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Epimorphisms and Dominions. II[1]

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This is a sequel to an earlier paper [*3*]; the account will, however, be almost self-contained. Our object here is to present certain results about dominions in the category of semigroups, in particular about absolutely closed and saturated semigroups.

Recalling the principal definitions in [*3*], we say that a subsemigroup A of a semigroup B *dominates* an element d in B if, for an arbitrary semigroup C and arbitrary homomorphisms $f, g : B \to C$, $f(a) = g(a)$ for every a in A implies $f(d) = g(d)$. The set of elements of B dominated by A is a subsemigroup of B containing A, which we call the *dominion* of A. If the dominion of A is the whole of B we say that A is *epimorphically* embedded in B (for the inclusion mapping is an epimorphism in the usual categorical sense of being right cancellable). If a semigroup S is its own dominion in whatever semigroup it is embedded we call it *absolutely closed*; if S cannot be (properly) epimorphically embedded in any semigroup we call it *saturated*. It is shown in [*3*] (Example 3.3) that a saturated semigroup need not be absolutely closed.

The key to all the results in this paper is the "zigzag" theorem (2.3) in [*3*]. The commutative analog, which is not a corollary, is proved in Section 1. The proof in the commutative case is in fact a good deal simpler, being free of any appeal to topology.

Section 2 deals with absolutely closed semigroups. It follows from a result in [*2*] that groups are absolutely closed. Here we show that certain broader

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classes of regular semigroups (including inverse semigroups and full transformation semigroups) are absolutely closed. It does not seem unreasonable to hope for a complete theory of absolute closure for commutative semigroups, but we are far from achieving such an end. Theorem 3.9 in [3] and Theorem 2.6 in the present paper give (respectively) necessary conditions and sufficient conditions for a commutative semigroup to be absolutely closed.

In Section 3 we study saturated semigroups. All the results concern commutative semigroups; about the noncommutative case almost nothing is known, and the example (3.6) in [3], of a finite idempotent semigroup that is not saturated, indicates that a theory of noncommutative saturated semigroups would look very different from the commutative theory. We can state a reasonably concise necessary and sufficient condition for a finitely generated commutative semigroup S to be saturated: it must be "inverse closed", which is to say that an element a in S has an inverse if (for x, y in S^1) $a^2x = a^2y$ implies $ax = ay$. (Here and elsewhere S^1 denotes the semigroup S with a unit adjoined if necessary.)

1. Zigzags

The zigzag theorem (2.3) of [3] carries over to commutative semigroups, but the commutative theorem is not a corollary, since it might *a priori* be easier for a subsemigroup of a commutative semigroup to dominate an element with respect to homomorphisms into *commutative* semigroups.

If A is a subsemigroup of a (not necessarily commutative) semigroup B, a system of equalities

$$d = a_0y_1\,, \qquad a_0 = x_1a_1\,,$$

$$a_{2i-1}y_i = a_{2i}y_{i+1}\,, \qquad x_ia_{2i} = x_{i+1}a_{2i+1} \qquad (i = 1, 2,\dots, m-1),$$

$$a_{2m-1}y_m = a_{2m}\,, \qquad x_ma_{2m} = d \tag{1}$$

with $a_0, a_1,\dots, a_{2m}$ in A and $x_1, x_2,\dots, x_m, y_1, y_2,\dots, y_m$ in B will be called a *zigzag* of *length m* in B over A with *value d*. By the *spine* of the zigzag we shall mean the set of elements $a_0, a_1,\dots, a_{2m}$ (in that order).

THEOREM 1.1. *A subsemigroup A of a commutative semigroup B dominates an element d in B if and only if either $d \in A$ or there exists a zigzag in B over A with value d.*

Proof. We use Lemma 1.1 of [3]. In commutative semigroups the free sum $S*T$ of two objects S and T can be described as follows: first form $S^{(1)}$

and $T^{(1)}$ by adjoining an extra identity element 1 to each of S and T (whether or not they already have identities); then form the direct product of $S^{(1)}$ and $T^{(1)}$; then remove the element $(1, 1)$.

