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Steady Nonlinear Double-Diffusive Convection in a Porous Medium Based upon the Brinkman–Forchheimer Model

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A theoretical study of the problem of steady nonlinear double-diffusive convection through a porous medium is presented. The Brinkman–Forchheimer model is used to represent the porous medium. A variational formulation is given to deal with the weak solution and the existence, regularity, and uniqueness results are discussed. © 1996 Academic Press, Inc.

1. INTRODUCTION

During the last three decades, the phenomenon of double-diffusive convection, in which two scalar fields, such as heat and salinity concentration, affect the density distribution in a fluid, has become increasingly important. The interesting effects in such problems arise from the fact that one substance diffuses more rapidly than the other and can thus modify the transport process considerably. While most studies deal primarily with the heat and mass transfer problem in a clear fluid layer, a new field, dealing with heat and mass transfer research in a fluid-saturated porous medium, has recently imerged. The review article by Trevisan and Bejan [14] covers the latest developments in this area of research. As it is pointed out in this article and in Murray and Chen [10], such studies have applications in geophysics, astrophysics, oceanography, and energy technology.

Much of the research in porous medium, however, has been concerned with the use of Darcy's law as a suitable model for the porous medium. This model takes into account the friction offered by the solid particles to

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the fluid and gives satisfactory results only when the porous medium is closely packed and the domain of consideration is infinite. Darcy's law, however, cannot account for the no-slip boundary conditions at an interface of a porous medium with solid boundary and the continuity of a porous medium in contact with a viscous fluid. It is believed that for the flow of a high porosity porous medium the Brinkman equation [3] removes some of the above deficiencies and gives preferrable results. Support for the use of the Brinkman equation, with appropriate care, over Darcy's law, may be found in the works of Tam [12], Lundgren [9], Slattery [11], and Vafai and Tien [15]. We also wish to mention the work of Allaire [1, 2] which indicates when the Brinkman equation or Darcy's law can be more effective, depending upon the length scale of the microstructure.

One of the basic questions which should be answered concerning any applied problem is whether it is well set, that is, whether the solutions exist and whether they are unique. In the present paper, we employ the Brinkman–Forchheimer model to discuss the existence, regularity, and uniqueness of weak solutions, via a variational formulation, for steady double-diffusive convection in a porous medium. Following the lead of several investigations, Givler and Altobelli [5] have recently determined experimentally the effective viscosity for the Brinkman–Forchheimer model for steady flow through a wall-bounded porous medium. Recognizing that this model will soon become popular, we employ it here along with the equations of energy and concentration statements. The method we employ to handle these equations is similar to the methods expounded in Ladyzhenskaya [7] and Temam [13] for Navier–Stokes equations. In addition we also take recourse to some of the ideas and results of Hopf [6], Lions [8], and Gilbarg and Trudinger [4].

We conclude this section with the remark that the Brinkman–Forchheimer model is not a universally valid model for the flow through a porous media. In fact, it is useful for sparsely packed porous media and situations when the flow velocity is quite high so that fluid inertia cannot be neglected. For fine grained (high density) materials and for slow flows such as the flow through natural rocks and clays, etc., the above model has severe limitations.

2. THE GOVERNING EQUATIONS

We consider the problem of steady double-diffusive convection in a fluid saturated porous medium. We assume that the porous medium is in local thermal equilibrium and the Boussinesq approximation is applicable. Let Ω be an open bounded set in $\mathbf{R}^n$ ($n = 2$ or 3) with boundary $\partial\Omega$ of class

C^2. The governing equations are

$$\nabla \cdot \mathbf{v} = 0 \qquad \text{in } \Omega, \tag{1}$$

$$c_0 \rho_0 k^{-1/2} |\mathbf{v}| \mathbf{v} = -\nabla P + \rho_0 [1 - \alpha_T (T - T_R) + \alpha_S (S - S_R)] \mathbf{g} + \tilde{\mu} \Delta \mathbf{v} - \mu \mathbf{M} \mathbf{v} \qquad \text{in } \Omega, \tag{2}$$

$$\mathbf{v} \cdot \nabla T = \nabla \cdot (\mathbf{N} \nabla T) \qquad \text{in } \Omega, \tag{3}$$

$$\mathbf{v} \cdot \nabla S = \nabla \cdot (\mathbf{Q} \nabla S) \qquad \text{in } \Omega, \tag{4}$$

where $\mathbf{v}, P, T, S, k, \rho_0, \mu, \tilde{\mu}, \alpha_T, \alpha_S, T_R, S_R$ are, respectively, the filtration velocity vector, pressure, temperature, concentration, permeability, density, viscosity, effective viscosity, thermal expansion coefficient, concentration expansion coefficient, reference temperature, and reference concentration, and c_0 is a constant coefficient. Also $\mathbf{g}$ is the potential type gravitational acceleration, $\mathbf{M}^{-1} = \mathbf{k}$ is the positive symmetric constant tensor of permeability, $\mathbf{N}$ is the positive constant tensor of thermal diffusion, and $\mathbf{Q}$ is the positive constant tensor of concentration diffusion.

The boundary conditions are

$$\mathbf{v} = \mathbf{a} \qquad \text{on } \partial\Omega, \tag{5}$$

$$T = \xi \qquad \text{on } \partial\Omega, \tag{6}$$

$$S = \eta \qquad \text{on } \partial\Omega. \tag{7}$$

Suppose that $\mathbf{a}$ can be extended inside Ω in the form $\mathbf{a} = \operatorname{curl} \mathbf{b}$ with $\mathbf{b} \in H^2(\Omega)$ and ξ and η can also be extended inside Ω such that $\xi, \eta \in H^1(\Omega)$.

We denote p by $p = P - \sum_{i=1}^{n} \rho_0 g_i x_i$ and introduce the following dimensionless variables,

$$\begin{aligned} &\mathbf{x}^* = L^{-1}\mathbf{x}, \qquad \mathbf{v}^* = (\alpha_T T_0 \rho_0 g)^{-1} \mu m_1 \mathbf{v}, \qquad \mathbf{a}^* = (\alpha_T T_0 \rho_0 g)^{-1} \mu m_1 \mathbf{a}, \\ &p^* = (\alpha_T T_0 \rho_0 g L)^{-1} p, \qquad T^* = T_0^{-1} T, \qquad T_R^* = T_0^{-1} T_R, \\ &S^* = S_0^{-1} S, \qquad S_R^* = S_0^{-1} S_R, \\ &\mathbf{M}^* = m_1^{-1}\mathbf{M}, \qquad \mathbf{N}^* = n_1^{-1}\mathbf{N}, \qquad \mathbf{Q}^* = q_1^{-1}\mathbf{Q}, \qquad \mathbf{g}^* = g^{-1}\mathbf{g}, \end{aligned} \tag{8}$$

where L is the length of edge of the n-cube in which Ω can be contained, $T_0 > 0$ is the constant temperature, $S_0 > 0$ is the constant concentration, $g = |\mathbf{g}|$, and m_1, n_1, q_1 are, respectively, the smallest eigenvalues of $\mathbf{M}, \mathbf{N}, \mathbf{Q}$.

Omitting the stars, Eqs. (1) to (7) are dimensionalized as

$$\nabla \cdot \mathbf{v} = 0 \qquad \text{in } \Omega, \tag{9}$$

$$\sigma D_a R_T |\mathbf{v}|\mathbf{v} + \nabla p - \mu_0 D_a \Delta \mathbf{v} + \mathbf{M}\mathbf{v} + (T - T_R)\mathbf{g}$$
$$- \frac{1}{\tau}(S - S_R)\mathbf{g} = 0 \qquad \text{in } \Omega, \tag{10}$$

$$R_T \mathbf{v} \cdot \nabla T - \nabla \cdot (\mathbf{N} \nabla T) = 0 \qquad \text{in } \Omega, \tag{11}$$

$$\tau R_S \mathbf{v} \cdot \nabla S - \nabla \cdot (\mathbf{Q} \nabla S) = 0 \qquad \text{in } \Omega, \tag{12}$$

$$\mathbf{v} = \mathbf{a} \qquad \text{on } \partial\Omega, \tag{13}$$

$$T = \xi \qquad \text{on } \partial\Omega, \tag{14}$$

$$S = \eta \qquad \text{on } \partial\Omega, \tag{15}$$

where $R_T = \alpha_T T_0 \rho_0 g L/(\mu m_1 n_1)$ and $R_S = \alpha_S S_0 \rho_0 g L/(\mu m_1 q_1)$ are, respectively, the thermal Rayleigh number and the solute Rayleigh number, $D_a = (L^2 m_1)^{-1}$ is the Darcy number, and $\sigma = \rho_0 n_1 L c_0/(\mu k^{1/2})$, $\tau = \alpha_T T_0/(\alpha_S S_0)$, $\mu_0 = \tilde{\mu}/\mu$.

