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Solving quantum stochastic differential equations with unbounded coefficients

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Abstract

We demonstrate a method for obtaining strong solutions to the right Hudson–Parthasarathy quantum stochastic differential equation

$$dU_t = F^\alpha_\beta U_t \, d\Lambda^\beta_\alpha(t), \quad U_0 = 1$$

where U is a contraction operator process, and the matrix of coefficients $[F^\alpha_\beta]$ consists of unbounded operators. This is achieved whenever there is a positive self-adjoint reference operator C that behaves well with respect to the F^α_β, allowing us to prove that Dom $C^{1/2}$ is left invariant by the operators U_t, thereby giving rigorous meaning to the formal expression above.

We give conditions under which the solution U is an isometry or coisometry process, and apply these results to construct unital *-homomorphic dilations of (quantum) Markov semigroups arising in probability and physics.

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0. Introduction

A quantum Markovian cocycle is a family of *-homomorphisms $(j_t:\mathscr{A}\to\mathscr{C})_{t\geqslant 0}$ between two operator algebras, $\mathscr{A}\subset\mathscr{C}$, where $\mathscr{C}$ is equipped with a semigroup $(\sigma_t)_{t\geqslant 0}$ of *-homomorphisms, and such that

$$j_{s+t} = \hat{j}_s\circ\sigma_s\circ j_t \quad \text{for all } s,t\geqslant 0, \tag{0.1}$$

where $\hat{j}_s$ denotes an extension of j_s whose domain includes $\sigma_s(\bigcup_{t\geqslant 0} j_t(\mathscr{A}))$. Such perturbations of the semigroup evolution law occur naturally in stochastic settings, where $\mathscr{A}$ and $\mathscr{C}$ are commutative algebras of bounded measurable functions. By no longer insisting that $\mathscr{A}$ and $\mathscr{C}$ be commutative, we obtain a quantum stochastic process (in the sense of [AFL]) which generalises the classical notion of a flow to a form more suitable for modelling situations in quantum physics.

The most commonly studied examples of cocycles are those for which $\mathscr{A}$ is a *-subalgebra of $B(\mathfrak{h})$, the algebra of all bounded operators on some Hilbert space $\mathfrak{h}$, and $\mathscr{C}$ is of the form $\mathscr{A}''\otimes B(\mathscr{F})$, where $\mathscr{A}''$ is the von Neumann algebra generated by $\mathscr{A}$ and $\mathscr{F}$ is the symmetric Fock space over $L^2(\mathbb{R}_+;\mathsf{k})$, the square integrable functions on $\mathbb{R}_+$ taking values in some Hilbert space k. Fock space here plays the role of Wiener space in the classical theory, and is equipped with a semigroup $(\sigma_t)_{t\geqslant 0}$ induced by the natural time shift on $L^2(\mathbb{R}_+;\mathsf{k})$. That this is an appropriate choice for the image algebra $\mathscr{C}$ can be justified by a limiting procedure motivated by physical arguments (see, for example, [AAFL]). Suppose $\mathscr{A}=\mathscr{A}''$ and let $\mathbb{E}:\mathscr{C}\to\mathscr{A}$ be the vacuum conditional expectation. Then $(\mathscr{T}_t := \mathbb{E}\circ j_t)_{t\geqslant 0}$ is a semigroup of completely positive maps on $\mathscr{A}$ that describes the reduced dynamics of an open quantum system, and j is a dilation of this *quantum dynamical semigroup* (QDS).

One method of constructing such cocycles is to solve a quantum stochastic differential equation (QSDE) of Evans–Hudson type [EvH,LW1],

$$dj_t = j_t\circ\theta^\alpha_\beta\, d\Lambda^\beta_\alpha(t), \quad j_0(a) = a\otimes 1, \tag{0.2}$$

where $\theta=[\theta^\alpha_\beta]_{\alpha,\beta\geqslant 0}$ is a matrix of linear maps on $\mathscr{A}$, $\Lambda=[\Lambda^\beta_\alpha]_{\alpha,\beta\geqslant 0}$ are the fundamental noise processes of Hudson–Parthasarathy quantum stochastic calculus [HuP,Mey,Par], and summation over repeated indices is understood. Conversely, it can be shown that all sufficiently well-behaved cocycles arise in this manner [AcM,LW2]. However for applications to physics [Bar,Be1,Sin], and references therein], and for the realisation of classical stochastic processes in the quantum setting [F4,F5], it is usually required that the components θ^α_β of the matrix θ consist of unbounded maps. Proving the existence of a solution to the EH equation is then a highly non-trivial problem [FSi,Be2]. An alternative route exists for the construction of cocycles when $\mathscr{A}=B(\mathfrak{h})$, the full algebra, by considering the subclass of *inner cocycles* obtained by conjugation. A family $U=(U_t)_{t\geqslant 0}$ of bounded operators on

$\mathsf{H} := \mathfrak{h} \otimes \mathscr{F}$ is a *right cocycle* if

$$U_0 = 1; \quad U_{s+t} = \sigma_s(U_t)U_s, \quad s, t \geqslant 0,$$

and a *left cocycle* if the adjoint family is a right cocycle. If U is a right cocycle for which each U_t is a coisometry then defining j by

$$j_t(X) := U_t^*(X \otimes 1)U_t, \quad X \in B(\mathfrak{h}) \tag{0.3}$$

produces a cocycle of the form (0.1). As above, right and left operator-valued cocycles can be constructed by solving the *right* and *left* Hudson–Parthasarathy equations:

$$dU_t = F^\alpha_\beta U_t \, d\Lambda^\beta_\alpha(t); \qquad dV_t = V_t G^\alpha_\beta \, d\Lambda^\beta_\alpha(t), \tag{0.4}$$

where $F = [F^\alpha_\beta]$ and $G = [G^\alpha_\beta]$ are matrices of operators on $\mathfrak{h}$, and again all sufficiently regular right and left cocycles arise this way [F3,LW2]. Furthermore, if U is the solution to the right equation and each F^α_β is bounded then j defined by (0.3) satisfies the EH equation (0.2) for the matrix of maps θ defined by

$$\theta^\alpha_\beta(X) = XF^\alpha_\beta + (F^\beta_\alpha)^* X + \sum_{i \geqslant 1} (F^i_\alpha)^* X F^i_\beta. \tag{0.5}$$

Eqs. (0.4) are written in the form usually encountered in the literature on quantum stochastic calculus, but to give rigorous meaning to these equations the operators F^α_β and G^α_β should really be defined as operators on the whole space H rather than just the first component $\mathfrak{h}$ of the tensor product. When they are bounded operators they can be identified with $F^\alpha_\beta \otimes 1$ and $G^\alpha_\beta \otimes 1$, the unique continuous extensions of the algebraic tensor products with the identity operator on Fock space, and then no serious difficulty occurs. Similarly, when seeking solutions to the left equation with unbounded coefficients, since processes are only ever defined, in the HP calculus, on the algebraic tensor product of some dense subspace of $\mathfrak{h}$ and $\mathscr{E}$ (the linear span of the exponential vectors in Fock space) we can identify G^α_β with its algebraic ampliation. To solve the right equation for unbounded F^α_β, it is clearly necessary to obtain information about the range of the operators $(U_t)_{t \geqslant 0}$. Solutions have been found in [App,F1,Vin], and in all cases this was achieved by assuming that there is a dense subspace of $\bigcap_{\alpha,\beta} \operatorname{Dom} F^\alpha_\beta$ consisting of vectors satisfying certain analyticity conditions. Subsequently Fagnola [F3] and Mohari [Mo1] focused on the left equation and obtained existence results for that equation under far less stringent hypotheses on the coefficient matrix since the analytical difficulties are considerably less.

In this paper, we present a new method for solving the right equation that allows us to incorporate advances made in the study of the left equation. In particular the domains of the coefficients are no longer required to contain a common dense invariant subspace. Two advantages of dealing the right equation are that the proof that the solution is an isometry process is very much easier (cf. Corollary 2.4), and that the inner cocycle j defined through conjugation by (0.3) is seen to possess an infinitesimal generator in that it satisfies the EH equation on some *-algebra $\mathscr{A}$ for the maps θ^α_β defined by (0.5). The germ of the idea is as follows: if the coefficients G^α_β for the left equation are bounded, and the solution V is a contraction process, then the adjoint process V^* is the solution to the adjoint right equation:

$$dV_t^* = (G_\alpha^\beta)^* V_t^* \, d\Lambda_\alpha^\beta(t), \quad V_0^* = 1.$$

The main result of the paper, Theorem 2.3, extends this principle to the case of unbounded generators by hypothesising the existence of a positive self-adjoint "reference operator" satisfying a form inequality that can be written heuristically as

$$\theta(C) \leqslant b(C \otimes 1).$$

This enables us to obtain a priori estimates on the continuity of each V_t^* with respect to the graph norm of $C^{1/2}$, and hence obtain information about the range of V_t^*. The method was inspired by the techniques developed in [ChF,CGQ] for proving the conservativity of QDSs, a problem that has intimate connections with proving that solutions to the left equation are isometric (see Proposition 2.5).

The plan of the paper is as follows. Section 1 contains some general results about closable operators and their ampliations, and one-parameter contraction semigroups. These allow us to define precisely what we mean by a solution of the right equation at the start of Section 2, before going on to establish our main result. This is then exploited in Section 3 to give simplified conditions under which it is possible to construct isometric solutions to the right equation when there is only one dimension of quantum noise, that is, when $\mathsf{k} = \mathbb{C}$. Finally, in Section 4, we apply these results to realise classical birth and death processes as quantum flows, prove the existence and unitarity of a solution to a QSDE that arises in models of superradiation (via an alternative approach to that used in [Wal]), and construct unitary right cocycles that enable us to dilate QDSs of diffusion type (see, for example, [AlF,F4]), as well as realising classical diffusion processes as quantum flows in Fock space.

Tensor product and summation conventions: We shall use the symbol $\odot$ to denote the algebraic tensor product of vector spaces and linear maps, reserving $\otimes$ for the Hilbert space tensor product of Hilbert spaces and their vectors. If S and T are closable operators on Hilbert spaces h and k, respectively, then we denote the closure of $S \odot T$ by $S \otimes T$ (see Lemma 1.1 below). Thus, if $S \in B(\mathsf{h})$, $T \in B(\mathsf{k})$, then $S \otimes T$ is the unique continuous extension to the Hilbert space $\mathsf{h} \otimes \mathsf{k}$ of the bounded operator $S \odot T$ whose domain is the inner product space $\mathsf{h} \odot \mathsf{k}$. At times we will follow the trends prevalent in the literature and identify *bounded* operators with their ampliations, but only when this does not lead to confusion.

We shall adopt the Einstein summation convention and sum over repeated indices; greek indices will run from 0 to d, and roman indices from 1 to d, where d is the number of dimensions of quantum noise (see the start of Section 2).

1. Operator theory preliminaries

In this section we collect together a number of results on closable operators and one-parameter semigroups that we shall need later in the paper.

Lemma 1.1. *Let S and T be closable operators on Hilbert spaces* h *and* k, *respectively. Then the operator $S \odot T$ is closable.*

Proof. This follows from the obvious operator inclusion $(S \odot T)^* \supset S^* \odot T^*$. $\square$

Remark. Since we denote the closure of $S \odot T$ by $S \otimes T$, we have that $\bar{S} \otimes \bar{T} = S \otimes T$.

The main use we make of the above result is to ampliate closable operators, that is taking T to be the identity. In particular, we shall need to consider pairs of closed or closable operators, the domain of one lying inside the domain of the other, and these behave well under such ampliations.

Lemma 1.2. *Let S and T be closable operators on a Hilbert space* h *such that* $\operatorname{Dom} \bar{S} \subset \operatorname{Dom} \bar{T}$. *Then* $\operatorname{Dom} S \otimes 1_{\mathsf{k}} \subset \operatorname{Dom} T \otimes 1_{\mathsf{k}}$ *for every Hilbert space* k, *and moreover there exist constants $a, b \geqslant 0$ such that*

$$||(T \otimes 1_{\mathsf{k}})\xi||^2 \leqslant a||(S \otimes 1_{\mathsf{k}})\xi||^2 + b||\xi||^2 \tag{1.1}$$

holds for all choices of k *and* $\xi \in \operatorname{Dom} S \otimes 1_{\mathsf{k}}$.

Note. The inequality holds in particular for the case $\mathsf{k} = \mathbb{C}$, when $S \otimes 1_{\mathsf{k}} = \bar{S}$ and $T \otimes 1_{\mathsf{k}} = \bar{T}$.

Proof. The inclusion map $\operatorname{Dom} \bar{S} \hookrightarrow \operatorname{Dom} \bar{T}$ is closed and everywhere defined, when these spaces are equipped with their respective graph norms, and hence bounded, giving existence of the constants a and b when $\mathsf{k} = \mathbb{C}$.