It is a routine matter to show that if a zigzag exists with value d, then $(i_1(d), i_2(d))$ belongs to the congruence $\mathfrak{A}^*$ on $B*B$ generated by

$$\mathfrak{A} = \{(i_1(a), i_2(a)); a \in A\}.$$

Conversely, suppose that A dominates d, so that $(i_1(d), i_2(d)) \in \mathfrak{A}^*$. Thus there is a sequence

$$(1, d) \to \cdots \to (d, 1) \tag{2}$$

of elementary $\mathfrak{A}$-transitions (in the sense of Clifford and Preston ([*1*], Section 1.4)) connecting $(1, d)$ and $(d, 1)$. Now, if

$$(x, y) \to (z, t) \tag{3}$$

is an $\mathfrak{A}$-transition, then either

$$(x, y) = (p, q)\,(a, 1)\,(r, s) \qquad \text{and} \qquad (z, t) = (p, q)\,(1, a)\,(r, s)$$

or

$$(x, y) = (p, q)\,(1, a)\,(r, s) \qquad \text{and} \qquad (z, t) = (p, q)\,(a, 1)\,(r, s)$$

$$(p, q, r, s \in S^{(1)}).$$

Let us call an $\mathfrak{A}$-transition of the first type an *r-step* (since the a moves right); one of the second type will be called an *l-step*. By commutativity we have that $x = za$ and $ay = k$ if the $\mathfrak{A}$-transition (3) is an r-step; and $xa = z$ and $y = at$ if it is an l-step. It is clear that two r-steps (corresponding to a and a', respectively) performed in succession can be collapsed to a single r-step (corresponding to $a'a$); a similar remark applies to l-steps. Hence we may assume that r- and l-steps occur alternately in the sequence (2). Since the element 1 has no divisors in A, the first and last $\mathfrak{A}$-transitions of the sequence (2) must be l-steps. There must therefore be an odd number (say $2m + 1$) of steps, the corresponding factorizations being necessarily of the form

$$d = a_0 y_1, \qquad a_0 = x_1 a_1,$$

$$a_1 y_1 = a_2 y_2, \qquad x_1 a_2 = x_2 a_3,$$

$$\cdots$$

$$a_{2m-3} y_{m-1} = a_{2m-2} y_m, \qquad x_{m-1} a_{2m-2} = x_m a_{2m-1},$$

$$a_{2m-1} y_m = a_{2m}, \qquad x_m a_{2m} = d,$$

with all a_i in A. This completes the proof of the theorem.

We end this section with two remarks on zigzags, which we shall have occasion to use later.

LEMMA 1.2. *Let A be a subsemigroup of a semigroup B and suppose that A dominates an element d in $B\backslash A$. Let (1) be a zigzag of minimum length with value d. Then*

(i) *$x_1a_2 \notin A$ and $a_{2m-2}y_m \notin A$; in particular, a_2 is neither equal to nor left-divisible by a_1, and a_{2m-2} is neither equal to nor right-divisible by a_{2m-1};*

(ii) *neither of the following two configurations can arise ($a', a'' \in A^1$):*

(a) $a_{2i-1} = a_{2i}a', \quad a''a_{2i} = a_{2i+1} \quad (i = 2, 3, \ldots, m-1)$;

(b) $a_{2i} = a'a_{2i+1}, \quad a_{2i+1}a'' = a_{2i+2} \quad (i = 1, 2, \ldots, m-2)$.

Proof. (i) If $x_1a_2 \in A$, we can clearly begin a shorter zigzag with $d = (x_1a_2)y_2$ instead of $d = a_0y_1$. Similarly, if $a_{2m-2}y_m \in A$, we can end a shorter zigzag with $x_{m-1}(a_{2m-2}y_m) = d$.

(ii) If we have the equalities (a), it follows easily that

$$x_{i-1}a_{2i-2} = x_{i+1}a^*, \qquad a^*y_i = a_{2i+2}y_{i+2},$$

where $a^* = a''a_{2i-1} = a_{2i+1}a' = a''a_{2i}a'$. Thus the zigzag can be shortened. This is also the case if we have the equalities (b).