Remark. We can assume $\tau \geq 1$. Since if $\tau < 1$, we replace $\alpha_T T_0 \rho_0 g$ by $\alpha_S S_0 \rho_0 g$ in introducing the dimensionless variables, and Eqs. (10) to (12) take the form

$$\sigma_1 D_a R_S |\mathbf{v}|\mathbf{v} + \nabla p - \mu_0 D_a \Delta \mathbf{v} + \mathbf{M}\mathbf{v} + \tau(T - T_R)\mathbf{g}$$
$$-(S - S_R)\mathbf{g} = 0 \qquad \text{in } \Omega, \tag{10.1}$$

$$\frac{R_T}{\tau} \mathbf{v} \cdot \nabla T - \nabla \cdot (\mathbf{N} \nabla T) = 0 \qquad \text{in } \Omega, \tag{11.1}$$

$$R_S \mathbf{v} \cdot \nabla S - \nabla \cdot (\mathbf{Q} \nabla S) = 0 \qquad \text{in } \Omega, \tag{12.1}$$

where $\sigma_1 = \rho_0 q_1 L c_0/(\mu k^{1/2})$.

With (10)–(12) replaced by (10.1) to (12.1), the entire procedure that follows can be carried through.

3. VARIATIONAL PROBLEM

We first list some function spaces which will be used later. Let $D(\Omega)$ be the space of C^∞ functions with compact support contained in Ω and $\mathbf{V}$ be defined as

$$\mathbf{V} = \{\mathbf{u} \in D(\Omega) : \nabla \cdot \mathbf{u} = 0\}. \tag{16}$$

The closures of $\mathbf{V}$ in $\mathbf{L}^2(\Omega)$ and $\mathbf{H}_0^1(\Omega)$ are two basic spaces in the study of the present problem. The characterizations of these two spaces are

$$\mathbf{H} = \{\mathbf{u} \in \mathbf{L}^2(\Omega) : \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega, \quad \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}, \tag{17}$$

where $\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega}$ should be understood as $\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = \lim_{m\to\infty} \mathbf{u}_m \cdot \mathbf{n}|_{\partial\Omega} = 0$, if $\mathbf{u} = \lim_{m\to\infty} \mathbf{u}_m$ in $\mathbf{L}^2(\Omega)$ for $\mathbf{u}_m \in \mathbf{V}$, and

$$\tilde{\mathbf{V}} = \{\mathbf{u} \in \mathbf{H}_0^1(\Omega) : \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega\}. \tag{18}$$

The scalar products and norms in $L^2(\Omega)$ and $H^m(\Omega)$ are, respectively, denoted by

$$(u, v) = \int_\Omega uv\,dx, \qquad \|u\|_{L^2(\Omega)} = (u, u)^{1/2},$$

$$((u, v))_{H^m(\Omega)} = \sum_{|j| \le m} (D^j u, D^j v), \qquad \|u\|_{H^m(\Omega)} = (u, u)^{1/2},$$

with

$$D^j = \frac{\partial^{|j|}}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}}, \qquad |j| = j_1 + j_2 + \cdots + j_n.$$

The norms in Banach spaces $L^p(\Omega)$ and $W^{m,p}(\Omega)$ are denoted by

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p\,dx\right)^{1/p},$$

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|j|\le m} \|D^j u\|_{L^p(\Omega)}\right)^{1/p}.$$

In the Hilbert space $H_0^1(\Omega)$, we choose an equivalent norm

$$\|u\|_{H_0^1(\Omega)} = \left(\sum_{i=1}^n \|D_i u\|_{L^2(\Omega)}^2\right)^{1/2}, \tag{19}$$

where $D_i = \partial/\partial x_i$.

The product Hilbert space $\tilde{\mathbf{V}} \times H_0^1(\Omega)^2$ is equipped with the usual scalar product,

$$((\mathbf{u}, T, S), (\mathbf{v}, t, s)) = (\mathbf{u}, \mathbf{v})_{\mathbf{H}_0^1(\Omega)} + (T, t)_{H_0^1(\Omega)} + (S, s)_{H_0^1(\Omega)}, \tag{20}$$

where $H_0^1(\Omega)^2 = H_0^1(\Omega) \times H_0^1(\Omega)$.

We also need the function which was introduced by Hopf [6] over fifty years ago. That is, for any $\epsilon > 0$, as $\partial\Omega$ is of class C^2, there exists a function $\theta_\epsilon \in C^2(\overline{\Omega})$ such that

(i) $\theta_\epsilon = 1$ in some neighbourhood of $\partial\Omega$ (which depends on ϵ),

(ii) $\theta_\epsilon = 0$ if $\rho(x) \geq 2\exp(-1/\epsilon)$,

(iii) $|\partial\theta_\epsilon/\partial x_k| \leq \epsilon/\rho(x)$ if $\rho(x) < 2\exp(-1/\epsilon)$, $k = 1, 2, \ldots, n$,

where $\rho(x) = \operatorname{dist}(x, \partial\Omega)$.

We now define $\mathbf{a}_\epsilon, \xi_\epsilon, \eta_\epsilon$, respectively, as

$$\mathbf{a}_\epsilon = \operatorname{curl}(\theta_\epsilon \mathbf{b}), \qquad \xi_\epsilon = \theta_\epsilon \xi, \qquad \eta_\epsilon = \theta_\epsilon \eta, \tag{21}$$

and let

$$\mathbf{u} = \mathbf{v} - \mathbf{a}_\epsilon, \qquad \theta = T - \xi_\epsilon, \qquad \gamma = S - \eta_\epsilon, \tag{22}$$

with $\mathbf{u} \in \tilde{\mathbf{V}}$, $\theta, \gamma \in H_0^1(\Omega)$.

It is straightforward to verify that (9)–(15) hold provided that $\mathbf{u}$, p, θ, and γ satisfy

$$\nabla \cdot \mathbf{u} = 0, \tag{23}$$

$$\sigma D_a R_T |\mathbf{u} + \mathbf{a}_\epsilon|(\mathbf{u} + \mathbf{a}_\epsilon) + \boldsymbol{\nabla} p - \mu_0 D_a \Delta(\mathbf{u} + \mathbf{a}_\epsilon) + \mathbf{M}(\mathbf{u} + \mathbf{a}_\epsilon)$$
$$+ (\theta + \psi_\epsilon)\mathbf{g} - \frac{1}{\tau}(\gamma + \beta_\epsilon)\mathbf{g} = 0, \tag{24}$$

$$R_T(\mathbf{u} + \mathbf{a}_\epsilon) \cdot \boldsymbol{\nabla}(\theta + \psi_\epsilon) - \boldsymbol{\nabla} \cdot \big(\mathbf{N}\boldsymbol{\nabla}(\theta + \psi_\epsilon)\big) = 0, \tag{25}$$

$$\tau R_S(\mathbf{u} + \mathbf{a}_\epsilon) \cdot \boldsymbol{\nabla}(\gamma + \beta_\epsilon) - \boldsymbol{\nabla} \cdot \big(\mathbf{Q}\boldsymbol{\nabla}(\gamma + \beta_\epsilon)\big) = 0, \tag{26}$$

$$\mathbf{u} = 0, \qquad \theta = 0, \qquad \gamma = 0 \text{ on } \partial\Omega, \tag{27}$$

where $\psi_\epsilon = \xi_\epsilon - T_R$, $\beta_\epsilon = \eta_\epsilon - S_R$.