For general k, note that $\operatorname{Dom} \bar{S} \odot \mathsf{k}$ is a core for $S \otimes 1_{\mathsf{k}}$, and that any element ξ of this space can be written as $\sum_i u_i \otimes v_i$ where $u_i \in \operatorname{Dom} \bar{S}$ and $\{v_i\}$ is an orthonormal set. It is then straightforward to check that (1.1) remains valid for such ξ and the same a and b, from which the result then follows. $\square$

To define quantum stochastic integrals we work with square-integrable Hilbert-space-valued functions, and when dealing with the right HP equation we must apply closed operators to such functions and determine if the resulting map is again square-integrable. The following settles the measurability part of the question.

Lemma 1.3. *Let T be a closed operator on a Hilbert space h, and let $f: X \to \mathsf{h}$ be a strongly measurable function on some measure space X satisfying $f(X) \subset \operatorname{Dom} T$. Then the map $g: x \mapsto Tf(x)$ is strongly measurable.*

Proof. Let $T = U|T|$ be the polar decomposition of T and define maps $g_n : X \to \mathsf{h}$ by $g_n(x) = U|T|\mathbf{1}_{[0,n]}(|T|)f(x)$, where $\mathbf{1}_{[0,n]}$ is the indicator function of $[0, n]$. Then each g_n is strongly measurable and (g_n) converges to g pointwise. □

When applying the lemmas above the operator S will usually be the generator of a strongly continuous one-parameter semigroup of operators, $(P_t)_{t \geqslant 0}$ say. Then, for any other Hilbert space k, the family of ampliations $(P_t \otimes 1_{\mathsf{k}})_{t \geqslant 1}$ is a strongly continuous one-parameter semigroup. If we denote its generator by $\tilde{S}$ then clearly $S \odot 1 \subset \tilde{S}$. But $\operatorname{Dom} S \odot \mathsf{k}$ is a dense subspace of $\mathsf{h} \otimes \mathsf{k}$ that is left invariant by the semigroup, and thus is a core for $\tilde{S}$ ([Dav], Theorem 1.9). Hence, $\tilde{S} = S \otimes 1$.

The particular example that we need later is given by taking a positive self-adjoint operator C on h, and letting Q be the contraction semigroup generated by $-C$. So then $(Q_t \otimes 1)_{t \geqslant 0}$ is generated by $-C \otimes 1$. We will make repeated use of the following *variant* of the Yosida approximation:

$$C_\epsilon := R_\epsilon C R_\epsilon \quad \text{where } R_\epsilon = (1 + \epsilon C)^{-1} \text{ for each } \epsilon > 0.$$

The spectral theorem implies that $C_\epsilon \in B(\mathsf{h})$, and that $u \in \mathsf{h}$ is in $\operatorname{Dom} C^{1/2}$ if and only if $\lim_{\epsilon \downarrow 0} \|(C_\epsilon)^{1/2} u\| < \infty$. Moreover, for any other Hilbert space k, $(1_{\mathsf{h}} + \epsilon C) \odot 1_{\mathsf{k}}$ is a bijection onto $\mathsf{h} \odot \mathsf{k}$ and a restriction of $1_{\mathsf{h} \otimes \mathsf{k}} + \epsilon C \otimes 1_{\mathsf{k}}$. Thus $(1 + \epsilon C \otimes 1)^{-1}|_{\mathsf{h} \odot \mathsf{k}} = R_\epsilon \odot 1_{\mathsf{k}}$, hence $(C \otimes 1_{\mathsf{k}})_{\epsilon | \mathsf{h} \odot \mathsf{k}} = C_\epsilon \odot 1_{\mathsf{k}}$, and so $(C \otimes 1_{\mathsf{k}})_\epsilon = C_\epsilon \otimes 1_{\mathsf{k}}$ by continuity. Thus, we can identify C_ϵ with $C_\epsilon \otimes 1_{\mathsf{k}}$ in what follows without causing serious harm, since we are actually working with the Yosida approximation of the generator of the ampliated semigroup.

The reason for using this variant of the Yosida approximation is that the unboundedness of the coefficients F^α_β is controlled by multiplying by $(C_\epsilon)^{1/2}$, and so we need a greater power of C in the denominator than the numerator.

Lemma 1.4. *Let C and T be operators on the Hilbert space h, with C positive, invertible and self-adjoint, and T closed. The following are equivalent*:

(i) $T(C_\epsilon)^{1/2}$ *is densely defined and bounded for all* $\epsilon > 0$;
(ii) $T(C_\epsilon)^{1/2}$ *is everywhere defined and bounded for all* $\epsilon > 0$;
(iii) $\operatorname{Dom} C^{1/2} \subset \operatorname{Dom} T$.

Proof. (i $\Rightarrow$ ii) $T(C_\epsilon)^{1/2}$ is closed since T is closed and $(C_\epsilon)^{1/2}$ is bounded, and so the result follows by the Closed Graph Theorem.

(ii $\Rightarrow$ iii) Writing $T(C_\epsilon)^{1/2}$ as the product $C^{1/2}R_\epsilon$, it is clear that it maps h bijectively onto Dom $C^{1/2}$, which is thus contained in Dom T.

(iii $\Rightarrow$ i) This follows from Lemma 1.2, since $C^{1/2}(C_\epsilon)^{1/2} \in B(\mathsf{h})$. $\square$

Remark. The implications (iii $\Rightarrow$ i $\Rightarrow$ ii) remain valid when 0 is in the spectrum of C. However, 0 must be in the resolvent of C for (ii $\Rightarrow$ iii)—consider $C = 0$.

2. Fock space and the right and left HP equations

Quantum stochastic integrals: Fix a Hilbert space $\mathfrak{h}$, called the *initial space*, and an integer $d \geqslant 1$, the number of dimensions of quantum noise. Let $\mathsf{H} = \mathfrak{h} \otimes \mathscr{F}$, the Hilbert space tensor product of the initial space and $\mathscr{F} = \Gamma(L^2(\mathbb{R}_+; \mathbb{C}^d))$, the symmetric Fock space over $L^2(\mathbb{R}_+; \mathbb{C}^d)$. Put

$$\mathbb{M} = L^2(\mathbb{R}_+; \mathbb{C}^d) \cap L^\infty_{\text{loc}}(\mathbb{R}_+; \mathbb{C}^d) \quad \text{and} \quad \mathscr{E} = \text{Lin}\{\varepsilon(f) : f \in \mathbb{M}\},$$

where $\varepsilon(f) = ((n!)^{-1/2} f^{\otimes n})$ is the exponential vector associated to the test function f. The elementary tensor $u \otimes \varepsilon(f)$ will usually be abbreviated to $u\varepsilon(f)$. The notion of adaptedness plays a crucial role in the theory of quantum stochastic calculus as developed by Hudson and Parthasarathy [HuP]. This is expressed through the continuous tensor product factorisation property of Fock space: for each $t > 0$ let

$$\mathscr{F}_t = \Gamma(L^2([0, t[; \mathbb{C}^d)), \quad \mathscr{F}^t = \Gamma(L^2([t, \infty[; \mathbb{C}^d)).$$

Then $\mathscr{F} = \mathscr{F}_t \otimes \mathscr{F}^t$ via the continuous linear extension of the isometric map $\varepsilon(f) \mapsto \varepsilon(f|_{[0,t[}) \otimes \varepsilon(f|_{[t,\infty[})$; $\mathscr{F}_t$ and $\mathscr{F}^t$ embed naturally into $\mathscr{F}$ as subspaces by tensoring with the vacuum vector $\varepsilon(0)$. Let $\mathfrak{D}$ be a dense subspace of $\mathfrak{h}$. An *operator process on* $\mathfrak{D}$ is a family $X = (X_t)_{t \geqslant 0}$ of operators on H satisfying:

(i) $\mathfrak{D} \odot \mathscr{E} \subset \bigcap_{t \geqslant 0} \text{Dom}\, X_t$,
(ii) $t \mapsto \langle u\varepsilon(f), X_t v\varepsilon(g) \rangle$ is measurable,
(iii) $X_t v\varepsilon(g|_{[0,t[}) \in \mathfrak{h} \otimes \mathscr{F}_t$, and $X_t v\varepsilon(g) = [X_t v\varepsilon(g|_{[0,t[})] \otimes \varepsilon(g|_{[t,\infty[})$,

for all $u \in \mathfrak{h}$, $v \in \mathfrak{D}$, $f, g \in \mathbb{M}$ and $t > 0$. Families of operators satisfying (iii) are called *adapted*. Any process satisfying the further condition

(iv) $t \mapsto X_t v\varepsilon(g)$ is strongly measurable and $\int_0^t \|X_s v\varepsilon(g)\|^2 \, ds < \infty \ \forall t > 0$

is called *stochastically integrable on* $\mathfrak{D}$. It is for these processes that Hudson and Parthasarathy defined the stochastic integral $\int_0^t X_s \, d\Lambda^\alpha_\beta(s)$ for each of the fundamental noise processes Λ^α_β which are defined with respect to the standard basis of $\mathbb{C}^d$. The resulting family $(\int_0^t X_s \, d\Lambda^\alpha_\beta(s))$ is a process on $\mathfrak{D}$, and moreover the map $t \mapsto \int_0^t X_s \, d\Lambda^\alpha_\beta(s)$ is strongly continuous on $\mathfrak{D} \odot \mathscr{E}$. The action of such integrals is given in (2.1), and their interaction with each other, the quantum Itô formula, is given in (2.2) below. However, rather than give these for a single integral we shall work with matrices of processes: if $M = [M^\alpha_\beta]^d_{\alpha,\beta=0}$ is a matrix of stochastically integrable processes on $\mathfrak{D}$ then we can set $I^M_t = \int_0^t M^\alpha_\beta(s)\, d\Lambda^\beta_\alpha(s)$, the sum of $(d+1)^2$ integrals, to produce another (continuous) process on $\mathfrak{D}$. Moreover, for all $u \in \mathfrak{h}$, $v \in \mathfrak{D}$, f, $g \in \mathbb{M}$ and $t > 0$

$$\langle u\varepsilon(f), I^M_t v\varepsilon(g)\rangle = \int_0^t f_\alpha(s) g^\beta(s) \langle u\varepsilon(f), M^\alpha_\beta(s) v\varepsilon(g)\rangle\, ds, \tag{2.1}$$

where $f^1, \dots, f^d$ are the components of the $\mathbb{C}^d$-valued function f, $f^0 \equiv 1$, $f_\alpha(s) = \overline{f^\alpha(s)}$, and our summation convention is in force. If $N = [N^\alpha_\beta]$ is another matrix of stochastically integrable processes on some other dense subspace $\mathfrak{D}'$ and we put $I^N_t = \int_0^t N^\alpha_\beta(s)\, d\Lambda^\beta_\alpha(s)$ then

$$\begin{aligned}\langle I^M_t u\varepsilon(f), I^N_t v\varepsilon(g)\rangle = \int_0^t f_\alpha(s) g^\beta(s) \{&\langle I^N_s u\varepsilon(f), N^\alpha_\beta(s) v\varepsilon(g)\rangle \\ &+ \langle M^\beta_\alpha(s) u\varepsilon(f), I^M_s v\varepsilon(g)\rangle + \langle M^i_\alpha(s) u\varepsilon(f), N^i_\beta(s) v\varepsilon(g)\rangle\}\, ds\end{aligned} \tag{2.2}$$

for all $u \in \mathfrak{D}$, $v \in \mathfrak{D}'$, f, $g \in \mathbb{M}$ and $t > 0$.

Finally, a process $X = (X_t)_{t \geqslant 0}$ on $\mathfrak{D}$ has a *strong stochastic integral representation* if there is a matrix $[M^\alpha_\beta]$ of stochastically integrable processes on $\mathfrak{D}$ such that

$$X_t = X_0 + \int_0^t M^\alpha_\beta(s)\, d\Lambda^\beta_\alpha(s).$$

It follows readily from (2.1) and (2.2) that

$$\begin{aligned}||X_t u\varepsilon(f)||^2 = ||X_0 u\varepsilon(f)||^2 + \int_0^t \Bigg\{ &2\,\mathrm{Re}\langle f^\alpha(s) X_s u\varepsilon(f), f^\beta(s) M^\alpha_\beta(s) u\varepsilon(f)\rangle \\ &+ \sum_{i=1}^d ||f^\alpha(s) M^i_\alpha(s) u\varepsilon(f)||^2 \Bigg\}\, ds\end{aligned} \tag{2.3}$$

for all $u \in \mathfrak{D}$, $f \in \mathbb{M}$ and $t > 0$.