2. ABSOLUTELY CLOSED SEMIGROUPS

Two zigzags in a semigroup B over a subsemigroup A will be called *equivalent* if they have the same spine. Two such zigzags must in fact have the same value; for if

$$d' = a_0t_1, \qquad a_0 = z_1a_1,$$
$$a_{2i-1}t_i = a_{2i}t_{i+1}, \qquad z_ia_{2i} = z_{i+1}a_{2i+1} \qquad (i = 1, 2, \ldots, m-1),$$
$$a_{2m-1}t_m = a_{2m}, \qquad z_ma_{2m} = d' \tag{4}$$

is a zigzag equivalent to the zigzag (1), then

$$d = a_0y_1 = z_1a_1y_1 = z_1a_2y_2 = z_2a_3y_2 = \cdots = z_ma_{2m} = d'.$$

A zigzag (1) will be called *left-inner* if $x_1, x_2, \ldots, x_m \in A$. Clearly in such a case $d = x_ma_{2m} \in A$. A useful notion in the investigation of absolutely closed semigroups is that of a *left-isolated* semigroup, that is, a semigroup A with the property that any zigzag over it (in any containing semigroup B)

is equivalent to a left-inner zigzag. Obviously, by virtue of the zigzag theorem in [*3*]:

LEMMA 2.1. *Left-isolated semigroups are absolutely closed.*

It turns out to be fairly easy to show that certain classes of semigroups are left-isolated; hence by the lemma they are absolutely closed. Much of this discussion is not symmetric; it is of course the case that the left-right duals of our theorems also hold.

First, consider a *left-simple* semigroup, that is, a semigroup A in which for every a, b, in A there exists a solution in A of the equation $xa = b$. Any zigzag (1) over A is equivalent to a left-inner zigzag (4), where $t_i = y_i$ for all i, z_1 is any solution in A of the equation $xa_1 = a_0$, z_2 is any solution in A of $xa_3 = z_1a_2$, and so on. Thus we have

THEOREM 2.2. *Left-simple semigroups are absolutely closed.*

Less trivial is the case of an *inverse semigroup*, defined as a semigroup A in which for every a there exists a unique x (called the *inverse* of a and in what follows denoted by $\bar{a}$) such that

$$axa = a, \qquad xax = x.$$

It is known (see [*1*, Section 1.9] for this and other standard results on inverse semigroups) that idempotents commute in such a semigroup. Also, $a\bar{a}$ and $\bar{a}a$ are idempotent,

$$\bar{\bar{a}} = a, \qquad \overline{ab} = \bar{b}\bar{a},$$

$\bar{e} = e$ if e is idempotent, and $\bar{a}\ e\ a$ is idempotent for any element a and any idempotent e.

THEOREM 2.3. *Inverse semigroups are absolutely closed.*

Proof. We show that any zigzag (1) over an inverse semigroup A is equivalent to a left inner zigzag (4), in which $t_i = y_i$ for every i. For $r = 1, 2, \ldots, m$, let $z_r = a_0u_r$, where

$$u_r = \bar{a}_1a_2\bar{a}_3a_4 \cdots \bar{a}_{2r-3}a_{2r-2}\bar{a}_{2r-1}\,.$$

Then clearly

$$a_0 = x_1a_1 = x_1a_1\bar{a}_1a_1 = a_0\bar{a}_1a_1 = z_1a_1\,.$$

We will show that $z_ra_{2r} = z_{r+1}a_{2r+1}$ for $r = 1, 2, \ldots, m - 1$. First we show inductively that

$$z_r = x_r\bar{u}_ru_r \qquad (r = 1, 2, \ldots, m - 1). \tag{5}$$

The result is immediate for $r = 1$. Also,

$$
\begin{aligned}
z_r &= z_{r-1}a_{2r-2}\bar{a}_{2r-1} \\
&= x_{r-1}\bar{u}_{r-1}u_{r-1}a_{2r-2}\bar{a}_{2r-1} \\
&= x_{r-1}(\bar{u}_{r-1}u_{r-1})\,(a_{2r-2}\bar{a}_{2r-2})\,a_{2r-2}\bar{a}_{2r-1} \\
&= x_{r-1}a_{2r-2}\bar{a}_{2r-2}\bar{u}_{r-1}u_{r-1}a_{2r-2}\bar{a}_{2r-1} && \text{(since idempotents commute)} \\
&= x_r a_{2r-1}\bar{a}_{2r-2}\bar{u}_{r-1}u_{r-1}a_{2r-2}\bar{a}_{2r-1} && \text{[by (1)];} \\
&= x_r\bar{u}_r u_r
\end{aligned}
$$

hence formula (5) is proved.