To motivate the variational problem, we assume that the smooth solutions $\mathbf{u}, p, \theta, \gamma$ exist for (23)–(27) and that $\mathbf{a}_\epsilon, \psi_\epsilon, \beta_\epsilon$ are also smooth. On taking scalar products of (24), (25), (26) with the functions $\mathbf{w} \in \mathbf{V}$, $t \in D(\Omega)$, $s \in D(\Omega)$, respectively, and integrating by parts, we obtain

$$\sigma D_a R_T\big(|\mathbf{u} + \mathbf{a}_\epsilon|(\mathbf{u} + \mathbf{a}_\epsilon), \mathbf{w}\big) + \mu_0 D_a\big(\boldsymbol{\nabla}(\mathbf{u} + \mathbf{a}_\epsilon), \boldsymbol{\nabla}\mathbf{w}\big) + \big(\mathbf{M}(\mathbf{u} +_\epsilon), \mathbf{w}\big)$$
$$+ \big((\theta + \psi_\epsilon)\mathbf{g}, \mathbf{w}\big) - \frac{1}{\tau}\big((\gamma + \beta_\epsilon)\mathbf{g}, \mathbf{w}\big) = 0, \tag{28}$$

$$R_T\big((\mathbf{u} + \mathbf{a}_\epsilon) \cdot \boldsymbol{\nabla}(\theta + \psi_\epsilon), t\big) + \big(\mathbf{N}\boldsymbol{\nabla}(\theta + \psi_\epsilon), \boldsymbol{\nabla} t\big) = 0, \tag{29}$$

$$\tau R_S\big((\mathbf{u} + \mathbf{a}_\epsilon) \cdot \boldsymbol{\nabla}(\gamma + \beta_\epsilon), s\big) + \big(\mathbf{Q}\boldsymbol{\nabla}(\gamma + \beta_\epsilon), \boldsymbol{\nabla} s\big) = 0. \tag{30}$$

Since $\mathbf{V}$ is dense in $\tilde{\mathbf{V}}$ and $D(\Omega)$ is dense in $H_0^1(\Omega)$, a continuity argument shows that (28)–(30) still hold if $(\mathbf{u}, \theta, \gamma) \in \tilde{\mathbf{V}} \times H_0^1(\Omega)^2$, $\mathbf{a}_\epsilon \in \mathbf{H}^1(\Omega)$, $\psi_\epsilon, \beta_\epsilon \in H^1(\Omega)$ and for $(\mathbf{w}, t, s) \in \tilde{\mathbf{V}} \times H_0^1(\Omega)^2$.

We define a mapping $G(\cdot, \cdot, \cdot)$ from $\tilde{\mathbf{V}} \times H_0^1(\Omega)^2$ into itself by

$$\begin{aligned}
\langle G(\mathbf{u}, \theta, \gamma), (\mathbf{w}, t, s)\rangle
&= \sigma D_a R_T\big(|\mathbf{u} + \mathbf{a}_\epsilon|(\mathbf{u} + \mathbf{a}_\epsilon), \mathbf{w}\big) + \mu_0 D_a\big(\nabla(\mathbf{u} + \mathbf{a}_\epsilon), \nabla \mathbf{w}\big) \\
&\quad + \big(\mathbf{M}(\mathbf{u} + \mathbf{a}_\epsilon), \mathbf{w}\big) + \big((\theta + \psi_\epsilon)\mathbf{g}, \mathbf{w}\big) - \frac{1}{\tau}\big((\gamma + \beta_\epsilon)\mathbf{g}, \mathbf{w}\big) \\
&\quad + R_T\big((\mathbf{u} + \mathbf{a}_\epsilon) \cdot \nabla(\theta + \psi_\epsilon), t\big) + \big(\mathbf{N}\nabla(\theta + \psi_\epsilon), \nabla t\big) \\
&\quad + \tau R_S\big((\mathbf{u} + \mathbf{a}_\epsilon) \cdot \nabla(\gamma + \beta_\epsilon), s\big) + \big(\mathbf{Q}\nabla(\gamma + \beta_\epsilon), \nabla s\big). \quad (31)
\end{aligned}$$

Thus the variational problem associated with (23)–(27) is to find $(\mathbf{u}, \theta, \gamma) \in \tilde{\mathbf{V}} \times H_0^1(\Omega)^2$ such that

$$\langle G(\mathbf{u}, \theta, \gamma), (\mathbf{w}, t, s)\rangle = 0 \qquad \forall(\mathbf{w}, t, s) \in \tilde{\mathbf{V}} \times H_0^1(\Omega)^2. \quad (32)$$

Conversely, if $(\mathbf{u}, \theta, \gamma) \in \tilde{\mathbf{v}} \times H_0^1(\Omega)^2$ satisfies (32), then (28), (29), and (30) hold for any $\mathbf{w} \in \tilde{\mathbf{V}}$, $t \in H_0^1(\Omega)$, and $s \in H_0^1(\Omega)$ by choosing $t = 0$, $s = 0$ or $\mathbf{w} = 0$, $s = 0$ or $\mathbf{w} = 0$, $t = 0$ in (32), respectively.

Propositions 1.1 and 1.2 in Temam [13, Chap. 1] assert that for $\mathbf{f} = \{f_1, f_2, \dots, f_n\}$ with $f_i \in D'(\Omega)$ $(i = 1, 2, \dots, n)$ the following results are true:

(i) A necessary and sufficient condition that $\mathbf{f} = \nabla p$ for some $p \in D'(\Omega)$ is that $(\mathbf{f}, \mathbf{w}) = 0$ $\forall \mathbf{w} \in \mathbf{V}$.

(ii) Let Ω be a bounded Lipschitz open set in $\mathbf{R}^n$. If a distribution p has all its first derivatives $D_i p$ $(1 \le i \le n)$ in $H^{-1}(\Omega)$, then $p \in L^2(\Omega)$.

It follows from (28) that there exists a distribution $p \in L^2(\Omega)$ such that (24) holds in the distribution sense in Ω. Also, (29) and (30) imply that (25) and (26) are true in the distribution sense in Ω and (23), (27) are satisfied in the distribution sense in Ω and in the trace sense on $\partial\Omega$, respectively.

4. THE EXISTENCE OF SOLUTIONS

With the use of above argument we now prove the existence of solutions of (9)–(15). To do so, we note that it is enough to show that variational problem (32) has a solution in $\tilde{\mathbf{V}} \times H_0^1(\Omega)^2$.

We first prove the following lemma.

LEMMA 1. *If G is a mapping from* $\tilde{\mathbf{V}} \times H_0^1(\Omega)^2$ *into itself defined by* (31), *then*

(i) *G is continuous,*

(ii) *there exists a* $r > 0$ *such that*

$$\langle G(\mathbf{Y}), \mathbf{Y}\rangle > 0 \qquad \forall \mathbf{Y} \in \tilde{\mathbf{V}} \times H_0^1(\Omega)^2 \text{ with } \|\mathbf{Y}\|_{\tilde{\mathbf{V}} \times H_0^1(\Omega)^2} = r.$$

Proof. Let $(\mathbf{u}^k, \theta^k, \gamma^k) \to (\mathbf{u}, \theta, \gamma)$ strongly in $\tilde{\mathbf{V}} \times H_0^1(\Omega)^2$ as $k \to \infty$ and m_l, n_l, q_l be the largest eigenvalues of matrices $\mathbf{M}, \mathbf{N}, \mathbf{Q}$, respectively. Then for any $(\mathbf{w}, t, s) \in \tilde{\mathbf{V}} \times H_0^1(\Omega)^2$ we have