Differential equations: In this paper we are concerned with the *right* and *left* HP equations:

$$dU_t = F^\alpha_\beta U_t\, d\Lambda^\beta_\alpha(t), \quad U_0 = 1, \tag{R}$$

$$dV_t = V_t G^\alpha_\beta\, d\Lambda^\beta_\alpha(t), \quad V_0 = 1, \tag{L}$$

where $F = [F^\alpha_\beta]^d_{\alpha,\beta=0}$ and $G = [G^\alpha_\beta]^d_{\alpha,\beta=0}$ are matrices of operators on $\mathfrak{h}$. Given any such matrix F of operators for which each F^α_β is densely defined (respectively closable) let F^* (resp. $\bar{F}$) denote the matrix $[(F^\beta_\alpha)^*]$ of adjoints (resp. $[\overline{F^\alpha_\beta}]$ of closures). Associated to any such matrix, we define the following subspace of $\mathfrak{h}$

$$\mathrm{Dom}[F] := \bigcap_{\alpha,\beta} \mathrm{Dom}\, F^\alpha_\beta. \tag{2.4}$$

Note that F gives rise naturally to an operator on $\oplus^{(d+1)}\mathfrak{h}$ by the prescription $(u^\gamma) \mapsto (F^\alpha_\beta u^\beta)$, which has domain $\{(u^\gamma) \in \oplus^{(d+1)}\mathfrak{h} : u^\gamma \in \bigcap_\alpha \mathrm{Dom}\, F^\alpha_\gamma \text{ for each } \gamma\}$. The subspace $\mathrm{Dom}[F]$ is the largest subspace $\mathfrak{D} \subset \mathfrak{h}$ such that $\oplus^{(d+1)}\mathfrak{D}$ is contained in this maximal domain.

We will only consider solutions that are *contraction processes*, that is processes U or V for which each U_t or V_t is a contraction. Let $\mathfrak{D} \subset \mathfrak{h}$ be a dense subspace. A contraction process V is a *weak solution of* (L) *on* $\mathfrak{D}$ *for the operator matrix* G if the following hold:

(Li) $\mathfrak{D} \subset \mathrm{Dom}[G]$,

(Lii) for all $u \in \mathfrak{h}$, $v \in \mathfrak{D}$, $f, g \in \mathbb{M}$ and $t > 0$,

$$\langle u\varepsilon(f), (V_t - 1)v\varepsilon(g)\rangle = \int_0^t f_\alpha(s)g^\beta(s)\langle u\varepsilon(f), V_s G^\alpha_\beta v\varepsilon(g)\rangle\, ds. \tag{2.5}$$

Note that any weak solution is necessarily weakly continuous. The process V is a *strong solution of* (L) *on* $\mathfrak{D}$ *for* G if, in addition,

(Liii) $t \mapsto V_t\xi$ is strongly measurable for all $\xi \in \mathsf{H}$.

The effect of this extra condition is that the processes $(V_t(G^\alpha_\beta \odot 1))_{t \geqslant 0}$ on $\mathfrak{D}$ are stochastically integrable, since V is assumed to be a contraction process, and so now by (2.1) and (2.5) it follows that

$$V_t = 1 + \int_0^t V_s(G^\alpha_\beta \odot 1)\, d\Lambda^\beta_\alpha(s).$$

For the right equation (R), the situation is in general more complex since there is no reason to expect that for any solution U the range of each U_t should lie in an algebraic tensor product of the form $\mathfrak{D}' \odot \mathscr{F}$. For this reason, we only define solutions of (R) when each component F^α_β of the matrix F is closable. Let $F \otimes 1$ denote the matrix $[F^\alpha_\beta \otimes 1]$ of closed operators on H (so that $\mathrm{Dom}[F \otimes 1] \subset \mathsf{H}$), and let $\mathfrak{D}$ be as above. A contraction process U is a *weak solution of* (R) *on* $\mathfrak{D}$ *for the operator matrix* F if the following hold:

(Ri) $\bigcup_{t \geqslant 0} U_t(\mathfrak{D} \odot \mathscr{E}) \subset \mathrm{Dom}\,[F \otimes 1]$,

(Rii) for all $u \in \mathfrak{h}, v \in \mathfrak{D}, f, g \in \mathbb{M}$ and $t > 0$

$$\langle u\varepsilon(f), (U_t - 1)v\varepsilon(g)\rangle = \int_0^t f_\alpha(s)g^\beta(s)\langle u\varepsilon(f), (F^\alpha_\beta \otimes 1)U_s v\varepsilon(g)\rangle\, ds.$$

Note that since F^α_β is assumed to be closable the measurability of the integrand follows by taking adjoints and using standard approximation arguments. A *strong solution of* (R) *on* $\mathfrak{D}$ is any weak solution that satisfies the further condition.

(Riii) each process $(F^\alpha_\beta \otimes 1)U$ on $\mathfrak{D}$ is stochastically integrable.

For such U, we have

$$U_t = 1 + \int_0^t (F^\alpha_\beta \otimes 1) U_s \, d\Lambda^\beta_\alpha(s),$$

and so in particular U is strongly continuous.

If $\mathfrak{h}$ is separable then any H-valued weakly measurable function is also strongly measurable by Pettis' Theorem, and thus any weak solution to (L) is necessarily a strong solution. However, the same need not be true for solutions of (R). A notion of mild solution for (R) has been introduced in [FW]. There it is shown that any strong solution is also a mild solution, and that there exist coefficient matrices F for which mild solutions exist, but for which there are no strong solutions.

Let $G = [G^\alpha_\beta]$ be a matrix of operators on $\mathfrak{h}$, T a positive, self-adjoint operator on $\mathfrak{h}$, and $\mathfrak{D}$ a subspace of $\mathfrak{h}$ such that

$$\mathfrak{D} \subset \mathrm{Dom}\, T \cap \mathrm{Dom}[G] \quad \text{and} \quad G^i_\beta(\mathfrak{D}) \subset \mathrm{Dom}\, T^{1/2} \quad \forall i \geqslant 1, \;\; \beta \geqslant 0.$$

Then we can define a real quadratic form $\theta_G(T)$ and a matrix of sesquilinear forms $[\theta_G(T)^\alpha_\beta]$ by

$$\theta_G(T)(\mathbf{u}) = 2\,\mathrm{Re}\langle Tu^\alpha, G^\alpha_\beta u^\beta \rangle + \sum_{i=1}^d \| T^{1/2} G^i_\beta u^\beta \|^2,$$

and

$$\theta_G(T)^\alpha_\beta(u,v) = \langle Tu, G^\alpha_\beta v \rangle + \langle G^\beta_\alpha u, Tv \rangle + \langle T^{1/2} G^i_\alpha u, T^{1/2} G^i_\beta v \rangle$$

for $\mathbf{u} = (u^\alpha) \in \bigoplus^{(d+1)} \mathfrak{D}$ and $u, v \in \mathfrak{D}$. It follows that

$$\theta_G(T)(\mathbf{u}) = \theta_G(T)^\alpha_\beta(u^\alpha, u^\beta).$$

We say that $\theta_G(T)$ is defined as a *form on* $\mathfrak{D}$ whenever we need to make the domain of definition precise; if T is bounded then $\theta_G(T)$ is defined as a form on the subspace $\mathrm{Dom}[G]$ of $\mathfrak{h}$. If $\theta_G(T)$ is in fact bounded then we shall also use $\theta_G(T)$ to denote the corresponding bounded self-adjoint operator.

Proposition 2.1 ([F3], [MoP]). *Let $G = [G^\alpha_\beta]$ be a matrix of operators on $\mathfrak{h}$, and let $\mathfrak{D}$ be a dense subspace of $\mathfrak{h}$. Suppose that there exists a contraction process V that is a strong solution to* (L) *on $\mathfrak{D}$ for this G. Then $\theta_G(1) \leqslant 0$ as a form on $\mathfrak{D}$. If V is an isometry process then $\theta_G(1) = 0$ on $\mathfrak{D}$.*

Proof. Let $\xi = \sum_p u_p \varepsilon(f_p \mathbf{1}_{[0,T]})$ for some $T > 0$ and some finite family $\{(u_p, f_p)\}$ in $\mathfrak{D} \times \mathbb{M}$ in which each f_p is continuous. Then contractivity of V implies that

$$\begin{aligned} 0 &\geqslant \|V_t \xi\|^2 - \|\xi\|^2 \\ &= \int_0^t \left\{ 2\,\mathrm{Re}\langle V_s y^\alpha(s), V_s G^\alpha_\beta y^\beta(s)\rangle + \sum_{i=1}^d \|V_s G^i_\alpha y^\alpha(s)\|^2 \right\} ds \end{aligned} \tag{2.6}$$

by (2.3), where $y^\alpha(s) = \sum_p f^\alpha_p(s) u_p \varepsilon(f_p \mathbf{1}_{[0,T]})$. Differentiating at 0 and letting $T \to 0$ gives

$$0 \geqslant \theta_G(1)(\mathbf{y}).$$

where $\mathbf{y} = (\sum_p f^\alpha_p(0) u_p)$. Varying the f_p and u_p then gives the result, and note that if V is an isometry process then the inequality in (2.6) becomes an equality. □

Remarks. (a) If G is a matrix of operators on $\mathfrak{h}$ such that the inequality $\theta_G(1) \leqslant 0$ holds on some dense subspace $\mathfrak{D}$ then $[\delta^i_j 1 + G^i_j]^d_{i,j=1}$ defines a contraction from $\oplus^{(d)}\mathfrak{D}$ to $\oplus^{(d)}\mathfrak{h}$, and so in particular each G^i_j has a unique continuous extension to an element of $B(\mathfrak{h})$. If $\theta_G(1) = 0$ then $[\delta^i_j 1 + G^i_j]^d_{i,j=1}$ is an isometry.

(b) If all the components in the matrix G (respectively F) are bounded then there is always a unique strong solution V of (L) (resp. a solution U of (R)), although it may be an unbounded process on $\mathfrak{h}$. In this situation $\theta_G(1) \leqslant 0$ is not only a necessary condition for contractivity of V but also sufficient one. Similarly, U will be a contraction process if and only if $\theta_F(1) \leqslant 0$. The original proofs of this characterisation are contained in [F3,Mo2]; an alternative line of proof is given in [LiP,LW1] that makes use of the characterisation of the generators of completely positive contraction flows. In this context, it makes sense to regard θ_F as the linear map $B(\mathfrak{h}) \to M_{d+1}(B(\mathfrak{h}))$ given by

$$\theta_F(X) = (X \otimes 1)F + F^*(X \otimes 1) + F^* \Delta(X) F,$$

where $\Delta(X) = \mathrm{diag}\{0, X, \ldots, X\}$, rather than just restricting it to the cone of positive self-adjoint operators.

Taking adjoints: Suppose that V is a contraction process that is a weak solution to (L) on $\mathfrak{D}$ for some operator matrix G, and also that each $G^\alpha_\beta \in B(\mathfrak{h})$. Then it follows from (2.1) that V^* is a weak solution to the QSDE $dV^* = (G^\beta_\alpha)^* V^* \, d\Lambda^\beta_\alpha$ on $\mathfrak{h}$. Our main result in this section shows how to extend this procedure to a class of generators G for which the G^α_β are no longer bounded. In particular, we must obtain information about the range of each V^*_t. Our arguments make use of the quantum Itô formula (2.2), which is valid for processes that have strong stochastic integral representations, and thus our standing hypothesis is the existence of a *strong* solution to (L), from which we will prove the existence of a strong solution to (R). In Section 3, we give conditions that guarantee the existence of this solution to (L).

As part of the proof, we will require that the adjoint process V^* be strongly measurable, and Proposition 2.2 below gives some sufficient conditions for this to be the case. In fact we shall show that it is strongly right continuous by first showing it is a Markovian cocycle and then adapting standard arguments of semigroup theory. For each $t \in \mathbb{R}$ let s_t be the unitary right shift operator on $L^2(\mathbb{R};\mathbb{C}^d)$, defined by $(s_t f)(r) = f(r-t)$ for $f \in L^2(\mathbb{R};\mathbb{C}^d)$. Let S_t be the second quantisation and ampliation of s_t, that is $S_t u\varepsilon(f) = u\varepsilon(s_t f)$. Then the map

$$B(\mathfrak{h} \otimes \Gamma(L^2(\mathbb{R};\mathbb{C}^d))) \ni Y \mapsto S_t Y S_t^* \in B(\mathfrak{h} \otimes \Gamma(L^2(\mathbb{R};\mathbb{C}^d)))$$

is a normal automorphism, and the collection of these for all $t \in \mathbb{R}$ is an ultraweakly continuous one-parameter group of such maps. Now let $X \in B(\mathsf{H})$, then ampliating with 1_-, the identity of $\Gamma(L^2(]-\infty,0[;\mathbb{C}^d))$, we get $X \otimes 1_- \in B(\mathfrak{h} \otimes \Gamma(L^2(\mathbb{R};\mathbb{C}^d)))$. If $t \geqslant 0$ it follows that there is some $\sigma_t(X) \in B(\mathsf{H})$ such that

$$\sigma_t(X) \otimes 1_- = S_t(X \otimes 1_-)S_t^*.$$

The family $(\sigma_t)_{t\geqslant 0}$ so defined is an ultraweakly continuous one-parameter semigroup of unital, normal *-homomorphisms of $B(\mathsf{H})$. A family $W = (W_t)_{t\geqslant 0} \subset B(\mathsf{H})$ is a *left cocycle* if it satisfies the following:

(i) The family W is adapted.
(ii) $W_0 = 1$.
(iii) $W_{s+t} = W_s \sigma_s(W_t)$ for all $s, t \geqslant 0$.