It now follows that

$$
\begin{aligned}
z_r a_{2r} &= x_r\bar{u}_r u_r a_{2r} \\
&= x_r(\bar{u}_r u_r)\,(a_{2r}\bar{a}_{2r})\,a_{2r} \\
&= x_r a_{2r}\bar{a}_{2r}\bar{u}_r u_r a_{2r} \\
&= x_{r+1}a_{2r+1}\bar{a}_{2r}\bar{u}_r u_r a_{2r} && \text{[by (1)].} \\
&= x_{r+1}a_{2r+1}(\bar{a}_{2r+1}a_{2r+1})\,(\bar{a}_{2r}\bar{u}_r u_r a_{2r}) \\
&= x_{r+1}a_{2r+1}\bar{a}_{2r}\bar{u}_r u_r a_{2r}\bar{a}_{2r+1}a_{2r+1} \\
&= x_{r+1}\bar{u}_{r+1}u_{r+1}a_{2r+1} \\
&= z_{r+1}a_{2r+1}
\end{aligned}
$$

This completes the proof.

The example (3.6) in [3] shows that not every regular semigroup is absolutely closed. Indeed it shows much more than this; to find a regular semigroup that is not absolutely closed one need look no further than the 2×2 rectangular band (see [1], p. 25). This will follow from Theorem 2.9.

The *full transformation semigroup* on a ground set G is defined to consist of all mappings of G into itself, with composition of mappings as the semigroup operation.

THEOREM 2.4. *Full transformation semigroups are absolutely closed.*

Proof. Again we show that if A, a full transformation semigroup (with ground set G) is embedded in an arbitrary semigroup B, then any zigzag in B over A is equivalent to a left-inner zigzag. Suppose we have a zigzag (1). Then, for $i = 1, 2, \ldots, m$, from the existence of $b_i, b_{i-1}, \ldots, b_1, c_i, c_{i-1}, \ldots, c_1$ in A such that

$$
\begin{array}{cc}
\multicolumn{2}{c}{a_{2i-1}b_i = a_{2i-1}c_i\,,} \\
a_{2i-2}b_i = a_{2i-3}b_{i-1}\,, & a_{2i-2}c_i = a_{2i-3}c_{i-1}\,, \\
a_{2i-4}b_{i-1} = a_{2i-5}b_{i-2}\,, & a_{2i-4}c_{i-1} = a_{2i-5}c_{i-1}\,, \\
\multicolumn{2}{c}{\cdots} \\
a_2b_2 = a_1b_1\,, & a_2c_2 = a_1c_1\,,
\end{array}
\tag{6}
$$

we can deduce that $a_0b_1 = a_0c_1$. For, beginning with $a_{2i-1}b_i = a_{2i-1}c_i$, we can successively deduce

$$x_ia_{2i-1}b_i = x_ia_{2i-1}c_i ,$$
$$x_{i-1}a_{2i-2}b_i = x_{i-1}a_{2i-2}c_i ,$$
$$x_{i-1}a_{2i-3}b_{i-1} = x_{i-1}a_{2i-3}c_{i-1} ,$$
$$x_{i-2}a_{2i-4}b_{i-1} = x_{i-2}a_{2i-4}c_{i-1} ,$$
$$\dots$$
$$x_1a_1b_1 = x_1a_1c_1 ,$$
$$a_0b_1 = a_0c_1 .$$