$$\begin{aligned}
&\left|\left\langle G(\mathbf{u}^k, \theta^k, \gamma^k) - G(\mathbf{u}, \theta, \gamma), (\mathbf{w}, t, s)\right\rangle\right| \\
&\quad \le \sigma D_a R_T \Big\{ \left| \left(|\mathbf{u}^k + \mathbf{a}_\epsilon| (\mathbf{u}^k - \mathbf{u}), \mathbf{w}\right)\right| \\
&\qquad\qquad + \left| \left(\left(|\mathbf{u}^k + \mathbf{a}_\epsilon| - |\mathbf{u} + \mathbf{a}_\epsilon| \right)(\mathbf{u} + \mathbf{a}_\epsilon), \mathbf{w}\right)\right| \Big\} \\
&\qquad + \mu_0 D_a \left|\left(\nabla(\mathbf{u}^k - \mathbf{u}), \nabla \mathbf{w}\right)\right| + \left|\left(\mathbf{M}(\mathbf{u}^k - \mathbf{u}), \mathbf{w}\right)\right| + \left|\left((\theta^k - \theta)\mathbf{g}, \mathbf{w}\right)\right| \\
&\qquad + \frac{1}{\tau}\left|\left((\gamma^k - \gamma)\mathbf{g}, \mathbf{w}\right)\right| + R_T \left|\left((\mathbf{u}^k + \mathbf{a}_\epsilon) \cdot \nabla(\theta^k - \theta), t\right)\right| \\
&\qquad + R_T \left|\left((\mathbf{u}^k - \mathbf{u}) \cdot (\theta + \psi_\epsilon), t\right)\right| \\
&\qquad + \left|\left(\mathbf{N}\nabla(\theta^k - \theta), \nabla t\right)\right| + \tau R_S \left|\left((\mathbf{u}^k + \mathbf{a}_\epsilon) \cdot \nabla(\gamma^k - \gamma), s\right)\right| \\
&\qquad + \tau R_s \left|\left((\mathbf{u}^k - \mathbf{u}) \cdot \nabla(\gamma + \beta_\epsilon), s\right)\right| + \left|\mathbf{Q}\nabla(\gamma^K - \gamma), \nabla s\right)\right| \\
&\quad \le \sigma D_a R_T \|\mathbf{u}^{\mathbf{k}} - \mathbf{u}\|_{L^2} \|\mathbf{u}^k + \mathbf{a}_\epsilon\|_{L^4} \|\mathbf{w}\|_{L^4} \\
&\qquad + \sigma D_a R_T \|\mathbf{u} + \mathbf{a}_\epsilon\|_{L^4} \|\mathbf{u}^k - \mathbf{u}\|_{L^2} \|\mathbf{w}\|_{L^4} \\
&\qquad + \mu_0 D_a \|\mathbf{u}^k - \mathbf{u}\|_{\tilde{V}} \|\mathbf{w}\|_{\tilde{V}} + m_l \|\mathbf{u}^k - \mathbf{u}\|_H \|\mathbf{w}\|_H + \|\theta^k - \theta\|_{L^2} \|\mathbf{w}\|_H \\
&\qquad + \frac{1}{\tau}\|\gamma^k - \gamma\|_{L^2} \|\mathbf{w}\|_H + R_T \|\theta^k - \theta\|_{H_0^1} \|\mathbf{u}^k - \mathbf{u}\|_{L^4} \|t\|_{L^4} \\
&\qquad + R_T \|t\|_{H_0^1} \|\mathbf{u}^k - \mathbf{u}\|_{L^4} \|\theta + \psi_\epsilon\|_{L^4} + n_l \|\theta^k - \theta\|_{H_0^1} \|t\|_{H_0^1} \\
&\qquad + \tau R_S \|\gamma^k - \gamma\|_{H_0^1} \|\mathbf{u}^k + \mathbf{a}_\epsilon\|_{L^4} \|s\|_{L^4} \\
&\qquad + \tau R_S \|s\|_{H_0^1} \|\mathbf{u}^K - \mathbf{u}\|_{L^4} \|\gamma + \beta_\epsilon\|_{L^4} + q_l \|\gamma^k - \gamma\|_{H_0^1} \|s\|_{H_0^1}.
\end{aligned} \tag{33}$$

The continuity of G follows from (33) and from Sobolev's imbedding theorems as well as from the boundedness of $(\mathbf{u}^k, \theta^k, \gamma^k)$ in $\tilde{\mathbf{V}} \times H_0^1(\Omega)^2$ (cf. Gilbarg and Trudinger [4]).

To prove the second part we note that for any $(\mathbf{u}, \theta, \gamma) \in \tilde{\mathbf{V}} \times H_0^1(\Omega)^2$ we have

$$
\begin{aligned}
&\langle G(\mathbf{u}, \theta, \gamma), (\mathbf{u}, \theta, \gamma)\rangle \\
&\quad \geq \mu_0 D_a \|\mathbf{u}\|_{\tilde{V}}^2 + \|\mathbf{u}\|_H^2 + \|\theta\|_{H_0^1}^2 + \|\gamma\|_{H_0^1}^2 \\
&\qquad + \sigma D_a R_T\big(|\mathbf{u} + \mathbf{a}_\epsilon|(\mathbf{u} + \mathbf{a}_\epsilon), \mathbf{u}\big) + \mu_0 D_a(\nabla \mathbf{a}_\epsilon, \nabla \mathbf{u}) \\
&\qquad + (\mathbf{M}\mathbf{a}_\epsilon, \mathbf{u}) + \big((\theta + \psi_\epsilon)\mathbf{g}, \mathbf{u}\big) - \frac{1}{\tau}\big((\gamma + \beta_\epsilon)\mathbf{g}, \mathbf{u}\big) \\
&\qquad + R_T\big((\mathbf{u} + \mathbf{a}_\epsilon) \cdot \nabla(\theta + \psi_\epsilon), \theta\big) + (\mathbf{N}\nabla\psi_\epsilon, \nabla\theta) \\
&\qquad + \tau R_S\big((\mathbf{u} + \mathbf{a}_\epsilon) \cdot \nabla(\gamma + \beta_\epsilon), \gamma\big) + (\mathbf{Q}\nabla\beta_\epsilon, \nabla\gamma) \\
&\quad \geq \mu_0 D_a \|\mathbf{u}\|_{\tilde{V}}^2 + \|\mathbf{u}\|_H^2 + \|\theta\|_{H_0^1}^2 + \|\gamma\|_{H_0^1}^2 \\
&\qquad + \sigma D_a R_T\big(|\mathbf{u} + \mathbf{a}_\epsilon|, |\mathbf{u}|^2\big) - \sigma D_a R_T\big|\big(|\mathbf{u} + \mathbf{a}_\epsilon|\mathbf{a}_\epsilon, \mathbf{u}\big)\big| \\
&\qquad - \mu_0 D_a \|\nabla \mathbf{a}_\epsilon\|_{L^2}\|\mathbf{u}\|_{\tilde{V}} - m_l\|\mathbf{a}_\epsilon\|_{L^2}\|\mathbf{u}\|_H - \|\theta\|_{L^2}\|\mathbf{u}\|_H \\
&\qquad - \|\psi_\epsilon\|_{L^2}\|\mathbf{u}\|_H - \frac{1}{\tau}\|\gamma\|_{L^2}\|\mathbf{u}\|_H - \frac{1}{\tau}\|\beta_\epsilon\|_{L^2}\|\mathbf{u}\|_H \\
&\qquad - R_T\big(\|\mathbf{u}\|_H + \|\mathbf{a}_\epsilon\|_{L^2}\big)\|\theta\nabla\psi_\epsilon\|_{L^2} - n_l\|\nabla\psi_\epsilon\|_{L^2}\|\theta\|_{H_0^1} \\
&\qquad - \tau R_S\big(\|\mathbf{u}\|_H + \|\mathbf{a}_\epsilon\|_{L^2}\big)\|\gamma\nabla\beta_\epsilon\|_{L^2} - q_l\|\nabla\beta_\epsilon\|_{L^2}\|\gamma\|_{H_0^1}. \qquad (34)
\end{aligned}
$$

We now estimate the terms $|(|\mathbf{u} + \mathbf{a}_\epsilon|\mathbf{a}_\epsilon, \mathbf{u})|$, $\|\theta\nabla\psi_\epsilon\|_{L^2}$, and $\|\gamma\nabla\beta_\epsilon\|_{L^2}$ successively.

