6), (7), (8), and (9) are changed only in that \((1^o)\) the integrals are taken over \((X, \infty)\) instead of \((0, X)\) and \((2^o)\) \(m - 1\) is replaced by \(1 - m\). Finally \((3)\) is obtained by making \(X \to 0\).

6. Further Theorems

For suitable functions \(r_n(t), z(t),\) and \(f(t)\) on \((0, \infty), x > 0,\) let \(K_n\) and \(L_n\) be the operators
\[
K_n f(x) = \int_{0}^{x} \frac{r_n(t)}{r_n(x) z(x) \log z(t)} d\tau,
\]
\[
L_n f(x) = \int_{x}^{\infty} \frac{r_n(t)}{r_n(x) z(x) \log z(t)} d\tau.
\]

From here on we shall write \(l(t)\) for \(\log z(t)\).

**Theorem 3.** Let \(m > 1, p > 1,\) and all hypotheses involving \(n\) hold for \(n = 1, 2, \ldots, N.\) Let \(r_n(x)\) and \(w(x)\) be positive and locally absolutely continuous in \((0, \infty).\) Let \(z(x)\) be differentiable in \((0, \infty)\) with \(z'(x) > 0\) and \(z(0+) > 1.\) With \(l(x) = \log z(x)\) let
\[
0 < \frac{1}{\beta_n} \leq \text{ess inf}_{0 < x < \infty} \left\{ \frac{1}{m - 1} \left( \frac{r_n'(x)}{r_n(x) - w'(x)} \right) \frac{z(x)}{z'(x)} l(x) \right\}. \tag{12}
\]

If \(f(x)\) is non-negative measurable on \((0, \infty), G_0(x) = z(x) l(x) f(x),\)
\[G_n(x) = z(x) K_n K_{n-1} \cdots K_1 f(x)\]
each exist for some positive \(x\) and hence for all, and
\[
\frac{w(x)}{l(x)^{m-1}} G_n(x)^p \to 0 \quad \text{as} \quad x \to 0+,
\]
then
\[
\left( \int_{0}^{\infty} \frac{z'(x)^p w(x)^p}{z(x)^p l(x)^{mp}} G_{\alpha}(x)^p dx \right)^{1/p} \leq B \left( \int_{0}^{\infty} \frac{z'(x)^p w(x)^p}{z(x)^p l(x)^{mp}} G_{\alpha}(x)^p dx \right)^{1/p} \tag{13}
\]
where
\[ B = \left( \frac{p}{|m-1|} \right)^N \prod_{n=1}^N \beta_n. \]

**Theorem 4.** This has the same hypotheses and conclusion as Theorem 3 except for replacement of
\[ m > 1 \quad \text{by} \quad m < 1, \]
\[ z(0+) > 1 \quad \text{by} \quad 1 \leq z(0+) < z(+\infty) < \infty, \]
\[ G_n(x) = z(x)K_nK_{n-1}\cdots K_1 f(x) \quad \text{by} \quad G_n(x) = z(x)L_nL_{n-1}\cdots L_1 f(x). \]

\[ \frac{w(x)}{l(x)^{m-1}} G_n(x)^{p} \to 0 \quad \text{as} \quad x \to 0+ \quad \text{by} \]
\[ \frac{w(x)}{l(x)^{m-1}} G_n(x)^{p} \to 0 \quad \text{as} \quad x \to +\infty. \]

7. **Proof of Theorem 3**

This proceeds in very much the same way as the proof of Theorem 1, with \( x_n, F_n, I_n, \) and \( A \) replaced by \( \beta_n, G_n, K_n, \) and \( B, \) respectively. Instead of (4) we have
\[ (m - 1) \int_0^x \frac{z'(x)}{z(x)l(x)^m} w(x) G_n(x)^{p} \, dx = \left[ -\frac{w(x)}{l(x)^{m-1}} G_n(x)^{p} \right]_0^x \]
\[ + \int_0^x \left\{ \frac{w'(x)}{l(x)^{m-1}} G_n(x)^{p} + \frac{w(x)}{l(x)^{m-1}} pG_n(x)^{p-1} \right. \]
\[ \times \left( \frac{z'(x)}{z(x)l(x)} G_{n-1}(x) - \frac{r_n(x)}{r_n(x)} G_n(x) \right) \right\} \, dx \quad (14) \]
the integral on the left existing because
\[ \frac{z'(x)}{z(x)l(x)} \quad \text{and} \quad \frac{w(x)}{l(x)^{m-1}} G_n(x)^{p} \]
are respectively integrable and bounded (because continuous) on \([0, X].\)