Now let p be some fixed element of G. For $i = 1, 2,\dots, m$, define $z_i \in A$ as follows: if there exist $g_i, g_{i-1},\dots, g_1$ in G such that

$$g = a_{2i-1}(g_i), \qquad a_{2i-2}(g_i) = a_{2i-3}(g_{i-1}),$$
$$a_{2i-4}(g_{i-1}) = a_{2i-5}(g_{i-2}),\dots, a_2(g_2) = a_1(g_1), \tag{7}$$

then $z_i(g) = a_0(g_1)$; otherwise $z_i(g) = p$. Then z_i is a well-defined mapping of G into itself. For suppose that $h_i, h_{i-1},\dots, h_1$ is another sequence of elements in G satisfying the conditions (7). Then for $k = 1, 2,\dots, i$ define the elements b_k, c_k of A by

$$b_k(g) = g_k , \qquad c_k(g) = h_k \qquad \text{for every } g \text{ in } G.$$

We obtain the equalities (6), and so it follows that $a_0b_1 = a_0c_1$, i.e., that $a_0(g_1) = a_0(h_1)$.

Also, $z_{i+1}a_{2i+1}(g) = a_0(g_1)$ if there exist $g_i, g_{i-1},\dots, g_1$ such that

$$a_{2i}(g) = a_{2i-1}(g_i), \qquad a_{2i-2}(g_i) = a_{2i-3}(g_{i-1}),$$
$$a_{2i-4}(g_{i-1}) = a_{2i-5}(g_{i-2}),\dots, a_2(g_2) = a_1(g_1), \tag{8}$$

and equals p otherwise. But Eqs. (8) constitute exactly the condition under which $z_ia_{2i}(g) = a_0(g_1)$. Hence

$$z_ia_{2i} = z_{i+1}a_{2i+1} \qquad (i = 1, 2,\dots, m-1),$$

and so (taking $t_i = y_i$ for every i) we have a left-inner zigzag (4) equivalent to the original zigzag (1). This completes the proof.

Corollary 2.5 (cf. [3], Corollary 1.8). *Every finite semigroup is embeddable in a finite absolutely closed semigroup.*

To describe the next class of absolutely closed semigroups we require some preliminary definitions. Some of these are already in [3], but we shall repeat them here for convenience. If c and d are two distinct elements of a semigroup S, then, following Šutov [5], we shall call c a *potential left divisor* of d if, for every, a, b, in S^1,

$$ac = bc \qquad \text{implies} \qquad ad = bd.$$

S will be called *left-division-ordered* if all potential left-divisors are actual divisors, and *left-totally division-ordered* if it is left-division-ordered and if, for any two distinct elements x and y in S, either x is a left-divisor of y or y is a left-divisor of x. The dual definitions are obvious. A semigroup will be called *totally division-ordered* if it is both left- and right-totally division-ordered.

We know ([3], Theorems 3.9 and 3.10) that absolutely closed commutative semigroups are division-ordered, but that not all division-ordered commutative semigroups are absolutely closed. The next theorem identifies a class of (not necessarily commutative) division-ordered semigroups that are absolutely closed.

THEOREM 2.6. *Totally division-ordered semigroups are absolutely closed.*

Proof. Let A, a totally division-ordered semigroup, be embedded in a semigroup B, and let d be an element in the dominion of A. Suppose that $d \in B \backslash A$ and that (1) is a zigzag of minimum length with value d.

By Lemma 1.2 (i), a_2 is neither equal to nor left-divisible by a_1; hence a_1 is left-divisible by a_2. By part (ii) of the same lemma it follows that a_3 is neither equal to nor right-divisible by a_2; hence a_2 is right-divisible by a_3. Hence a_4 is neither equal to nor left-divisible by a_3; hence a_3 is left-divisible by a_4; and so on. We end with the statement that a_{2m} is neither equal to nor left-divisible by a_{2m-1}. But $a_{2m} = a_{2m-1}y_m$, so that a_{2m-1} is a potential (and hence an actual) left-divisor of a_{2m}. This is a contradiction and so A is absolutely closed.