We first note that since $b_i \in H^2(\Omega)$ $(1 \leq i \leq n)$, the Sobolev's imbedding theorems imply that $b_i \in L^\infty(\Omega)$ $(1 \leq i \leq n)$. This gives

$$
|\mathbf{a}_\epsilon| \leq c_1\{|\nabla \mathbf{b}| + \epsilon|\mathbf{b}|/\rho(x)\} \leq c_2\{\epsilon/\rho(x) + |\nabla \mathbf{b}|\}, \qquad (35)
$$

where c_1, c_2 are constants. Now

$$
|(|\mathbf{u} + \mathbf{a}_\epsilon|\mathbf{a}_\epsilon, \mathbf{u})| \leq \|\mathbf{u} + \mathbf{a}_\epsilon\|_{L^2}\|\mathbf{u} \cdot \mathbf{a}_\epsilon\|_{L^2(\Omega_\epsilon)}, \qquad (36)
$$

where $\Omega_\epsilon = \{x \in \Omega : \rho(x) < 2\exp(-1/\epsilon)\}$ and

$$
\begin{aligned}
\|\mathbf{u} \cdot \mathbf{a}_\epsilon\|_{L^2(\Omega_\epsilon)} &= \left(\sum_{i=1}^n \int_{\Omega_\epsilon} |u_i a_\epsilon^i|^2\, dx\right)^{1/2} \\
&\leq \left(\int_{\Omega_\epsilon} |\mathbf{u}|^2 |\mathbf{a}_\epsilon|^2\, dx\right)^{1/2} \\
&\leq 2c_2\left\{\epsilon\|\mathbf{u}/\rho\|_{L^2} + \left(\int_{\Omega_\epsilon} |\nabla \mathbf{b}|^4\, dx\right)^{1/4} \|\mathbf{u}\|_{L^4}\right\}. \qquad (37)
\end{aligned}
$$

With use of the Hardy inequality (Hopf [6]) and the Sobolev inequality

$$\|u/\rho\|_{L^2} \leq \text{const}\|u\|_{H_0^1}, \qquad \|u\|_{L^4} \leq \text{const}\|u\|_{H_0^1} \quad \forall u \in H_0^1(\Omega), \quad (38)$$

the inequality (37) becomes

$$\|\mathbf{u}\cdot\mathbf{a}_\epsilon\|_{L^2(\Omega_\epsilon)} \leq c_3\lambda(\epsilon)\|\mathbf{u}\|_{\tilde{V}}, \tag{39}$$

where $\lambda(\epsilon) = \max\{\epsilon, (\int_{\Omega_\epsilon}|\nabla\mathbf{b}|^4\,dx)^{1/4}\} \to 0$ as $\epsilon \to 0$.

From (36)–(39) we conclude that for any $\delta > 0$ we can choose ϵ sufficiently small such that

$$\left|\left(|\mathbf{u}+\mathbf{a}_\epsilon|\mathbf{a}_\epsilon, \mathbf{u}\right)\right| \leq \delta\|\mathbf{u}\|_{\tilde{V}}^2 + \delta\|\mathbf{u}\|_{\tilde{V}}\|\mathbf{a}_\epsilon\|_{L^2}. \tag{40}$$

In a similar way we can show that for any $\delta > 0$ we can choose ϵ sufficiently small such that

$$\|\theta\nabla\psi_\epsilon\|_{L^2} \leq \delta\|\theta\|_{H_0^1}, \qquad \|\gamma\nabla\beta_\epsilon\|_{L^2} \leq \delta\|\gamma\|_{H_0^1}. \tag{41}$$

We now return to (34). By applying the well-known inequality (Ladyzhenskaya [7])

$$\|u\|_{L^2} \leq \frac{1}{\pi n^{1/2}}\|u\|_{H_0^1} \qquad \forall u \in H_0^1(\Omega), \tag{42}$$

and inequalities

$$\|\mathbf{u}\|_H\|\theta\|_{L^2} \leq \frac{1}{2}\|\mathbf{u}\|_H^2 + \frac{1}{2}\|\theta\|_{L^2}^2, \qquad \|\mathbf{u}\|_H\|\gamma\|_{L^2} \leq \frac{\tau}{2}\|\mathbf{u}\|_H^2 + \frac{1}{2\tau}\|\gamma\|_{L^2}^2$$

together with (40) and (41) in (34) we obtain

$$\begin{aligned}
&\left\langle G(\mathbf{u},\theta,\gamma),(\mathbf{u},\theta,\gamma)\right\rangle \\
&\quad\geq \left\{D_a(\mu_0 - \sigma\delta R_T) - \frac{R_T\delta + \tau R_S\delta}{2\pi n^{1/2}}\right\}\|\mathbf{u}\|_{\tilde{V}}^2 \\
&\qquad + \left\{1 - \frac{1}{2\pi n^{1/2}} - \frac{R_T\delta}{2}\right\}\|\theta\|_{H_0^1}^2 \\
&\qquad + \left\{1 - \frac{1}{2\pi\tau^2 n^{1/2}} - \frac{\tau R_S\delta}{2}\right\}\|\gamma\|_{H_0^1}^2 \\
&\qquad - \left\{\delta\sigma D_a R_T\|\mathbf{a}_\epsilon\|_{L^2} + \mu_0 D_a\|\nabla\mathbf{a}_\epsilon\|_{L^2} + \frac{m_l}{\pi n^{1/2}}\|\mathbf{a}_\epsilon\|_{L^2} + \frac{1}{\pi n^{1/2}}\|\psi_\epsilon\|_{L^2}\right. \\
&\qquad\qquad \left. + \frac{1}{\tau\pi n^{1/2}}\|\beta_\epsilon\|_{L^2}\right\}\|\mathbf{u}\|_{\tilde{V}} - \left\{\delta R_T\|\mathbf{a}_\epsilon\|_{L^2} + n_l\|\nabla\psi_\epsilon\|_{L^2}\right\}\|\theta\|_{H_0^1} \\
&\qquad - \left\{\tau\delta R_S\|\mathbf{a}_\epsilon\|_{L^2} + q_l\|\nabla\beta_\epsilon\|_{L^2}\right\}\|\gamma\|_{H_0^1}.
\end{aligned} \tag{43}$$

Since $\tau \geq 1$, by choosing $0 < \delta \leq \delta_0 < \mu_0/(\sigma R_T)$ we have

$$\begin{aligned}
&\left\langle G(\mathbf{u}, \theta, \gamma), (\mathbf{u}, \theta, \gamma)\right\rangle \\
&\quad \geq \left\{\min\left\{D_a(\mu_0 - \sigma\delta_0 R_T), 1 - \frac{1}{2\pi n^{1/2}}\right\} - \frac{R_T\delta + \tau R_S\delta}{2}\right\} \\
&\qquad \times\left(\|\mathbf{u}\|_{\tilde{V}}^2 + \|\theta\|_{H_0^1}^2 + \|\gamma\|_{H_0^1}^2\right) \\
&\qquad -\left\{\delta\sigma D_a R_T\|\mathbf{a}_\epsilon\|_{L^2} + \mu_0 D_a\|\nabla\mathbf{a}_\epsilon\|_{L^2}\right. \\
&\qquad\quad +\left(\frac{m_l}{\pi n^{1/2}} + \delta R_T + \delta\tau R_S\right)\|\mathbf{a}_\epsilon\|_{L^2} \\
&\qquad\quad \left.+\frac{1}{\pi n^{1/2}}\|\psi_\epsilon\|_{L^2} + \frac{1}{\tau\pi n^{1/2}}\|\beta_\epsilon\|_{L^2} + n_l\|\nabla\psi_\epsilon\|_{L^2} + q_l\|\nabla\beta_\epsilon\|_{L^2}\right\} \\
&\qquad\quad \times\left(\|\mathbf{u}\|_{\tilde{V}}^2 + \|\theta\|_{H_0^1}^2 + \|\gamma\|_{H_0^1}^2\right)^{1/2}. \qquad (44)
\end{aligned}$$

We now choose

$$0 < \delta \leq \min\left\{\delta_0, \min\left\{D_a(\mu_0 - \sigma\delta_0 R_T), 1 - 1/2\pi n^{1/2}\right\}\frac{2}{R_T + \tau R_S}\right\}, \qquad (45)$$

$$\begin{aligned}
r \geq \frac{1}{\kappa}&\left\{n^{1/2}\sigma D_a R_T\|\mathbf{a}_\epsilon\|_{L^4} + \mu_0 D_a\|\nabla\mathbf{a}_\epsilon\|_{L^2}\right. \\
&\quad +\left(\frac{m_l}{\pi n^{1/2}} + \delta R_T + \delta\tau R_S\right)\|\mathbf{a}_\epsilon\|_{L^2} + \frac{1}{\pi n^{1/2}}\|\psi_\epsilon\|_{L^2} \\
&\quad \left.+\frac{1}{\tau\pi n^{1/2}}\|\beta_\epsilon\|_{L^2} + n_l\|\nabla\psi_\epsilon\|_{L^2} + q_l\|\nabla\beta_\epsilon\|_{L^2}\right\}, \qquad (46)
\end{aligned}$$

where

$$\kappa = \min\left\{D_a(\mu_0 - \sigma\delta_0 R_T), 1 - 1/2\pi n^{1/2}\right\} - \delta(R_T + \tau R_S)/2.$$

The above choices for δ and r lead to

$$\left\langle G(\mathbf{Y}), \mathbf{Y}\right\rangle > 0 \qquad \text{with } \|\mathbf{Y}\|_{\tilde{V}\times H_0^1(\Omega)^2} = r,$$

which proves the second part of the lemma.