Similarly, W is a *right cocycle* if $W^* = (W_t^*)_{t\geqslant 0}$ is a left cocycle.

Proposition 2.2. *Let G be a matrix of operators on $\mathfrak{h}$, and suppose that there exists a contraction process V that is a strong solution to* (L) *on some dense subspace $\mathfrak{D} \subset \mathfrak{h}$. If V is the unique strong solution for this G and $\mathfrak{D}$ then V is strongly continuous and a left cocycle. Furthermore, V^* is a right cocycle that is strongly right continuous.*

Proof. That V is strongly continuous is a consequence of its strong stochastic integral representation as noted earlier. So now fix $t > 0$ and consider the process V^t defined by

$$V_s^t = \begin{cases} V_s, & s \leqslant t, \\ V_t \sigma_t(V_{s-t}), & s > t. \end{cases}$$

It follows that V^t is a strong solution to (L) on $\mathfrak{D} \odot \mathscr{E}$ for G, and so by uniqueness V is a left cocycle. Thus, V^* is a right cocycle by definition.

Now for any $s, t \geqslant 0$ and $\xi \in \mathsf{H}$, we have

$$||(V^*_{s+t} - V^*_t)\xi||^2 = ||(\sigma_t(V^*_s) - 1)V^*_t\xi||^2$$
$$\leqslant 2||V^*_t\xi||^2 - 2\,\mathrm{Re}\langle V^*_t\xi, \sigma_t(V_s)V^*_t\xi\rangle$$

since $\sigma_t(V^*_s)$ is a contraction. The right-hand side converges to zero as $s \to 0$ by strong continuity of V and normality of σ_t, and the result follows. □

Remark. Mohari proved the following uniqueness result in [Mo1]: let $G = [G^\alpha_\beta]$ be an operator matrix with G^0_0 the generator of a strongly continuous contraction semigroup. If $\mathfrak{D} \subset \mathrm{Dom}[G]$ is a core for G^0_0 then there is at most one *weak* solution V to (L) on $\mathfrak{D}$ for this G. In fact in [Mo1] it is assumed from the outset that $\mathfrak{h}$ is separable and so there is no distinction between weak and strong solutions. However, using this uniqueness result, the arguments of the proof above can be adapted to show that if a weak solution to (L) does exist then it is a cocycle, hence it is strongly continuous, and so it must actually be a strong solution.

Given any positive self-adjoint operator T on $\mathfrak{h}$ let $\imath(T)$ denote the form $\mathbf{u} \mapsto \sum_{\alpha=0}^d ||T^{1/2}u^\alpha||^2$, defined for each $\mathbf{u} \in \oplus^{(d+1)}\mathrm{Dom}\, T^{1/2}$. Also, recall the notation $\mathrm{Dom}[F]$ introduced in (2.4).

Theorem 2.3. *Suppose that U is a contraction process, F is an operator matrix, C is a positive, self-adjoint operator on $\mathfrak{h}$, and $\delta > 0$ and $b_1, b_2 \geqslant 0$ are constants such that the following hold*:

(i) *There is a dense subspace $\mathfrak{D} \subset \mathfrak{h}$ such that the adjoint process U^* is a strong solution of $dU^* = U^*(F^\beta_\alpha)^*\, d\Lambda^\beta_\alpha$ on $\mathfrak{D}$, and is the unique strong solution for this $F^* = [(F^\beta_\alpha)^*]$ and $\mathfrak{D}$.*

(ii) *For each $0 < \epsilon < \delta$ there is a dense subspace $\mathfrak{D}_\epsilon \subset \mathfrak{D}$ such that $(C_\epsilon)^{1/2}(\mathfrak{D}_\epsilon) \subset \mathfrak{D}$ and each $(F^\alpha_\beta)^*(C_\epsilon)^{1/2}|_{\mathfrak{D}_\epsilon}$ is bounded.*

(iii) $\mathrm{Dom}\, C^{1/2} \subset \mathrm{Dom}[\bar{F}]$.

(iv) $\mathrm{Dom}[F]$ *is dense in $\mathfrak{h}$, and for all $0 < \epsilon < \delta$ the form $\theta_F(C_\epsilon)$ on* $\mathrm{Dom}[F]$ *satisfies the inequality*

$$\theta_F(C_\epsilon) \leqslant b_1\imath(C_\epsilon) + b_2 1$$

on some dense subspace of $\mathrm{Dom}[F]$.

Then U is a strong solution to the right equation (R) *on* $\mathrm{Dom}\, C^{1/2}$ *for the operator matrix F.*

Note. By (i) it follows that each F^α_β is closable, hence the matrix $\bar{F}$ is defined, and so (iii) makes sense.

Proof. First note that $(F^{\alpha}_{\beta})^*(C_\epsilon)^{1/2}$ is bounded and everywhere defined by (ii) and Lemma 1.4. Taking adjoints we have

$$B(\mathfrak{h}) \ni [(F^{\alpha}_{\beta})^*(C_\epsilon)^{1/2}]^* \supset (C_\epsilon)^{1/2}\overline{F^{\alpha}_{\beta}} \supset (C_\epsilon)^{1/2}F^{\alpha}_{\beta}.$$

Thus, the form $\theta_F(C_\epsilon)$ on Dom$[F]$ is defined in terms of bounded operators that have a dense common domain of definition. Using the operators $[(F^{\alpha}_{\beta})^*(C_\epsilon)^{1/2}]^*$, we can define an extension of $\theta_F(C_\epsilon)$ to a bounded form on all of $\mathfrak{h}$, and so we shall treat it as a bounded operator on $\oplus^{(d+1)}\mathfrak{h}$, also identifying it with its ampliation to an operator on $\oplus^{(d+1)}\mathsf{H}$. Moreover, the inequality in (iv) is now valid as an *operator inequality*.

Now by (i) the process U^* satisfies

$$U^*_t = 1 + \int_0^t U^*_s(F^{\beta}_{\alpha})^*\, d\Lambda^{\beta}_{\alpha}(s)$$

on the domain $\mathfrak{D}\odot\mathscr{E}$, and so it follows that the process $(U^*_t(C_\epsilon)^{1/2})_{t\geqslant 0}$ has the stochastic integral representation

$$U^*_t(C_\epsilon)^{1/2} = (C_\epsilon)^{1/2} + \int_0^t U^*_s(F^{\beta}_{\alpha})^*(C_\epsilon)^{1/2}\, d\Lambda^{\beta}_{\alpha}(s)$$

on $\mathfrak{D}_\epsilon\odot\mathscr{E}$, which extends to all of $\mathfrak{h}\odot\mathscr{E}$ by continuity. By (i) and Proposition 2.2 it follows that U^* is a strongly continuous left cocycle and so U is a strongly (right) continuous right cocycle. Thus, we can take the adjoint of the above, since the resulting integrands are stochastically integrable, to get

$$(C_\epsilon)^{1/2}U_t = (C_\epsilon)^{1/2} + \int_0^t [(F^{\alpha}_{\beta})^*(C_\epsilon)^{1/2}]^*U_s\, d\Lambda^{\beta}_{\alpha}(s)$$

on $\mathfrak{h}\odot\mathscr{E}$. Applying (2.3) gives

$$\begin{aligned}||(C_\epsilon)^{1/2}U_t u\varepsilon(f)||^2 &= ||(C_\epsilon)^{1/2}u\varepsilon(f)||^2 \\ &\quad + \int_0^t \Bigg\{ 2\,\mathrm{Re}\langle f^{\alpha}(s)(C_\epsilon)^{1/2}U_s u\varepsilon(f), f^{\beta}(s)[(F^{\alpha}_{\beta})^*(C_\epsilon)^{1/2}]^*U_s u\varepsilon(f)\rangle \\ &\quad + \sum_{i=1}^d ||f^{\alpha}(s)[(F^{i}_{\alpha})^*(C_\epsilon)^{1/2}]^*U_s u\varepsilon(f)||^2 \Bigg\}\, ds\end{aligned}$$

for all $u\in\mathfrak{h}, f\in\mathbb{M}$. Collecting together the terms making up $\theta_F(C_\epsilon)$, we have

$$||(C_\epsilon)^{1/2}U_t u\varepsilon(f)||^2 = ||(C_\epsilon)^{1/2}u\varepsilon(f)||^2 + \int_0^t \langle x(s), \theta_F(C_\epsilon)x(s)\rangle\, ds,$$

where $x^\alpha(s) := f^\alpha(s) U_s u\varepsilon(f)$. The inequality in (iv) implies that

$$\|(C_\epsilon)^{1/2} U_t u\varepsilon(f)\|^2 \leqslant \|(C_\epsilon)^{1/2} u\varepsilon(f)\|^2 + \int_0^t (b_1 \|(C_\epsilon)^{1/2} U_s u\varepsilon(f)\|^2 + b_2 \|U_s u\varepsilon(f)\|^2)\, d\nu_f(s),$$

where $\nu_f(t) = \int_0^t (1 + \|f(s)\|^2)\, ds$, and so the Gronwall inequality gives

$$\|(C_\epsilon)^{1/2} U_t u\varepsilon(f)\|^2 \leqslant \left\{ \|(C_\epsilon)^{1/2} u\varepsilon(f)\|^2 + b_2 \int_0^t \|U_s u\varepsilon(f)\|^2 d\nu_f(s) \right\} \exp\{b_1 \nu_f(t)\}.$$

Letting $\epsilon \to 0$ we see that $U_t((\text{Dom}\, C^{1/2}) \odot \mathscr{E}) \subset \text{Dom}\, C^{1/2} \otimes 1$, and condition (iii) in conjunction with Lemma 1.2 implies that $\text{Dom}\, C^{1/2} \otimes 1 \subset \text{Dom}\, F^\alpha_\beta \otimes 1$ for all α, β, and hence U is a weak solution of (R) on $\text{Dom}\, C^{1/2}$.

Now by Lemma 1.3 the functions $t \mapsto (F^\alpha_\beta \otimes 1) U_t u\varepsilon(f)$ are strongly measurable for all $0 \leqslant \alpha, \beta \leqslant d$, $u \in \text{Dom}\, C^{1/2}$ and $f \in \mathbb{M}$. Also the above inequality (in the limit as $\varepsilon \to 0$) shows that $t \mapsto \|(C^{1/2} \otimes 1) U_t u\varepsilon(f)\|$ is locally bounded, and so Lemma 1.2 and (iii) imply that the processes $((F^\alpha_\beta \otimes 1) U_t)_{t \geqslant 0}$ are stochastically integrable on $\text{Dom}\, C^{1/2}$. Hence U is a strong solution as required. □

Remarks. (a) The requirement that U^* be the *unique* strong solution to the adjoint left equation allowed us to conclude that the processes $\{[(F^\alpha_\beta)^*(C_\epsilon)^{1/2}]^* U\}$ are stochastically integrable and hence $(C_\epsilon)^{1/2} U$ has a strong stochastic integral representation. If $\mathfrak{h}$ is separable then the stochastic integrability of this family of processes is guaranteed by the equivalence of strong and weak measurability for functions taking values in a separable Hilbert space, and so the uniqueness requirement in part (i) of the hypothesis can be dropped without affecting the result.

(b) By Lemma 1.4, a sufficient condition for the boundedness of each $(F^\beta_\alpha)^*(C_\epsilon)|_{\mathfrak{D}_\epsilon}$ is $\text{Dom}\, C^{1/2} \subset \text{Dom}[F^*]$, and indeed this is necessary if 0 lies in the resolvent of C. This observation provides an important guide as to what would be a suitable choice for C. As an illustration, in the diffusion example in Section 4 we have that $(F^0_0)^*$ is a second-order differential operator, and so for C we take $\partial^4 + 1$. However, it is still important to check that $(C_\epsilon)^{1/2}$ maps some dense subspace $\mathfrak{D}_\epsilon$ into $\mathfrak{D}$, the subspace for which U^* satisfies the QSDE (L).

(c) The proof of the above result remains valid if we replace C_ϵ by other variants of the Yosida approximation which raises the possibility of using a "less unbounded" reference operator C. Indeed we could used CR^4_ϵ or $C(1 + \epsilon C^2)^{-2}$ instead of C_ϵ, and then we would be able to use C of the same order as $(F^0_0)^*$. However, if we adopt these variants then proving the analogous result to Proposition 3.4 below becomes much harder.