COROLLARY 2.7. *Finite monothetic semigroups are absolutely closed.*

Proof. If A is such a semigroup, then (see [1], Section 1.6]) A is commutative and has distinct elements

$$x, x^2, \dots, x^r, x^{r+1}, \dots, x^{r+m-1},$$

where

$$x^r = x^{r+m}.$$

The set of elements $\{x^r, x^{r+1}, \dots, x^{r+m-1}\}$ is a subgroup K of A. It is clear that if y and z are two distinct elements of A, then either y divides z or z divides y. It remains to verify that a semigroup of this type is division-ordered.

Let $c = x^k$ and $d = x^l$ be two distinct elements of A, and suppose that c is a potential divisor of d:

$$ac = bc \quad \text{implies} \quad ad = bd$$

for all a, b in A^1. If $k \geqslant r$, take $a = 1$ and $b = x^m$. Then $ac = bc$ and so $ad = bd$; that is, $x^l = x^{l+m}$. It follows that $l \geqslant r$ and so both c and d are in the subgroup K, where divisibility is automatic. If $k < r$, take $a = x^{r-k}$ and $b = x^{r-k+m}$. Then $ac = bc$ and so $ad = bd$: that is, $x^{r-k+l} = x^{r-k+l+m}$. It follows that $r - k + l \geqslant r$ and so (k and l being, by assumption, distinct) $l > k$. Thus c is a divisor of d.

Notice that the *infinite* monothetic semigroup is certainly not absolutely closed, being epimorphically embeddable in an infinite cyclic group. However, in ([*3*], Example 3.4), the infinite monothetic semigroup is embedded as a retract in a semigroup A which is not only (as stated in [*3*]) saturated, but even (as is easily verified) totally division-ordered. We conclude that *a retract of an absolutely closed semigroup need not even be saturated.* We do not know whether a direct product of two absolutely closed semigroups must be saturated, but it need not be absolutely closed: the direct product of two (monothetic) 2-element zero semigroups is a 4-element zero semigroup, which is not division ordered.

In a semigroup S we shall call an element u a *unit* if $uS = Su = S$. Clearly if S is finite then the set of nonunits in S is a subsemigroup.

THEOREM 2.8. *If the subsemigroup of nonunits of a finite semigroup S is absolutely closed, then so is S.*

Proof. Suppose that S is embedded in a semigroup T, that $d \in T\backslash S$ is in the dominion of S, and that (1) is a zigzag of minimum length with value d. By Lemma 1.2 (ii), no a_j ($j = 2, 3, \ldots, 2m - 2$) can have the property $a_jS = Sa_j = S$. By part (i) of the same lemma, a_2 is not left-divisible by a_1 and so $a_1S \neq S$; similarly, $Sa_{2m-1} \neq S$. Finally, $a_0S = x_1a_1S$ cannot contain any more elements than a_1S and so must be properly contained in S; similarly, $Sa_{2m} \neq S$.

We have shown that the zigzag (1) must in fact be a zigzag over the subsemigroup of nonunits of S. The result follows.

The next theorem implies in particular that the 2×2 rectangular band is not absolutely closed.

THEOREM 2.9. *If a semigroup S contains elements a_1, a_2, a_3 such that $a_1S \cap a_2S = Sa_2 \cap Sa_3 = \emptyset$, then S is not absolutely closed.*

Proof. Consider the free semigroup $\{x_1, x_2, y_1, y_2\}$ and let

$P = S*\{x_1, x_2, y_1, y_2\}$. Let T be the factor semigroup of P by the congruence generated by

$$\mathfrak{R} = \{(a_1, x_1a_1), (a_1y_1, a_2y_2), (x_1a_2, x_2a_3), (a_3y_2, a_3)\}.$$

We show that S is embedded in T with dominion properly containing S. Consider a sequence of elementary $\mathfrak{R}$-transitions conducted in P and beginning on an element of S. Any element of P obtained by performing such a sequence is of the type

$$w^1z^1w^2z^2 \cdots w^n,$$

where $w^1, w^2,..., w^n$ are (perhaps empty) elements in S and each of $z^1, z^2,..., z^{n-1}$ is either x_1 or y_2; moreover if $z^i = x_1$ and w^{i+1} is nonempty, then $w^{i+1} \in a_1S$, and if $z^i = y_2$ and w^i is nonempty, then $w^i \in Sa_3$. It is not hard to prove this by induction: the crucial point is that the second and third relations in $\mathfrak{R}$ can never be used, since

$$a_1S \cap a_2S = Sa_2 \cap Sa_3 = \emptyset.$$

As a consequence, two elements p and q in S are equal (in S) if they are connected by a sequence of elementary $\mathfrak{R}$-transitions in P; for we can produce an "image" sequence of equalities in S simply by leaving out all x_1's and y_2's. Hence $S \subset T$.