Besides the above lemma the following lemma is needed to obtain the existence result (Lions [8]).

LEMMA 2. *Let $\mathscr{H}$ be a finite-dimensional Hilbert space with scalar product $\langle \cdot , \cdot \rangle$ and norm $\| \cdot \|$, and let G be a continuous mapping from $\mathscr{H}$ into itself such that*

$$\langle G(x), x\rangle > 0 \qquad \text{for } \|x\| = r_0 > 0.$$

Then there exists $x \in \mathscr{H}$, with $\|x\| \le r_0$, such that

$$G(x) = 0.$$

We are now ready to obtain the main result of the section.

THEOREM 1. *The problem* (32) *has at least one solution* $(\mathbf{u}, \theta, \gamma) \in \tilde{\mathbf{V}} \times H_0^1(\Omega)^2$.

Proof. We employ the Galerkin method to prove this theorem. Since $\tilde{V}$ and $H_0^1(\Omega)$ are separable, there exist three sequences $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m$ of linearly independent elements in $\tilde{\mathbf{V}}$, $t_1, t_2, \dots, t_m$ of linearly independent elements in $H_0^1(\Omega)$, and $s_1, s_2, \dots, s_m$ of linearly independent elements in $H_0^1(\Omega)$. We define an approximate solution $(\mathbf{u}_m, \theta_m, \gamma_m)$ of (32) by

$$\mathbf{u}_m = \sum_{l=1}^{m} a_j \mathbf{w}_j, \qquad \theta_m = \sum_{l=1}^{m} b_j t_j, \qquad \gamma_m = \sum_{l=1}^{m} c_j s_j \tag{47}$$

$$\sigma D_a R_T\big(|\mathbf{u}_m + \mathbf{a}_\epsilon|(\mathbf{u}_m + \mathbf{a}_\epsilon), \mathbf{w}_j\big) + \mu_0 D_a\big(\nabla(\mathbf{u}_m + \mathbf{a}_\epsilon), \boldsymbol{\nabla} \mathbf{w}_j\big)$$
$$+ \big(\mathbf{M}(\mathbf{u}_m + \mathbf{a}_\epsilon), \mathbf{w}_j\big) + \big((\theta_m + \psi_\epsilon)\mathbf{g}, \mathbf{w}_j\big) - \frac{1}{\tau}\big((\gamma_m + \beta_\epsilon)\mathbf{g}, \mathbf{w}_j\big) = 0, \tag{48}$$

$$R_T\big((\mathbf{u}_m + \mathbf{a}_\epsilon) \cdot \boldsymbol{\nabla}(\theta_m + \psi_\epsilon), t_j\big) + \big(\mathbf{N}\boldsymbol{\nabla}(\theta_m + \psi_\epsilon), \boldsymbol{\nabla} t_j\big) = 0, \tag{49}$$

$$\tau R_S\big((\mathbf{u}_m + \mathbf{a}_\epsilon) \cdot \boldsymbol{\nabla}(\gamma_m + \beta_\epsilon), s_j\big) + \big(\mathbf{Q}\boldsymbol{\nabla}(\gamma_m + \beta_\epsilon), \boldsymbol{\nabla} s_j\big) = 0, \tag{50}$$

with

$$a_j, b_j, c_j \in \mathbf{R}, \qquad j = 1, 2, \dots, m.$$

Let X be the product space spanned by $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m$; $t_1, t_2, \dots, t_m$ and $s_1, s_2, \dots, s_m$. The scalar product on X is induced by $\tilde{\mathbf{V}} \times H_0^1(\Omega)^2$, and $G = G_m$ is defined by

$$\big\langle G_m(\mathbf{u}, \theta, \gamma), (\mathbf{w}, t, s)\big\rangle = \big\langle G(\mathbf{u}, \theta, \gamma), (\mathbf{w}, t, s)\big\rangle$$
$$= (31) \qquad \forall(\mathbf{u}, \theta, \gamma), (\mathbf{w}, t, s) \in X. \tag{51}$$

It is obvious that G_m satisfies the hypotheses of Lemma 1 and Lemma 2. Therefore, there exists a solution $(\mathbf{u}_m, \theta_m, \gamma_m) \in X$ such that

$$\big\langle G_m(\mathbf{u}_m, \theta_m, \gamma_m), (\mathbf{w}, t, s)\big\rangle = 0, \qquad \forall(\mathbf{w}, t, s) \in X. \tag{52}$$

In particular

$$\sigma D_a R_T\big(|\mathbf{u}_m + \mathbf{a}_\epsilon|(\mathbf{u}_m + \mathbf{a}_\epsilon), \mathbf{w}\big) + \mu_0 D_a\big(\nabla(\mathbf{u}_m + \mathbf{a}_\epsilon), \nabla\mathbf{w}\big) + \big(\mathbf{M}(\mathbf{u}_m + \mathbf{a}_\epsilon), \mathbf{w}\big) + \big((\theta_m + \psi_\epsilon)\mathbf{g}, \mathbf{w}\big) - \frac{1}{\tau}\big((\gamma_m + \beta_\epsilon)\mathbf{g}, \mathbf{w}\big) = 0 \quad \forall \mathbf{w} \in \tilde{\mathbf{V}} \cap X,$$

$$R_T\big((\mathbf{u}_m + \mathbf{a}_\epsilon) \cdot \nabla(\theta_m + \psi_\epsilon), t\big) + \big(\mathbf{N}\nabla(\theta_m + \psi_\epsilon), \nabla t\big) = 0 \quad \forall t \in H_0^1(\Omega) \cap X,$$

$$\tau R_S\big((\mathbf{u}_m + \mathbf{a}_\epsilon) \cdot \nabla(\gamma_m + \beta_\epsilon), s\big) + \big(\mathbf{Q}\nabla(\gamma_m + \beta_\epsilon), \nabla s\big) = 0 \quad \forall s \in H_0^1(\Omega) \cap X.$$

It follows that (48)–(50) are satisfied and a_j, b_j, c_j can be determined through (48)–(50). Multiplying (48), (49), (50) by, respectively, a_j, b_j, c_j and adding the equalities for $j = 1, 2, \ldots, m$ we obtain

$$\begin{aligned} 0 &= \big\langle G_m(\mathbf{u}_m, \theta_m, \gamma_m), (\mathbf{u}_m, \theta_m, \gamma_m)\big\rangle \\ &\geq \kappa\Big(\|\mathbf{u}_m\|_{\tilde{V}}^2 + \|\theta_m\|_{H_0^1}^2 + \|\gamma_m\|_{H_0^1}^2\Big) \\ &\quad - \Big\{\delta\sigma D_a R_T\|\mathbf{a}_\epsilon\|_{L^2} + \mu_0 D_a\|\nabla\mathbf{a}_\epsilon\|_{L^2} + \Big(\frac{m_l}{\pi n^{1/2}} + \delta R_T + \delta\tau R_S\Big)\|\mathbf{a}_\epsilon\|_{L^2} \\ &\quad + \frac{1}{\pi n^{1/2}}\|\psi_\epsilon\|_{L^2} + \frac{1}{\tau\pi n^{1/2}}\|\beta_\epsilon\|_{L^2} + n_l\|\nabla\psi_\epsilon\|_{L^2} + q_l\|\nabla\beta_\epsilon\|_{L^2}\Big\} \\ &\quad \times\Big(\|\mathbf{u}_m\|_{\tilde{V}}^2 + \|\theta_m\|_{H_0^1}^2 + \|\gamma_m\|_{H_0^1}^2\Big)^{1/2}. \end{aligned}$$

This gives

$$\begin{aligned} &\|(\mathbf{u}_m, \theta_m, \gamma_m)\|_{\tilde{V}\times H_0^1(\Omega)^2} \\ &\quad \leq r = \frac{1}{\kappa}\Big\{\delta\sigma D_a R_T\|\mathbf{a}_\epsilon\|_{L^2} + \mu_0 D_a\|\nabla\mathbf{a}_\epsilon\|_{L^2} \\ &\qquad + \Big(\frac{m_l}{\pi n^{1/2}} + \delta R_T + \delta\tau R_S\Big)\|\mathbf{a}_\epsilon\|_{L^2} + \frac{1}{\pi n^{1/2}}\|\psi_\epsilon\|_{L^2} \\ &\qquad + \frac{1}{\tau\pi n^{1/2}}\|\beta_\epsilon\|_{L^2} + n_l\|\nabla\psi_\epsilon\|_{L^2} + q_l\|\nabla\beta_\epsilon\|_{L^2}\Big\}. \end{aligned} \tag{53}$$