Having constructed a solution U to the right equation we now finish this section with two results that give conditions under which we can prove that U is an isometry or coisometry process. The first of these happens, at least formally, when $\theta_F(1) = 0$ (cf. the remark after Proposition 2.1), although we must take care when considering the form $\theta_F(1)$, defined on $\mathrm{Dom}[F]$, and its extension $\theta_{\bar{F}}(1)$ to $\mathrm{Dom}[\bar{F}]$.

Corollary 2.4. *Suppose that the conditions of* Theorem 2.3 *hold and let U be the strong solution to* (R) *for the given matrix F. If either*

(i) $\mathrm{Dom}\, C^{1/2} \cap \mathrm{Dom}[F]$ *is a core for* $C^{1/2}$ *and* $\theta_F(1) = 0$, *or*
(ii) $\theta_{\bar{F}}(1) = 0$,

then U is an isometry process.

Proof. If (i) holds then, using the condition (iii) from the hypotheses of Theorem 2.3 and Lemma 1.2, it is possible to find for each $u \in \mathrm{Dom}\, C^{1/2}$ a sequence (u_n) in this core such that $u_n \to u$ and $F^\alpha_\beta u_n \to \overline{F^\alpha_\beta} u$. It follows that the form $\theta_{\bar{F}}(1)$ when restricted to $\mathrm{Dom}\, C^{1/2}$ is equal to zero, which is also clearly the case if condition (ii) holds. So now let Θ_F denote the *sesquilinear* form defined by

$$((\xi^\gamma), (\eta^\gamma)) \mapsto \langle \xi^\alpha, (F^\alpha_\beta \otimes 1)\eta^\beta \rangle + \langle (F^\beta_\alpha \otimes 1)\xi^\alpha, \eta^\beta \rangle + \langle (F^i_\alpha \otimes 1)\xi^\alpha, (F^i_\beta \otimes 1)\eta^\beta \rangle$$

for $(\xi^\gamma), (\eta^\gamma) \in \oplus^{(d+1)} \mathrm{Dom}[F \otimes 1]$. By the above the restriction of this form to $\mathrm{Dom}\, C^{1/2} \odot \mathscr{F}$ is zero. But as noted in Section 1, $\mathrm{Dom}\, C^{1/2} \odot \mathscr{F}$ is a core for $C^{1/2} \otimes 1$, and so another application of the inequality (1.1) allows us to show that Θ_F is zero when restricted to $\mathrm{Dom}\, C^{1/2} \otimes 1$. Now since U is a strong solution to (R) we can apply (2.2) to get

$$\langle U_t u\varepsilon(f), U_t v\varepsilon(g) \rangle = \langle u\varepsilon(f), v\varepsilon(g) \rangle + \int_0^t \Theta_F(x(s), y(s))\, ds$$

for all $u, v \in \mathrm{Dom}\, C^{1/2}, f, g \in \mathbb{M}$, and where $x^\alpha(s) = f^\alpha(s) U_s u\varepsilon(f)$ and $y^\alpha(s) = g^\alpha(s) U_s v\varepsilon(g)$. But U_t maps $\mathrm{Dom}\, C^{1/2} \odot \mathscr{E}$ into $\mathrm{Dom}\, C^{1/2} \otimes 1$ by Theorem 2.3, hence the integrand is zero if either (i) or (ii) holds, and the result follows. □

A quantum dynamical semigroup (QDS) on $B(\mathfrak{h})$ is an ultraweakly continuous semigroup $\mathscr{T} = (\mathscr{T}_t)_{t \geqslant 0}$ of normal completely positive maps on $B(\mathfrak{h})$. It is *conservative* if $\mathscr{T}_t(1) = 1$ for all $t \geqslant 0$. Given the generator K of a strongly continuous contraction semigroup $\mathfrak{h}$ and operators $(L_l)_{l \geqslant 0}$ such that

$$\langle u, Ku \rangle + \langle Ku, u \rangle + \sum_{l \geqslant 1} \|L_l u\|^2 \leqslant 0, \tag{2.7}$$

for all u in some core $\mathfrak{D}$ for K, it is possible to construct the *minimal* QDS $\mathscr{T}$ (see [F5] and references therein) that has (formal) generator $\mathscr{L}$ given by

$$\langle u, \mathscr{L}(X)v \rangle = \langle u, XKv \rangle + \langle Ku, Xv \rangle + \sum_{l \geqslant 1} \langle L_l u, XL_l v \rangle, \quad u, v \in \mathfrak{D}.$$

That is, $\mathscr{T}$ is a QDS that satisfies

$$\langle u, \mathscr{T}_t(X)v \rangle = \langle u, Xv \rangle + \int_0^t \langle u, \mathscr{L}(\mathscr{T}_s(X))v \rangle ds,$$

and if $\mathscr{T}'$ is another QDS satisfying the above integral identity then $\mathscr{T}_t(X) \leqslant \mathscr{T}'_t(X)$ for all $t \geqslant 0$ and positive $X \in B(\mathfrak{h})$. Showing that the solution U to (R) constructed in Theorem 2.3 is a coisometry process is equivalent to showing that U^* is an isometry process, which (under favourable circumstances) is equivalent to showing that a related QDS is conservative:

Proposition 2.5 ([F3,F5]). *Suppose that the conditions of* Theorem 2.3 *hold and let U be the strong solution to* (R) *for the given matrix F. Suppose further that $(F_0^0)^*$ is the generator of a strongly continuous contraction semigroup, that the subspace $\mathfrak{D}$ is a core for $(F_0^0)^*$, and let $\mathscr{T}$ be the minimal QDS with generator*

$$\langle u, \mathscr{L}(X)v \rangle = \langle u, X(F_0^0)^* v \rangle + \langle (F_0^0)^* u, Xv \rangle + \sum_{i=1}^{d} \langle (F_i^0)^* u, X(F_i^0)^* v \rangle.$$

The following are equivalent:

(i) *U is a coisometry process.*
(ii) *$\theta_{F^*}(1) = 0$ on $\mathfrak{D}$ and $\mathscr{T}$ is conservative.*
(iii) *$[\delta^i_j 1 + F^i_j]_{i,j=1}^d$ is a coisometry on $\oplus_{i=1}^d \mathfrak{h}$ and $\mathscr{T}$ is conservative.*

Remark. By Proposition 2.1 the inequality (2.7) holds for $K = (F_0^0)^*$ and $L_l = (F_l^0)^*$, hence the minimal QDS $\mathscr{T}$ exists.

3. Special case: isometric solutions with one-dimensional noise

The results in the previous section provide a very general method for generating contraction solutions to the right HP equation, and one whose basic idea could be modified easily if necessary, for instance by using different regularisations to C_ϵ. In this section we refine our basic result in a number of ways. Firstly, in order to make use of known results on the existence of (strong) solutions to the QSDE (L) (and hence verify part (i) of Theorem 2.3) we shall assume *from now on* that the initial space $\mathfrak{h}$ is *separable*. Secondly, in order to simplify the form of the generator we shall set $d=1$, that is we work with only one dimension of quantum noise, and we now look for isometric solutions to the QSDE (R) (cf. Proposition 2.1).

The operator matrix for the rest of the section is specified by a triple of operators (L, H, S), where L and H are densely defined, with L closable and H symmetric, and S is an isometric element of $B(\mathfrak{h})$. The operator matrix is

$$F = \begin{bmatrix} -\frac{1}{2}L^*L - iH & -L^*S \\ L & S-1 \end{bmatrix},$$

and we assume throughout that $\mathrm{Dom}[F]$ is a dense subspace of $\mathfrak{h}$. Rather than work with F^*, the matrix of adjoints, we shall use the following matrix:

$$F^\dagger = \begin{bmatrix} -\frac{1}{2}L^*L + iH & L^* \\ -S^*L & S^*-1 \end{bmatrix},$$

whose components are restrictions of the components of F^*. Thus, the QSDEs that we are now working with are

$$dU_t = [(-\tfrac{1}{2}L^*L - iH)\,dt - L^*S\,dA_t + L\,dA_t^\dagger + (S-1)\,d\Lambda_t]U_t \qquad \text{(R)}'$$

and

$$dU_t^* = U_t^*[(-\tfrac{1}{2}L^*L + iH)\,dt + L^*\,dA_t - S^*L\,dA_t^\dagger + (S^*-1)\,d\Lambda_t]. \qquad \text{(L)}'$$

The formal generator of the related flow is

$$\theta_F(X) = \begin{bmatrix} -\frac{1}{2}XL^*L + L^*XL - \frac{1}{2}L^*LX + i[H,X] & [L^*,X]S \\ S^*[X,L] & S^*XS - X \end{bmatrix}, \qquad (3.1)$$

so in particular $\theta_F(1) = 0$ on some domain.

Proposition 3.1. *Let (L, H, S) be a triple of operators as above, and suppose that there is a dense subspace $\mathfrak{D}$ of $\mathfrak{h}$ such that*

(i) $\mathfrak{D} \subset \mathrm{Dom}\, L^*L \cap \mathrm{Dom}\, L^* \cap \mathrm{Dom}\, H$, *and*
(ii) *the closure of $(-\frac{1}{2}L^*L + iH)|_{\mathfrak{D}}$ is the infinitesimal generator of a strongly continuous contraction semigroup on $\mathfrak{h}$.*

Then there is a contraction process U^* that is a strong solution to (L)′ *on $\mathfrak{D}$, and furthermore it is the unique strong solution.*

Proof. The result follows immediately from the method given in [F3]. The form

$$\theta_{F^\dagger}(1) : \mathbf{u} \mapsto 2\,\mathrm{Re}\,\langle \mathbf{u}, F^\dagger \mathbf{u} \rangle + \langle F^\dagger \mathbf{u}, \Delta(1) F^\dagger \mathbf{u} \rangle \qquad (3.2)$$

is well-defined for $\mathbf{u} \in \mathfrak{D} \oplus \mathfrak{D}$ by (i), where $\Delta(1) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. By the construction of $F^\dagger$, we have

$$\theta_{F^\dagger}(1)(\mathbf{u}) = \langle Lu^0 - u^1, (SS^* - 1)(Lu^0 - u^1) \rangle \leqslant 0 \tag{3.3}$$

for all $\mathbf{u} = [u^0, u^1]^\top \in \mathfrak{D} \oplus \mathfrak{D}$, that is $F^\dagger$ satisfies the formal contractivity conditions (cf. the remark after Proposition 2.1). Let K denote the closure of $(-\frac{1}{2}L^*L + iH)|_{\mathfrak{D}}$, then by considering vectors of the form $\mathbf{u} = [u^0, 0]^\top$ in (3.2) we can extend $(-S^*L)|_{\mathfrak{D}}$ to all of Dom K by approximating elements of Dom K by sequences in $\mathfrak{D}$ that converge in graph norm and using the inequality (3.3). Thus, $\theta_{F^\dagger}(1)$ extends to Dom $K \oplus \mathfrak{D}$, and continues to satisfy (3.3) on this domain. Put $I_n = \operatorname{diag}\{n(n-K)^{-1}, 1\}$ and $F_n^\dagger = I_n^* F^\dagger I_n$ for each $n \geqslant 1$, then it follows that $F_n^\dagger$ is a bounded map satisfying

$$F_n^\dagger + (F_n^\dagger)^* + (F_n^\dagger)^* \Delta(1) F_n^\dagger \leqslant 0.$$

So for each n we can solve the equation

$$dU^{(n)*} = U^{(n)*} (F_n^\dagger)^\alpha_\beta \, d\Lambda^\beta_\alpha$$

and each $U^{(n)*}$ is contractive. In fact for each $u \in \mathfrak{h}$, $f \in \mathbb{M}$ and $\xi \in \mathsf{H}$ the family $\{\langle \xi, U_t^{(n)*} u\varepsilon(f) \rangle\}_{n=1}^\infty$ is equibounded and equicontinuous on each bounded interval. A diagonalisation argument and the Ascoli–Arzelà theorem can then be employed to show that there is a weakly convergent subsequence $\{U^{(n_k)*}\}$ whose limit is the required solution. The uniqueness follows by the result of Mohari. □

Remark. In the examples below we shall take S to be unitary since we will be constructing unitary cocycles. Thus S^*L is closable with $\overline{S^*L} = S^*\bar{L}$. Also, from (3.3), it follows that Dom $\bar{L} \supset$ Dom K, and that the extension of $(-S^*L)|_{\mathfrak{D}}$ to Dom K in the proof is nothing but the restriction of $-S^*\bar{L}$ to this domain.

The next result gives some *sufficient* conditions that imply that the solution U^* to (L) constructed above is an isometry process. The conditions are by no means optimal, in particular they are not necessary, but are written in such a way as to be easily applicable to our examples in the next section.