Moreover, we can conclude that the element $d = a_1y_1$ of T is not in S, since the corresponding element of P is not of the form described above. However,

$$\begin{aligned} d &= a_1y_1, & a_1 &= x_1a_1, \\ a_1y_1 &= a_2y_2, & x_1a_2 &= x_2a_3, \\ a_3y_2 &= a_3, & x_2a_3 &= d \end{aligned}$$

is a zigzag in T over S with value d. Thus d is dominated by S and so S is not absolutely closed.

3. Saturated Semigroups

In this section we shall be concerned exclusively with commutative semigroups.

In a commutative semigroup S, define S_1 to be S, S_2 to be the set of all elements having potential divisors in S^2, $S_{\alpha+1}$ as the set of all elements having potential divisors in $S_\alpha \cdot S$, and S_λ as $\bigcap \{S_\alpha : \alpha < \lambda\}$ if λ is a limit ordinal. Then $S_1 \supseteq S_2 \supseteq \cdots$ and so the descent must stabilize at some ordinal τ. We define $K = K(S)$ as $\bigcap \{S_\alpha : \alpha \leqslant \tau\}$. Then K is an ideal of S.

THEOREM 3.1. *If S is a commutative semigroup and $K(S)$ satisfies the minimum condition on principal ideals, then S is saturated.*

It is convenient to lay out the proof of this theorem in a series of lemmas, all of which hold under the hypothesis that S is properly epimorphically embedded in some semigroup U. We shall eventually derive a contradiction.

Before starting on the lemmas, we note that some power of each element a in K lies in a subgroup of K; for the descending sequence of principal ideals generated in turn by a, a^2, a^3,... must stabilize, and so for some n we have that a^n is a multiple $a^{2n}c$ of a^{2n}. It follows that a^n lies in the maximal subgroup containing the idempotent $a^n c$.

LEMMA 3.1.1. *If $d \in U \backslash S$, then every multiple of d that lies in S must necessarily lie in K.*

Proof. Suppose, by way of contradiction, that some $s = du$ is in S_β but not in $S_{\beta+1}$, and choose s to make β as small as possible. Since S dominates the whole of U by hypothesis, there is a zigzag over S with value d; in particular, $d = a_0 y_1 = x_1 a_1 y_1$. Since $d \in U \backslash S$ we can choose the zigzag to be as short as possible, in which case $x_1 \in U \backslash S$. Thus $x_1 a_1 = a_0$ is a multiple of x_1 lying in S; hence $x_1 a_1 = a_0 \in S_\beta$.

Now $s = a_0 \cdot y_1 u$, and either $y_1 u = a_0' \in S$ or $y_1 u = a_0' y_1'$ by the zigzag theorem (since the embedding of S in U is by assumption epimorphic). In either case $a_0 a_0'$ is a potential divisor of s, and $a_0 a_0' \in S_\beta \cdot S$. Hence $s \in S_{\beta+1}$, a contradiction.

LEMMA 3.1.2. *Every element b in $U \backslash S$ is a multiple of an idempotent f in K.*

Proof. By the zigzag theorem, $b = a_0 y_1 = x_1 a_1 y_1$, where $a_0, a_1 \in S$. In fact by the previous lemma, $a_0 = x_1 a_1 \in K$. If B is the set (non-empty, by virtue of the preceding remark) of elements in K dividing b, consider the set $\mathfrak{B}$ of principal ideals of K generated by the elements of B, and let k be any element in B for which the principal ideal generated by k is minimal in $\mathfrak{B}$. Then $b = kz$ for some z in $U \backslash S$.