Since the sequence $(\mathbf{u}_m, \theta_m, \gamma_m)$ is uniformly bounded in $\tilde{\mathbf{V}} \times H_0^1(\Omega)^2$, it follows that there exists a $(\mathbf{u}, \theta, \gamma) \in \tilde{\mathbf{V}} \times H_0^1(\Omega)^2$ and a subsequence

$m_k \to \infty$ (we still write m instead of m_k for the convenience) such that

$$(\mathbf{u}_m, \theta_m, \gamma_m) \to (\mathbf{u}, \theta, \gamma) \qquad \text{weakly in } \tilde{\mathbf{V}} \times H_0^1(\Omega)^2. \tag{54}$$

Moreover, the compactness imbedding theorem shows that

$$(\mathbf{u}_m, \theta_m, \gamma_m) \to (\mathbf{u}, \theta, \gamma) \qquad \text{strongly in } \mathbf{L}^4 \times L^4 \times L^4. \tag{55}$$

Taking the limit in (52) with $m \to \infty$ we get

$$\langle G(\mathbf{u}, \theta, \gamma), (\mathbf{w}, t, s) \rangle = 0 \qquad \forall (\mathbf{w}, t, s) \in X. \tag{56}$$

A continuity argument finally shows that (56) holds for any $(\mathbf{w}, t, s) \in \tilde{\mathbf{V}} \times H_0^1(\Omega)^2$ and $(\mathbf{u}, \theta, \gamma)$ is a solution of (32). This completes the proof.

5. REGULARITY AND UNIQUENESS

In this section, we discuss the regularity and uniqueness of the solution of (9)–(15). Here $\mathbf{a}_\epsilon, \psi_\epsilon, \beta_\epsilon$ are replaced, respectively, by $\mathbf{a}$, $\psi = \xi - T_R$, $\beta = \eta - S_R$. We assume that $\mathbf{a} \in \mathbf{H}^2(\Omega)$, $\psi, \beta \in H^2(\Omega)$.

THEOREM 2. *Let Ω be an open bounded set of class C^3 and $(\mathbf{v}, p, T, S) \in \mathbf{H}^1(\Omega) \times L^2(\Omega) \times H^1(\Omega)^2$ be a solution of* (9)–(15), *then* $(\mathbf{v}, p, T, S) \in \mathbf{H}^2(\Omega) \times H^1(\Omega) \times H^2(\Omega) \times H^2(\Omega)$.

Proof. Let $\mathbf{u} = \mathbf{v} - \mathbf{a}$, $\theta = T - \xi$, $\gamma = S - \eta$, then $(\mathbf{u}, p, \theta, \gamma)$ is a solution of (23)–(27). We write (24) as

$$-\mu_0 D_a \Delta \mathbf{u} + \nabla p = \mathbf{f}, \tag{57}$$

where

$$\mathbf{f} = \mu_0 D_a \Delta \mathbf{a} - \mathbf{M}(\mathbf{u} + \mathbf{a}) - (\theta + \psi)\mathbf{g} + \frac{1}{\tau}(\gamma + \beta)\mathbf{g}$$
$$- \sigma D_a R_T |\mathbf{u} + \mathbf{a}|(\mathbf{u} + \mathbf{a}).$$

Notice that $\mathbf{f} \in \mathbf{L}^2(\Omega)$, thus the regularity theory for the generalized Stokes problem (see [13, Proposition 2.2, Chap. I]) shows that

$$\mathbf{u} \in \mathbf{H}^2(\Omega) \qquad \text{and} \qquad p \in H^1(\Omega).$$

We consider the Dirichlet problem

$$-\nabla \cdot (\mathbf{N} \nabla \theta) = -R_T(\mathbf{u} + \mathbf{a}) \cdot \nabla(\theta + \psi) + \nabla \cdot (\mathbf{N} \nabla \psi), \tag{58}$$
$$\theta = 0 \qquad \text{on } \partial\Omega.$$

Since $-R_T(\mathbf{u}+\mathbf{a})\cdot\nabla(\theta+\psi)+\nabla\cdot(\mathbf{N}\nabla\psi)\in L^2(\Omega)$, the standard regularity theory of elliptic partial differential equations tells us $\theta\in H^2(\Omega)$. Similarly, we can show that $\gamma\in H^2(\Omega)$.

The further regularity results can be obtained by reiterating the same procedure as in the proof of Theorem 2, provided that the additional conditions are imposed on boundary $\partial\Omega$ and on boundary data $\mathbf{a}$, ξ, and η. We state without proof the following theorem:

THEOREM 3. *Let Ω be an open bounded set of class C^∞ and $\mathbf{a}\in\mathbf{C}^\infty$, $\xi,\eta\in C^\infty$, then any solution $(\mathbf{v},p,T,S)$ of* (9)–(15) *belongs to* $\mathbf{C}^\infty(\overline{\Omega})\times C^\infty(\Omega)^3$.

Finally, we establish a uniqueness result.

THEOREM 4. *If* $\|\mathbf{a}\|_{H^1}$, $\sup_{\partial\Omega}|\psi|$, *and* $\sup_{\partial\Omega}|\beta|$ *are small, then the solution of* (9)–(15) *is unique* (*as always, p is unique up to a constant*).

Proof. Let $(\mathbf{v}_1,p_1,T_1,S_1)$ and $(\mathbf{v}_2,p_2,T_2,S_2)$ be two solutions of (9)–(15), then $(\mathbf{u}_1=\mathbf{v}_1-\mathbf{a},p_1,\theta_1=T_1-\xi,\ \gamma_1=S_1-\eta)$ and $(\mathbf{u}_2=\mathbf{v}_2-\mathbf{a},p_2,\theta_2=T_2-\xi,\ \gamma_2=S_2-\eta)$ are two solutions of (23)–(27). It follows that $(\mathbf{u}_1,\theta_1,\gamma_1)$ and $(\mathbf{u}_2,\theta_2,\gamma_2)$ are two solutions of problem (32). We, therefore, have

$$
\begin{aligned}
0 &= \left\langle G(\mathbf{u}_1,\theta_1,\gamma_1)-G(\mathbf{u}_2,\theta_2,\gamma_2),(\mathbf{u}_1-\mathbf{u}_2,\theta_1-\theta_2,\gamma_1-\gamma_2)\right\rangle\\
&= \sigma D_a R_T\big(|\mathbf{u}_1+\mathbf{a}|(\mathbf{u}_1+\mathbf{a}),\mathbf{u}_1-\mathbf{u}_2\big)\\
&\quad -\sigma D_a R_T\big(|\mathbf{u}_2+\mathbf{a}|(\mathbf{u}_2+\mathbf{a}),\mathbf{u}_1-\mathbf{u}_2\big)\\
&\quad +\mu_0 D_a\big(\nabla(\mathbf{u}_1-\mathbf{u}_2),\nabla(\mathbf{u}_1-\mathbf{u}_2)\big)+\big(\mathbf{M}(\mathbf{u}_2-\mathbf{u}_2),\mathbf{u}_1-\mathbf{u}_2\big)\\
&\quad +\big((\theta_1-\theta_2)\mathbf{g},\mathbf{u}_1-\mathbf{u}_2\big)+\frac{1}{\tau}\big((\gamma_1-\gamma_2)\mathbf{g},\mathbf{u}_1-\mathbf{u}_2\big)\\
&\quad +R_T\big((\mathbf{u}_1+\mathbf{a})\cdot\nabla(\theta_1+\psi),\theta_1-\theta_2\big)\\
&\quad -R_T\big((\mathbf{u}_2+\mathbf{a})\cdot\nabla(\theta_2+\psi),\theta_1-\theta_2\big)\\
&\quad +\big(\mathbf{N}\nabla(\theta_1-\theta_2),\nabla(\theta_1-\theta_2)\big)\\
&\quad +\tau R_S\big((\mathbf{u}_1+\mathbf{a})\cdot\nabla(\gamma_1+\beta),\gamma_1-\gamma_2\big)\\
&\quad -\tau R_S\big((\mathbf{u}_2+\mathbf{a})\cdot\nabla(\gamma_2+\beta),\gamma_1-\gamma_2\big)\\
&\quad +\big(\mathbf{Q}\nabla(\gamma_1-\gamma_2),\nabla(\gamma_1-\gamma_2)\big)
\end{aligned}
$$