Corollary 3.2. *Let (L, H, S) be a triple of operators as above, and suppose that the conditions of* Proposition 3.1 *hold, with U^* the contraction process that is the strong solution to* (L)$'$. *Suppose further that the operator S is unitary, and that there is a positive self-adjoint operator M on $\mathfrak{h}$ and a constant $k \geqslant 0$ such that*:

(i) Dom $K \subset$ Dom $M^{1/2}$, *and* Dom K *is a core for* $M^{1/2}$,
(ii) $S^*\bar{L}(\text{Dom}\, K^2) \subset \text{Dom}\, M^{1/2}$,

(iii) $2\,\mathrm{Re}\langle M^{1/2}u, M^{1/2}Ku\rangle + \|M^{1/2}S^*\bar{L}u\|^2 \leqslant k\|M^{1/2}u\|^2\ \forall u \in \mathrm{Dom}\, K^2$, *and*
(iv) $\mathrm{Dom}\, M \subset \mathrm{Dom}\, L^*\bar{L}$, *and* $\|\bar{L}u\| \leqslant \|M^{1/2}u\|\ \forall u \in \mathrm{Dom}\, M$,

where K denotes the closure of $(-\frac{1}{2}L^*L + iH)|_{\mathfrak{D}}$. *Then* U^* *is an isometric process.*

Proof. Apply Theorem 4.4 of [ChF] to deduce that the minimal QDS associated to (L)′ is conservative, and so U^* is an isometry process by Proposition 2.5. □

Perhaps one of the more difficult things to check in order to be able to apply Theorem 2.3 is that the form inequality for $\theta_F(C_\epsilon)$ in (iv) holds for all values of ϵ in an interval of the form $(0, \delta)$. Proposition 3.4 below shows that this will be the case if the analogous form inequality holds for $\theta_F(C)$, and if the commutators of C with L and S are sufficiently well-behaved. The following lemma eases the algebraic burden of the proof of this result.

Lemma 3.3. *Let* $\mathscr{B}$ *be a unital associative algebra and suppose that* $c, r \in \mathscr{B}$ *satisfy* $\epsilon cr = \epsilon rc = 1 - r$ *for some* $\epsilon > 0$. *For any* $a, b \in \mathscr{B}$ *the linear maps* $\tau_{a,b}$ *and* $\omega_{a,b}$ *on* $\mathscr{B}$ *defined by*

$$\tau_{a,b}(x) = a[x,b] \quad \text{and} \quad \omega_{a,b}(x) = [a,x]b \qquad (x \in \mathscr{B})$$

satisfy

$$\begin{aligned}\tau_{a,b}(rcr) = {} & r^2\tau_{a,b}(c)r^2 - r(1-r)\tau_{a,b}(c)(1-r)r - \epsilon r[c,a]r^2[b,c]r \\ & + \epsilon r(1-r)[c,a]r[b,c](1-r)r - \epsilon r^2[c,a]r[b,c]r^2\end{aligned}$$

and

$$\begin{aligned}\omega_{a,b}(rcr) = {} & r^2\omega_{a,b}(c)r^2 - r(1-r)\omega_{a,b}(c)(1-r)r - \epsilon r[c,a]r^2[b,c]r \\ & + \epsilon r(1-r)[c,a]r[b,c](1-r)r - \epsilon r^2[c,a]r[b,c]r^2.\end{aligned}$$

Proof. The relations satisfied by c and r imply that $[x,r] = \epsilon r[c,x]r$ and hence $[x,r^2] = \epsilon r^2[c,x]r + \epsilon r[c,x]r^2$ for all $x \in \mathscr{B}$. These identities and those already given lead to the following chain of equalities:

$$\begin{aligned}\tau_{a,b}(rcr) &= a[rcr,b] = arc[r,b] + ar[c,b]r + a[r,b]cr \\ &= ar(1-r)[b,c]r + ar[c,b]r + ar[b,c](1-r)r \\ &= -r^2a[b,c]r - [a,r^2][b,c]r + ra[b,c](1-r)r + [a,r][b,c](1-r)r \\ &= r^2\tau_{a,b}(c)r - \epsilon r^2[c,a]r[b,c]r - \epsilon r[c,a]r^2[b,c]r \\ &\quad + r\tau_{a,b}(c)(r-1)r + \epsilon r[c,a]r[b,c](1-r)r\end{aligned}$$

$$= r^2\tau_{a,b}(c)r^2 - r(1-r)\tau_{a,b}(c)(1-r)r - \epsilon r[c,a]r^2[b,c]r$$
$$+ \epsilon r(1-r)[c,a]r[b,c](1-r)r - \epsilon r^2[c,a]r[b,c]r^2,$$

giving the identity for $\tau_{a,b}$. The one for $\omega_{a,b}$ follows by an almost identical proof. □

Remark. If $\mathscr{B}$ is assumed to be involutive and c and r are self-adjoint, then the identity for ω can be derived from that for τ since $\omega_{a,b}(x) = \tau_{b^*,a^*}(x^*)^*$.

Proposition 3.4. *Let (L, H, S) be a triple of operators as above, and suppose that there exists a positive, invertible, self-adjoint operator C, a dense subspace $\tilde{\mathfrak{D}}$ of $\mathfrak{h}$, and constants $0 < \delta < 1$ and $b_3, b_4 \geqslant 0$ such that the following hold:*

(i) *$R_\epsilon(\tilde{\mathfrak{D}}) \subset \tilde{\mathfrak{D}}$ for all $0 < \epsilon < \delta$, and $\tilde{\mathfrak{D}}$ is contained in the domain of the following operators:*

$$L^*L, L^*S, [C, L], [C, H], CS.$$

(ii) *The form $\theta_F(C)$ defined on $\tilde{\mathfrak{D}}$ satisfies the following inequality:*

$$-b_3\imath(C) \leqslant \theta_F(C) \leqslant b_3\imath(C).$$

(iii) *For all $u \in \tilde{\mathfrak{D}}$, $||C^{-1/2}[C, L]u|| \leqslant b_4||C^{1/2}u||$.*

(iv) *For all $u \in \tilde{\mathfrak{D}}$, $|||[C, S]u|| \leqslant b_4||C^{1/2}u||$.*

Then for all $0 < \epsilon < \delta$ the form $\theta_F(C_\epsilon)$ is well-defined on $\tilde{\mathfrak{D}}$ and satisfies

$$\theta_F(C_\epsilon) \leqslant 2(b_3 + b_4)\imath(C_\epsilon).$$

Proof. We shall use the preceding lemma to rewrite each component $\theta^\alpha_\beta(C_\epsilon) := \theta_F(C_\epsilon)^\alpha_\beta$ of the form (3.1). This is possible, since in the notation of the lemma (and ignoring domain problems for now) we have

$$\theta^0_0(X) = L^*[\tfrac{1}{2}L, X] + [\tfrac{1}{2}L^*, X]L + [iH, X]$$
$$= \tau_{L^*,\frac{1}{2}L}(X) + \omega_{\frac{1}{2}L^*,L}(X) + \omega_{iH,1}(X)$$

and, similarly,

$$\theta^0_1 = \omega_{L^*,S}, \quad \theta^0_1 = \tau_{S^*,L}, \quad \theta^1_1 = \tau_{S^*,\frac{1}{2}S} + \omega_{\frac{1}{2}S^*,S},$$

noting for θ^1_1 that $S^*S = 1$. Now note that the issue of domains is covered for us by condition (i). Indeed, each component $\theta^\alpha_\beta(C_\epsilon)$ is a well-defined sesquilinear form on $\tilde{\mathfrak{D}}$, and moreover so is each of the terms such as $\tau_{L^*,\frac{1}{2}L}(C_\epsilon)$. For example, we have

$$\theta_1^0(C_\epsilon) = \omega_{L^*,S}(R_\epsilon C R_\epsilon) = [L^*, R_\epsilon C R_\epsilon]S$$

which should really be thought of as the form

$$\tilde{\mathfrak{D}} \times \tilde{\mathfrak{D}} \ni (u, v) \mapsto \langle Lu, R_\epsilon C R_\epsilon S v \rangle - \langle R_\epsilon C R_\epsilon u, L^* S v \rangle,$$

which is well-defined since CR_ϵ is bounded and $\tilde{\mathfrak{D}} \subset \operatorname{Dom} L \cap \operatorname{Dom} L^*S$. Lemma 3.3 allows us to rewrite this as

$$\begin{aligned}\theta_1^0(C_\epsilon) = {} & R_\epsilon^2 \theta_1^0(C) R_\epsilon^2 - R_\epsilon(1 - R_\epsilon)\theta_1^0(C)(1 - R_\epsilon)R_\epsilon \\ & - \epsilon R_\epsilon [C, L^*] R_\epsilon^2 [S, C] R_\epsilon + \epsilon R_\epsilon (1 - R_\epsilon)[C, L^*] R_\epsilon [S, C](1 - R_\epsilon) R_\epsilon \\ & - \epsilon R_\epsilon^2 [C, L^*] R_\epsilon [S, C] R_\epsilon^2\end{aligned}$$

and again each of the terms on the right-hand side, plus those appearing in the derivation of the above, make good sense courtesy of condition (i). The adjoint identity holds for $\theta_0^1(C_\epsilon)$,

$$\begin{aligned}\theta_0^0(C_\epsilon) = {} & R_\epsilon^2 \theta_0^0(C) R_\epsilon^2 - R_\epsilon(1 - R_\epsilon)\theta_0^0(C)(1 - R_\epsilon)R_\epsilon \\ & - \epsilon R_\epsilon [C, L^*] R_\epsilon^2 [L, C] R_\epsilon + \epsilon R_\epsilon (1 - R_\epsilon)[C, L^*] R_\epsilon [L, C](1 - R_\epsilon) R_\epsilon \\ & - \epsilon R_\epsilon^2 [C, L^*] R_\epsilon [L, C] R_\epsilon^2,\end{aligned}$$

and the identity for $\theta_1^1(C_\epsilon)$ is got by changing θ_0^0 to θ_1^1 and L to S in the above. Thus, each component $\theta_\beta^\alpha(C_\epsilon)$ defines a sesquilinear form on $\tilde{\mathfrak{D}}$, and the quadratic form $\theta_F(C_\epsilon)$ is well-defined on $\tilde{\mathfrak{D}}$ with

$$\begin{aligned}\theta_F(C_\epsilon) = {} & \mathbf{R}_\epsilon^1 \theta_F(C) \mathbf{R}_\epsilon^1 - \mathbf{R}_\epsilon^2 \theta_F(C) \mathbf{R}_\epsilon^2 - \epsilon \mathbf{R}_\epsilon^1 \phi_\epsilon(C) \mathbf{R}_\epsilon^1 + \epsilon \mathbf{R}_\epsilon^2 \phi_\epsilon(C) \mathbf{R}_\epsilon^2 \\ & - \epsilon \begin{bmatrix} R_\epsilon [C, L^*] R_\epsilon^2 [L, C] R_\epsilon & R_\epsilon [C, L^*] R_\epsilon^2 [S, C] R_\epsilon \\ R_\epsilon [C, S^*] R_\epsilon^2 [L, C] R_\epsilon & R_\epsilon [C, S^*] R_\epsilon^2 [S, C] R_\epsilon \end{bmatrix},\end{aligned} \tag{3.4}$$

where

$$\mathbf{R}_\epsilon^1 = \begin{bmatrix} R_\epsilon^2 & 0 \\ 0 & R_\epsilon^2 \end{bmatrix}, \qquad \mathbf{R}_\epsilon^2 = \begin{bmatrix} R_\epsilon(1 - R_\epsilon) & 0 \\ 0 & R_\epsilon(1 - R_\epsilon) \end{bmatrix}$$

and

$$\phi_\epsilon(C) = \begin{bmatrix} [C, L^*] R_\epsilon [L, C] & [C, L^*] R_\epsilon [S, C] \\ [C, S^*] R_\epsilon [L, C] & [C, S^*] R_\epsilon [S, C] \end{bmatrix}$$

are all positive operator matrices or forms.