Instead of (5) and (6) we have
\[ \int_0^x \frac{z'(x)w(x)}{z(x)l(x)^m} G_n(x)^{p} \left\{ m - 1 - \frac{z(x)l(x)}{z'(x)} \left( \frac{w'(x)}{w(x)} - \frac{r_n(x)}{r_n(x)} \right) \right\} \, dx \]
\[ \leq p \int_0^x \frac{z'(x)w(x)}{z(x)l(x)^m} G_n(x)^{p-1} G_{n-1}(x) \, dx \quad (15) \]
and
\[
\int_0^x \left( \frac{z'(x)w(x)}{z(x)l(x)^m G_n(x)^p} \right)^{(p-1)/p} \left( \frac{z'(x)w(x)}{z(x)l(x)^m G_{n-1}(x)^p} \right)^{1/p} dx 
\leq \left( \int_0^x \frac{z'(x)w(x)}{z(x)l(x)^m G_n(x)^p} dx \right)^{(p-1)/p} 
\times \left( \int_0^x \frac{z'(x)w(x)}{z(x)l(x)^m G_{n-1}(x)^p} dx \right)^{1/p}.
\] (16)

Inequalities (7), (8), and (9) are changed in just the same way as (6) is changed into (16). For instance, (8) is replaced by
\[
\int_0^x \frac{z'(x)w(x)}{z(x)l(x)^m G_n(x)^p} dx \leq \frac{p \beta_n}{m-1} \left( \int_0^x \frac{z'(x)w(x)}{z(x)l(x)^m G_{n-1}(x)^p} dx \right)^{1/p}.
\]
The conclusion (13) is obtained from this inequality by iterating it and letting \(X \to \infty\), just as in the proof of Theorem 1.

8. PROOF OF THEOREM 4

This is a hybrid of the proofs of Theorems 2 and 3. Probably a sufficient indication of the detail is given by the counterpart of (11) and (15); this differs from (15) only in that \(G_n\) has the meaning appropriate to Theorem 4, the integrals are taken over \((X, \infty)\) instead of \((0, X)\), and the braced expression on the left is replaced by its negative.

9. REMARKS

In the special case \(N = 1\) and \(r_1(t) = 1\), the function \(F_1\) in Theorem 1 takes the form
\[
F_1(x) = \int_0^x f(t)z'(t) \, dt,
\]
and the inequality (3) (with \(z\) replaced by \(\phi\) and \(m\) by \(c\)) resembles the inequality given by Love in [6, Theorem 3.1]. Neither theorem includes the other, but it is worth noting that if \(w(t)\) is decreasing, as required in [6],
hypothesis (2) of Theorem 1 permits $z_1$ to be 1 and the two inequalities then coincide.

The further specialization that $w(t) = 1$ brings (3) into coincidence with Copson's [2, Theorem 1]; and the still further specialization that $z(t) = t$ reduces (3) to the case $m > 1$ of Hardy's Inequality (1).

Similar remarks can be made relating Theorem 2 with [6, Theorem 3.3], [2, Theorem 3], and the case $m < 1$ of Hardy's Inequality (1).

Again, similar remarks can be made relating Theorem 3 with [6, Theorem 6.1] and [2, Theorem 5], and relating Theorem 4 with [6, Theorem 6.3].

Finally, Theorems 3 and 4 with $N = 1$, $r_1(t) = 1$, $z(t) = t$, and $w(t)$ monotonic are closely related to [6, Theorems 5.1 and 5.3]. The latter, as pointed out in [6], are generalizations of some variants of Hardy's Inequality established by Kadlec and Kufner [4, Lemma 3(b)].

REFERENCES