$$\geq \mu_0 D_a \|\mathbf{u}_1 - \mathbf{u}_2\|_{\tilde{V}}^2 + \|\mathbf{u}_1 - \mathbf{u}_2\|_H^2 + \|\theta_1 - \theta_2\|_{H_0^1}^2 + \|\gamma_1 - \gamma_2\|_{H_0^1}^2$$
$$-\|\mathbf{u}_1 - \mathbf{u}_2\|_H \|\theta_1 - \theta_2\|_{L^2} - \frac{1}{\tau}\|\mathbf{u}_1 - \mathbf{u}_2\|_H \|\gamma_1 - \gamma_2\|_{L^2}$$
$$-R_T \|\mathbf{u}_1 - \mathbf{u}_2\|_H \|\theta_1 - \theta_2\|_{H_0^1} \|\theta_1 + \psi\|_{L^\infty}$$
$$-\tau R_S \|\mathbf{u}_1 - \mathbf{u}_2\|_H \|\gamma_1 - \gamma_2\|_{H_0^1} \|\gamma_1 + \beta\|_{L^\infty}$$
$$-\sigma D_a R_T \big| (|\mathbf{u}_1 + \mathbf{a}|(\mathbf{u}_1 - \mathbf{u}_2), \mathbf{u}_1 - \mathbf{u}_2) \big|$$
$$-\sigma D_a R_T \big| ((|\mathbf{u}_1 + \mathbf{a}| - |\mathbf{u}_2 + \mathbf{a}|)(\mathbf{u}_2 + \mathbf{a}), \mathbf{u}_1 - \mathbf{u}_2) \big|. \tag{59}$$

From Theorem 2 we know that $\theta_1 + \psi$ and $\gamma_1 + \beta$ are continuous. The weak maximum principle gives

$$\|\theta_1 + \psi\|_{L^\infty} \leq \sup_{\partial\Omega} |\psi| \qquad \text{and} \qquad \|\gamma_1 + \beta\|_{L^\infty} \leq \sup_{\partial\Omega} |\beta|. \tag{60}$$

Also by Sobolev imbedding theorem we have

$$\big| (|\mathbf{u}_1 + \mathbf{a}|(\mathbf{u}_1 - \mathbf{u}_2), \mathbf{u}_1 - \mathbf{u}_2) \big| \leq c(\|\mathbf{u}_1\|_H + \|\mathbf{a}\|_{L^2}) \|\mathbf{u}_1 - \mathbf{u}_2\|_{\tilde{V}}^2, \tag{61}$$

$$\big| ((|\mathbf{u}_1 + \mathbf{a}| - |\mathbf{u}_2 + \mathbf{a}|)(\mathbf{u}_2 + \mathbf{a}), \mathbf{u}_1 - \mathbf{u}_2) \big| \leq c(\|\mathbf{u}_2\|_H + \|\mathbf{a}\|_{L^2}) \|\mathbf{u}_1 - \mathbf{u}_2\|_{\tilde{V}}^2, \tag{62}$$

where c is Sobolev imbedding constant.

We now estimate $\|\mathbf{u}_1\|_{\tilde{V}}$ and $\|\mathbf{u}_2\|_{\tilde{V}}$. For any solution $(\mathbf{u}, \theta, \gamma)$ of problem (32) we have

$$\sigma D_a R_T (|\mathbf{u} + \mathbf{a}|(\mathbf{u} + \mathbf{a}), \mathbf{u}) + \mu_0 D_a (\nabla(\mathbf{u} + \mathbf{a}), \nabla \mathbf{u})$$
$$+ (\mathbf{M}(\mathbf{u} + \mathbf{a}), \mathbf{u}) + ((\theta + \psi)\mathbf{g}, \mathbf{u}) - \frac{1}{\tau}((\gamma + \beta)\mathbf{g}, \mathbf{u}) = 0.$$

It follows that

$$\mu_0 D_a \|\mathbf{u}\|_{\tilde{V}}^2 + \|\mathbf{u}\|_H^2$$
$$\leq c\sigma D_a R_T \|\mathbf{a}\|_{L^2} \|\mathbf{u}\|_{\tilde{V}}^2 + c\sigma D_a R_T \|\mathbf{a}\|_{L^4}^2 \|\mathbf{u}\|_{\tilde{V}} / (\pi n^{1/2})$$
$$+ \mu_0 D_a \|\nabla \mathbf{a}\|_{L^2} \|\mathbf{u}\|_{\tilde{V}} + m_l \|\mathbf{a}\|_{L^2} \|\mathbf{u}\|_H$$
$$+ \left(\sup_{\partial\Omega} |\psi| + \sup_{\partial\Omega} |\beta| / \tau \right) \|\mathbf{g}\|_{L^2} \|\mathbf{u}\|_H$$
$$\leq c\sigma D_a R_T \|\mathbf{a}\|_{L^2} \|\mathbf{u}\|_{\tilde{V}}^2$$
$$+ \Bigg\{ c\sigma D_a R_T \|\mathbf{a}\|_{L^4}^2 / (\pi n^{1/2}) + \left(\mu_0 D_a + \frac{m_l}{\pi n^{1/2}} \right) \|\mathbf{a}\|_{H^1}$$
$$+ \frac{1}{\pi n^{1/2}} \left(\sup_{\partial\Omega} |\psi| + \sup_{\partial\Omega} |\beta| / \tau \right) \|\mathbf{g}\|_{L^2} \Bigg\} \|\mathbf{u}\|_{\tilde{V}}.$$

This inequality implies that

$$\|\mathbf{u}\|_{\tilde{V}} \leq \frac{c\sigma D_a R_T \|\mathbf{a}\|_{H^1}^2 + (\pi n^{1/2}\mu_0 D_a + m_l)\|\mathbf{a}\|_{H^1} + (\sup_{\partial\Omega}|\psi| + \sup_{\partial\Omega}|\beta|/\tau)\|\mathbf{g}\|_{L^2}}{\pi n^{1/2} D_a(\mu_0 - c\sigma R_T\|\mathbf{a}\|_{L^2})}.$$

Employing the above inequality and (42), (60)–(62) in (59) we obtain

$$\begin{aligned} 0 \geq \Bigg\{ & D_a(\mu_0 - c\sigma R_T\|\mathbf{a}\|_{L^2})(\mu_0 - 2c\sigma R_T\|\mathbf{a}\|_{L^2}) \\ & -2c\sigma R_T\Bigg\{c\sigma D_a R_T\|\mathbf{a}\|_{H^1}^2 + \left(\pi n^{1/2}\mu_0 D_a + m_l\right)\|\mathbf{a}\|_{H^1} \\ & + \left(\sup_{\partial\Omega}|\psi| + \sup_{\partial\Omega}|\beta|/\tau\right)\|\mathbf{g}\|_{L^2}\Bigg\}\Bigg\} \frac{\|\mathbf{u}_1 - \mathbf{u}_2\|_{\tilde{v}}^2}{\pi n^{1/2}(\mu_0 - c\sigma R_T\|\mathbf{a}\|_{L^2})} \\ & + \left(1 - \frac{1}{n\pi^2} - R_T^2\left(\sup_{\partial\Omega}|\psi|\right)^2\right)\|\theta_1 - \theta_2\|_{H_0^1}^2 \\ & + \left(1 - \frac{1}{n\pi^2\tau^2} - \tau^2 R_S^2\left(\sup_{\partial\Omega}|\beta|^2\right)\right)\|\gamma_1 - \gamma_2\|_{H_0^1}^2. \end{aligned} \tag{63}$$

From (63) we know that if $\|\mathbf{a}\|_{H^1}$, $\sup_{\partial\Omega}|\psi|$, and $\sup_{\partial\Omega}|\beta|$ are small, then $\mathbf{u}_1 = \mathbf{u}_2$, $\theta_1 = \theta_2$, $\gamma_1 = \gamma_2$. These in turn imply that $p_1 - p_2 = \text{const}$.

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