Now R_ϵ and $1 - R_\epsilon$ are positive contractions, and $\epsilon R_\epsilon \leqslant C^{-1}$ for each $\epsilon > 0$. Thus for any $\mathbf{u} = [u^0, u^1]^\top \in \tilde{\mathfrak{D}} \oplus \tilde{\mathfrak{D}}$ and $0 < \epsilon < \delta < 1$,

$$\begin{aligned}\phi_\epsilon(C)(\mathbf{u}) &= \|R_\epsilon^{1/2}[L, C]u^0\|^2 + 2\,\mathrm{Re}\langle R_\epsilon^{1/2}[C, L]u^0, R_\epsilon^{1/2}[S, C]u^1\rangle \\ &\quad + \|R_\epsilon^{1/2}[S, C]u^1\|^2 \\ &\leqslant 2\{\|R_\epsilon^{1/2}[C, L]u^0\|^2 + \|[C, S]u^1\|^2\} \\ &\leqslant 2\epsilon^{-1}\{\|C^{-1/2}[C, L]u\|^2 + \|[C, S]v\|^2\} \\ &\leqslant 2\epsilon^{-1} b_4 \imath(C)(\mathbf{u}),\end{aligned}$$

by inequalities (iii) and (iv). Thus the last three terms of (3.4) are bounded above by $2b_4 \mathbf{R}_\epsilon^2 \imath(C) \mathbf{R}_\epsilon^2$. The result now follows since for all $0 < \epsilon < \delta$ we have $R_\epsilon^2 C R_\epsilon^2 \leqslant C_\epsilon$ and $R_\epsilon(1 - R_\epsilon)C(1 - R_\epsilon)R_\epsilon \leqslant C_\epsilon$. □

Theorem 3.5. *Let (L, H, S) be a triple of operators as above and suppose that there is a positive, invertible, self-adjoint operator C such that the hypotheses of* Propositions 3.1 *and* 3.4 *hold. Suppose also that the following conditions hold for some $0 < \delta' < \delta$:*

(i) *For each $0 < \epsilon < \delta'$ there is a dense subspace $\mathfrak{D}_\epsilon \subset \mathfrak{D}$ such that $(C_\epsilon)^{1/2}(\mathfrak{D}_\epsilon) \subset \mathfrak{D}$, and such that the restrictions of the operators $L^*(C_\epsilon)^{1/2}, L(C_\epsilon)^{1/2}$ and $(-\frac{1}{2}L^*L + iH)(C_\epsilon)^{1/2}$* to *$\mathfrak{D}_\epsilon$ are bounded.*

(ii) $\mathrm{Dom}\, C^{1/2} \subset \mathrm{Dom}\, \overline{-\frac{1}{2}L^*L - iH} \cap \mathrm{Dom}\, \bar{L} \cap \mathrm{Dom}\, L^*S$.

Then U, the adjoint of the process from Proposition 3.1, *is a strong solution to* $(\mathrm{R})'$ *on* $\mathrm{Dom}\, C^{1/2}$. *Moreover, if* $\mathrm{Dom}\, C^{1/2} \cap \mathrm{Dom}[F]$ *is a core for $C^{1/2}$ then U is isometric.*

Proof. Since the hypotheses of Propositions 3.1 and 3.4 and the additional conditions above hold it follows that all of the requirements of Theorem 2.3 are fulfilled, and so the contraction process U is a strong solution to $(\mathrm{R})'$ for the operator matrix F on the domain $\mathrm{Dom}\, C^{1/2}$. If $\mathrm{Dom}\, C^{1/2} \cap \mathrm{Dom}[F]$ is a core for $C^{1/2}$ then Corollary 2.4 can be applied to show that U is an isometry process since $\theta_F(1) = 0$. □

4. Examples

Birth and death processes: Let $\mathfrak{h} = l^2(\mathbb{Z})$, with standard orthonormal basis $(e_n)_{n \in \mathbb{Z}}$, and let W be the unitary right shift given by $We_n = e_{n+1}$. Let N be the number operator on $l^2(\mathbb{Z})$, that is

$$\mathrm{Dom}\, N = \left\{ (u_n) \colon\ u_n \in \mathbb{C}, \sum_{n \in \mathbb{Z}} n^2 |u_n|^2 < \infty \right\}, \quad N(u_n) = (nu_n).$$

Then for any function $\lambda : \mathbb{Z} \to \mathbb{C}$ the operator denoted $\lambda(N)$ is defined by

$$\mathrm{Dom}\, \lambda(N) = \left\{ (u_n) \colon\ u_n \in \mathbb{C}, \sum_{n \in \mathbb{Z}} |\lambda(n) u_n|^2 < \infty \right\}, \quad \lambda(N)(u_n) = (\lambda(n) u_n).$$

Let $\mathfrak{D}^0 = \mathrm{Lin}\{e_n\}$, so then $\mathfrak{D}^0 \subset \mathrm{Dom}\, \lambda(N)$ for any function λ, and is in fact a core for $\lambda(N)$.

Define the triple (L, H, S) and the reference operator C by

$$L = W\lambda(N), \quad H = 0, \quad S = W, \quad C = N^2 + 1,$$

and note that $\mathfrak{D}^0$ is an invariant subspace for all these.

Now $L^* L = |\lambda|^2(N)$, a positive self-adjoint operator for which $\mathfrak{D}^0$ is a core, so we can apply Proposition 3.1 and Corollary 3.2 (with $\mathfrak{D} = \mathfrak{D}^0, M = |\lambda|^2(N)$, and $k = 0$) to obtain an isometric process U^* that is a strong solution to $(\mathrm{L})'$ on $\mathfrak{D}^0$.

For any function $\varphi : \mathbb{Z} \to \mathbb{C}$, the form $\theta_F(\varphi(N))$ is well-defined on $\mathfrak{D}^0$. Indeed, it is actually possible to regard $\theta_F(\varphi(N))$ as an operator on $\mathfrak{h} \oplus \mathfrak{h}$ with domain $\mathfrak{D}^0 \oplus \mathfrak{D}^0$, and it can be written

$$\theta_F(\varphi(N)) = \begin{bmatrix} \bar{\lambda}(N) \\ 1 \end{bmatrix} (\varphi(N+1) - \varphi(N)) [\lambda(N) \quad 1].$$

We now restrict our attention to functions λ for which there is some $b > 0$ such that

$$|\lambda(n)|^2 \leqslant b(|n| + 1) \quad \forall n \in \mathbb{Z}.$$

Since

$$-(2|N| + 1) \leqslant [(N+1)^2 + 1] - [N^2 + 1] = 2N + 1 \leqslant 2|N| + 1,$$

it follows readily that the form $\theta_F(C)$ satisfies the inequality in part (ii) of Proposition 3.4 on the domain $\mathfrak{D}^0$. Also, note that on $\mathfrak{D}^0$

$$[C, L] = [N^2, W]\lambda(N) = W(W^* N^2 W - N^2)\lambda(N) = W(2N + 1)\lambda(N),$$

which is relatively bounded by $C^{3/4}$, so that (iii) follows, and similarly

$$[C, S] = W(W^* N^2 W - N^2) = W(2N + 1),$$

from which part (iv) follows. Thus, we can apply Proposition 3.4 to get that $\theta_F(C_\epsilon)$ satisfies a form inequality of the correct type on $\tilde{\mathfrak{D}} := \mathfrak{D}^0$.

Finally, the growth condition on λ implies that Dom $C^{1/2}$ is contained in the subspace $\mathrm{Dom}|\lambda|^2(N) \cap \mathrm{Dom}\,\lambda(N) \cap \mathrm{Dom}\,\bar{\lambda}(N)W^*$, so conditions (i) and (ii) of Theorem 3.5 are satisfied with $\mathfrak{D}_\epsilon = \mathfrak{D}^0$. Thus, there is a strong unitary solution to the QSDE

$$dU_t = [-\tfrac{1}{2}|\lambda|^2(N)\,dt - \bar{\lambda}(N)\,dA_t + W\lambda(N)\,dA_t^\dagger + (W-1)\,d\Lambda_t]U_t$$

on Dom $C^{1/2}$, since $\mathfrak{D}^0 \subset \mathrm{Dom}[F]$, and $\mathfrak{D}^0$ is a core for $C^{1/2}$.

If we consider the algebra $l^\infty(\mathbb{Z}) \subset B(\mathfrak{h})$ acting by pointwise multiplication, then

$$\theta_0^0(\varphi(N)) = |\lambda|^2(N)\{\varphi(N+1) - \varphi(N)\}, \quad \varphi \in l^\infty(\mathbb{Z}),$$

and so the flow $X \mapsto U_t^*(X \otimes 1)U$ gives a realisation of the classical pure birth process with intensity $|\lambda|^2$, since the generator θ_0^0 is of the appropriate form.

Replacing the triple above by

$$L = W^*\mu(N), \quad H = 0, \quad S = W^*,$$

with $\mu : \mathbb{Z} \to \mathbb{C}$ subject to the same growth condition, and setting $C = N^2 + 1$ once more, we obtain a strong unitary solution to the QSDE

$$dU_t = [-\tfrac{1}{2}|\mu|^2(N)\,dt - \bar{\mu}(N)\,dA_t + W^*\mu(N)\,dA_t^\dagger + (W^*-1)\,d\Lambda_t]U_t,$$

on Dom $C^{1/2}$. This time

$$\theta_0^0(\varphi(N)) = |\mu|^2(N)\{\varphi(N-1) - \varphi(N)\},$$

and so we have a realisation of the classical pure death process with intensity μ.

By increasing the number of noise dimensions to two we are able to realise a combined birth and death process. Let F be the operator matrix

$$F = \begin{bmatrix} -\tfrac{1}{2}|\lambda|^2(N) - \tfrac{1}{2}|\mu|^2(N) & -\bar{\lambda}(N) & -\bar{\mu}(N) \\ W\lambda(N) & W-1 & 0 \\ W^*\mu(N) & 0 & W^*-1 \end{bmatrix}.$$

Then for any function $\varphi : \mathbb{Z} \to \mathbb{C}$ the operator matrix $\theta_F(\varphi(N))$ (with domain containing $\mathfrak{D}^0 \oplus \mathfrak{D}^0 \oplus \mathfrak{D}^0$) decomposes as

$$\begin{bmatrix} |\lambda|^2(N) & \bar{\lambda}(N) & 0 \\ \lambda(N) & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{\imath}_{(+)} + \begin{bmatrix} |\mu|^2(N) & 0 & \bar{\mu}(N) \\ 0 & 0 & 0 \\ \mu(N) & 0 & 1 \end{bmatrix} \boldsymbol{\imath}_{(-)}, \tag{4.1}$$

where $\boldsymbol{\imath}_{(+)} = \imath(\varphi(N+1) - \varphi(N))$ and $\boldsymbol{\imath}_{(-)} = \imath(\varphi(N-1) - \varphi(N))$. Viewing each component as a 2×2 matrix, the estimates above together with Proposition 3.4 show that there is some $b' > 0$ such that $\theta_F(C_\epsilon) \leqslant b' \imath(C_\epsilon)$, where $C = N^2 + 1$ as before. So now appealing directly to Theorem 2.3 (rather than Theorem 3.5), Corollary 2.4 and Proposition 2.5 (or, rather, a two-dimensional version of Corollary 3.2), we can show that there is a unitary process U that is a strong solution to

$$\begin{aligned} dU_t = [&-\tfrac{1}{2}(|\lambda|^2(N) + |\mu|^2(N))\, d\Lambda_0^0(t) - \bar{\lambda}(N)\, d\Lambda_1^0(t) - \bar{\mu}(N)\, d\Lambda_2^0(t) \\ &+ W\lambda(N)\, d\Lambda_0^1(t) + W^*\mu(N)\, d\Lambda_0^2(t) + (W-1)\, d\Lambda_1^1(t) \\ &+ (W^* - 1)\, d\Lambda_2^2(t)] U_t. \end{aligned}$$

Then note from (4.1) that $\theta_0^0(\varphi(N))$ has the required form for the cocycle $X \mapsto U_t^*(X \otimes 1)U_t$ to be a realisation of the birth and death process with intensities λ and μ.

The inverse harmonic oscillator: Let $\mathfrak{h} = l^2(\mathbb{Z}_+)$, where $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, equipped with the standard basis $(e_n)_{n \geqslant 0}$. Let W be the *isometric* right shift, $We_n = e_{n+1}$, and $\mathfrak{D}^0 = \mathrm{Lin}\{e_n\}$. Define the triple (L, H, S) and reference operator C by

$$L = \lambda(N)W, \quad H = \mu(N), \quad S = 1, \quad C = N^2 + 1$$

for functions $\lambda : \mathbb{Z}_+ \to \mathbb{C}$ and $\mu : \mathbb{Z}_+ \to \mathbb{R}$, and assume that there is some $c > 0$ such that

$$\max\{|\lambda(n)|^2, |\mu(n)|\} \leqslant c(n+1) \quad \forall n \geqslant 0.$$

So now for any function $\varphi : \mathbb{Z}_+ \to \mathbb{C}$, the components of $\theta_F(\varphi(N))$ (whose domains contain $\mathfrak{D}^0$) are

$$\theta_0^0(\varphi(N)) = |\lambda|^2(N+1)\{\varphi(N+1) - \varphi(N)\}, \quad \theta_1^1(\varphi(N)) = 0,$$

$$\theta_0^1(\varphi(N)) = \lambda(N)[\varphi(N), W] = \theta_1^0(\bar{\varphi}(N))^*.$$

In particular, if we take $\varphi(N) = C = N^2 + 1$, then

$$\theta_0^0(C) = (2N+1)|\lambda|^2(N+1),$$

and

$$[C,L]=\theta_0^1(C)=\lambda(N)[N^2,W]=\lambda(N)(WW^*+P_0)N^2W-\lambda(N)WN^2$$
$$=\lambda(N)W(2N+1),$$

where $P_0=1-WW^*$ is the projection onto $\mathbb{C}e_0$, so that $P_0N^2W=0$. Thus $[C,L]$ is relatively bounded by $C^{3/4}$. These observations allow us to apply Propositions 3.1 and 3.4, and Theorem 3.5 to prove the existence of an isometric process U that is a strong solution to

$$dU_t=[(-\tfrac{1}{2}|\lambda|^2(N+1)-i\mu(N))\,dt-W^*\bar{\lambda}(N)\,dA_t+\lambda(N)W\,dA_t^\dagger]U_t.$$

If, further, we assume the existence of $0<c'\leqslant c$ such that

$$c'n\leqslant|\lambda(n)|^2\leqslant c(n+1)\quad\forall n\geqslant 0,$$

then $\operatorname{Dom}K=\operatorname{Dom}N$ and we can apply Corollary 3.2 with $M=c(N+1)$ and $k=c$ to deduce that U is a unitary process. As a particular example we take $\lambda(n)=-in^{1/2}$, $\mu(n)=0$. Then equation $(\mathrm{R})'$ reads

$$dU_t=[-\tfrac{1}{2}BB^\dagger\,dt-iB\,dA_t-iB^\dagger\,dA_t^\dagger]U_t,$$

where $B^\dagger=N^{1/2}W$ and $B=W^*N^{1/2}$ are the usual creation and annihilation operators on $\mathfrak{h}$. As shown in [Wal], this equation arises by considering the interaction of an inverse oscillator in a heat bath and taking the singular coupling limit. A strong solution of the QSDE is constructed in [Wal] by use of Maassen kernels, and analytical difficulties such as investigating the range of the U_t are surmountable there because of the simple algebraic structure of the equation for this special choice of λ and μ.

Perturbations of Hamiltonian evolutions: Let $\mathfrak{h}=L^2(\mathbb{R})$. We shall use the following vector spaces of functions on $\mathbb{R}$: $C_b^k(\mathbb{R})$, the space of bounded continuous functions with bounded continuous derivatives up to the order k; $C_c^\infty(\mathbb{R})$, the space of infinitely differentiable functions with compact support; and $H^k(\mathbb{R})$, the space of functions in $\mathfrak{h}$ that have weak derivatives up to order k in $\mathfrak{h}$. Let σ, ρ and ξ (resp. η) be $\mathbb{R}$-valued functions in $C_b^4(\mathbb{R})$ (resp. $C_b^5(\mathbb{R})$), m a positive constant, and define the triple (L,H,S) as operators on $H^2(\mathbb{R})$ by

$$Lu=\sigma u'+\rho u,\quad Hu=-\frac{1}{2m}u''-\frac{i}{2}(\eta u'+(\eta u)')+\xi u,\quad S=1$$

for $u\in H^2(\mathbb{R})$. So then L^*, restricted to $H^2(\mathbb{R})$, satisfies

$$L^*u=-\sigma u'+(\rho-\sigma')u.$$

Under the assumption $||\sigma^2||_\infty < m^{-1}$ it can be shown that $-\frac{1}{2}L^*L + iH$, restricted to $H^4(\mathbb{R})$, is closable and that its closure is the generator of a strongly continuous contraction semigroup [Kat, Theorem 2.7, p. 499]. Moreover, its domain coincides with $H^2(\mathbb{R})$. So we can apply Proposition 3.1 with $\mathfrak{D} = H^4(\mathbb{R})$ to obtain a contraction process U^* that is a solution to $(\mathrm{L})'$ on $\mathfrak{D}$.

Define the reference operator to be the positive self-adjoint operator C with domain $H^4(\mathbb{R})$ given by

$$Cu = u^{(4)} + u.$$

Lemma 4.1. *The hypotheses of* Proposition 3.4 *hold for the triple* (L, H, S) *and the reference operator* C *when we put* $\tilde{\mathfrak{D}} = H^8(\mathbb{R})$.

Proof. Since $\tilde{\mathfrak{D}} = \mathrm{Dom}\, C^2$ it follows that it is invariant under each R_ϵ, and clearly $\tilde{\mathfrak{D}}$ is contained in the domains of L^*L, L^*, $[C, L]$ and $[C, H]$, so that (i) holds. Also, (iv) holds trivially.

As a first step for verifying (ii) we compute the commutator $i[H, C]$. Denoting by ∂ the differentiation operator,

$$\begin{aligned}[\eta\partial + \partial\eta, C] &= [\eta, \partial^4]\partial + \partial[\eta, \partial^4] \\ &= -\eta'\partial^4 - 2\partial\eta'\partial^3 - 2\partial^2\eta'\partial^2 - 2\partial^3\eta'\partial - \partial^4\eta' \\ &= -3\partial\eta'\partial^3 - 2\partial^2\eta'\partial^2 - 3\partial^3\eta'\partial + \eta''\partial^3 - \partial^3\eta'' \\ &= -8\partial^2\eta'\partial^2 + 3\partial\eta''\partial^2 - 3\partial^2\eta''\partial - \partial^2\eta^{(3)} - \partial\eta^{(3)}\partial - \eta^{(3)}\partial^2 \\ &= -8\partial^2\eta'\partial^2 - 4\partial\eta^{(3)}\partial - \partial^2\eta^{(3)} - \eta^{(3)}\partial^2\end{aligned}$$

and

$$[\xi, C] = -2\partial\xi'\partial^2 - 2\partial^2\xi'\partial - \partial\xi^{(3)} - \xi^{(3)}\partial.$$

By the Cauchy–Schwarz inequality

$$\begin{aligned}|\langle u, [\eta\partial + \partial\eta, C]u\rangle| &\leqslant 8||\eta'||_\infty ||\partial^2 u||^2 + 4||\eta^{(3)}||_\infty ||\partial u||^2 + 2||\eta^{(3)}||_\infty ||u|| ||\partial^2 u|| \\ &\leqslant 8||\eta'||_\infty ||\partial^2 u||^2 + 6||\eta^{(3)}||_\infty ||\partial^2 u|| ||u|| \\ &\leqslant (8||\eta'||_\infty + 3||\eta^{(3)}||_\infty)\langle u, Cu\rangle\end{aligned}$$

and

$$2|\langle u, [\xi, C]u\rangle| \leqslant (6||\xi'||_\infty + 3||\xi^{(3)}||_\infty)\langle u, Cu\rangle$$

for all $u \in \tilde{\mathfrak{D}}$. Thus, we obtain

$$2|\langle u, i[H,C]u\rangle| \leqslant (8||\eta'||_\infty + 6||\xi'||_\infty + 3||\eta^{(3)}||_\infty + 3||\xi^{(3)}||_\infty)\langle u, Cu\rangle.$$

Similarly,

$$[C, L] = \sum_{0 \leqslant \alpha, \beta \leqslant 2} \partial^\alpha \varphi_{\alpha\beta} \partial^\beta,$$

where

$$\varphi_{22} = 4\sigma', \quad \varphi_{11} = \sigma^{(3)}, \quad \varphi_{10} = \varphi_{01} = \rho^{(3)}, \quad \varphi_{00} = 0,$$

$$\varphi_{12} = 2\rho' - 2\sigma'', \quad \varphi_{21} = 2\rho', \quad \varphi_{20} = 0, \quad \varphi_{02} = \sigma^{(3)},$$

so again even though C is a differential operator of order 4, and L a differential operator of order 1, their commutator has order 4. Taking adjoints, we have

$$[L^*, C] = \sum_{0 \leqslant \alpha, \beta \leqslant 2} (-1)^{\alpha+\beta} \partial^\beta \varphi_{\alpha\beta} \partial^\alpha.$$

Therefore, the term of order 5 in ∂ of the differential operator

$$\theta_0^0(C) = [L^*, C]L + L^*[C, L]$$

must come from

$$-\partial\sigma\partial^2\varphi_{22}\partial^2 + \partial^2\varphi_{22}\partial^2\sigma\partial,$$

and in fact its coefficient vanishes, so that $\theta_0^0(C)$ is actually of order at most 4. The same arguments used in the estimate of $|\langle u, i[H,C]u\rangle|$ yield the inequalities

$$|\langle u, \{[L^*, C]L + L^*[C, L]\}u\rangle| \leqslant c\langle u, Cu\rangle, \quad |\langle v, \theta_0^1(C)u\rangle| \leqslant c\langle \mathbf{u}, \imath(C)\mathbf{u}\rangle$$

for all u, $v \in \tilde{\mathfrak{D}}$ (and where we have set $\mathbf{u} = [u, v]^\mathsf{T}$), and for some constant c that depends only on σ, ρ and their derivatives up to the order 4. Thus, the inequality in part (ii) of Proposition 3.4 holds for

$$b_3 = 3c + 4||\eta'||_\infty + 3||\xi'||_\infty + \tfrac{3}{2}||\eta^{(3)}||_\infty + \tfrac{3}{2}||\xi^{(3)}||_\infty.$$

Finally, to show that (iii) holds, note that $\partial^{\alpha'} C^{-1} \partial^{\alpha}$ is a contraction for all $0 \leqslant \alpha, \alpha' \leqslant 2$, and so

$$\begin{aligned} \|C^{-1/2}[C,L]u\|^2 &= \sum_{0\leqslant \alpha,\beta,\alpha',\beta' \leqslant 2} \langle \partial^{\alpha'} \varphi_{\alpha'\beta'} \partial^{\beta'} u, C^{-1} \partial^{\alpha} \varphi_{\alpha\beta} \partial^{\beta} u \rangle \\ &= \sum_{0\leq \alpha,\beta,\alpha',\beta' \leqslant 2} (-1)^{\alpha'} \langle \varphi_{\alpha'\beta'} \partial^{\beta'} u, (\partial^{\alpha'} C^{-1} \partial^{\alpha}) \varphi_{\alpha\beta} \partial^{\beta} u \rangle \\ &\leqslant 9c' \sum_{0\leq \beta,\beta' \leqslant 2} \|\partial^{\beta'} u\| \|\partial^{\beta} u\|, \end{aligned}$$

where $c' = \max_{0\leq \alpha,\beta \leqslant 2} \|\varphi_{\alpha\beta}\|_{\infty}^2$. Thus

$$\|C^{-1/2}[C,L]u\|^2 \leqslant 9c' \left(\sum_{0\leqslant \beta \leqslant 2} \|\partial^{\beta} u\| \right)^2 \leqslant 27c' \sum_{0\leqslant \beta \leqslant 2} \|\partial^{\beta} u\|^2 \leqslant \tfrac{81}{2} c' \langle u, Cu \rangle.$$

This proves the lemma. □

Now note that $\mathrm{Dom}\, C^{1/2} \subset \mathrm{Dom}\, \overline{-\frac{1}{2}L^*L \pm iH} \cap \mathrm{Dom}\, L \cap \mathrm{Dom}\, L^*$, and so setting $\mathfrak{D}_{\epsilon} = H^4(\mathbb{R})$ for all ϵ we see that the conditions of Theorem 3.5 hold, noting part (b) of the remarks after Theorem 2.3. Thus, there is an isometric process U that is a strong solution to

$$dU_t = (L^*\, dA_t - L\, dA_t^{\dagger} + K\, dt) U_t$$

on $\mathrm{Dom}\, C^{1/2} = H^2(\mathbb{R})$. That U is a coisometry process can be shown by applying the arguments of [ChF], Section 5.1, where it is shown that the QDS associated to the model of heavy ion collision from [AlF] is conservative, this time taking M to be a multiple of $-\partial^2 + 1$.

The above argument can be modified by taking

$$Hu = -\frac{i}{2}(\eta u' + (\eta u)'),$$

and in this case we no longer need to impose the bound on $\|\sigma^2\|$ to show that K, the closure of $-\frac{1}{2}L^*L + iH$, is the generator of a strongly continuous contraction semigroup (see, for example, [F5], Theorem A.3). The rest of the calculation above can then be applied directly to prove the existence of an isometric solution to $(\mathrm{R})'$ for this new form of H. Moreover, it can be shown as in [F5, Chapter 4], that this solution is unitary. If we consider the algebra $L^{\infty}(\mathbb{R})$ acting by pointwise multiplication on $\mathfrak{h}$, then since we have

$$\theta_0^0(f) = \tfrac{1}{2}\sigma^2 f'' + (\sigma\sigma' - \sigma\rho + \eta) f',$$

we see that the flow $X \mapsto U_t^*(X \otimes 1)U_t$ gives a realisation of a diffusion process with covariance σ and drift $\sigma\sigma' - \sigma\rho + \eta$.

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