Notice now that we have shown incidentally that *every* element in $U \backslash S$ is a multiple of some element in K. Applying this to z, we find that $z = k'u$, where kk' generates a principal ideal in K no smaller than that generated by k; hence, by virtue of the minimality assumption on k, we have that $k = kk'l$ for some l in K. In fact $k = k(k'l)^r$ for $r = 1, 2,...$. Now for some n there exists k'' in K such that $(k'l)^{2n} k'' = (k'l)^n$. Thus $(k'l)^n$ is a multiple of an idempotent $f = (k'l)^n k''$ in K. Hence $k = k(k'l)^n$ is a multiple of f and so $b = fb$ as required.

We shall have occasion shortly to draw attention to the exact manner in which this divisor was found.

For two idempotents e, f in K we shall write $f \leqslant e$ if the principal ideal generated by f is contained in that generated by e, or (equivalently) if $ef = f$. There can be no infinite descending chain of idempotents in K.

LEMMA 3.1.3. *For each b in $U \backslash S$ there is a smallest idempotent e in K dividing b.*

Proof. If $e_1 b = b$ and $e_2 b = b$ then $e_1 e_2 b = b$. Thus the set F of idempotent divisors of b in K is a subsemilattice of the semilattice of all idempotents in K. Clearly F can have no infinite descending chain and so there must be a least element e in F.

Now fix b, and note that $eS = eK$ (since K is an ideal). Note also that eK is properly contained in eU, since $b \in eU \backslash eK$.

LEMMA 3.1.4. *The subsemigroup eK is epimorphically embedded in eU.*

Proof. Because of commutativity, if a zigzag exists in U over S with value d, then a zigzag exists in eU over eS $(= eK)$ with value ed.

In the final lemma, H_e denotes the maximal subgroup of K containing e.

LEMMA 3.1.5. *There exists an element z in $eU \backslash eK$ whose only divisors in eK are the elements of H_e.*

Proof. Applying the argument used to prove Lemma 3.1.2, we find an element k in eK and an element z in $eU \backslash eK$ such that $b = kz$. If k' is a divisor of z in eK, then (again as in the proof of Lemma 3.1.2) there exists k^* $(= (k')^{n-1} l^n k'')$ in K such that $k'k^* = f$, an idempotent in K. This f is an idempotent factor of b and so $e \leqslant f$. Hence $k'(k^* e) = fe = e$ and so $k' \in H_e$. Thus z is the element we require.

We can now complete the proof of the theorem. By the last lemma, any zigzag in eU over eK with value z must in fact be a zigzag over H_e. But H_e, being a group, is absolutely closed and so we have a contradiction.

We remark that a simplified version of this proof (in which Lemma 3.1.1 is unnecessary and in which S replaces K in the other lemmas) establishes

THEOREM 3.2. *A commutative semigroup satisfying the minimum condition on principal ideals is saturated.*

An interesting property of domination in commutative semigroups is given by the next theorem.

THEOREM 3.3. *If a commutative semigroup S is embedded epimorphically in a commutative semigroup U, then S^n dominates every element of $U \backslash S$, for every positive integer n.*

Proof. The result follows from two lemmas, in which all semigroups are assumed to be commutative.

LEMMA 3.3.1. *If U is the dominion of S in T, then S^n dominates U^n.*

Proof. Let $u = u^1u^2 \cdots u^n$ be an element of U^n (where $u^i \in U$ for each i). Then for $i = 1, 2,..., n$, we have a zigzag

$$\begin{aligned} u^i &= a_0{}^i y_1{}^i, & a_0{}^i &= x_1{}^i a_1{}^i, \\ a_1{}^i y_1{}^i &= a_2{}^i y_2{}^i, & x_1{}^i a_2{}^i &= x_2{}^i a_3{}^i, \\ & \cdots \\ a_{2m-1}^i y_m{}^i &= a_{2m}^i, & x_m{}^i a_{2m}^i &= u^i, \end{aligned}$$

since we can easily arrange for all the n zigzags to have the same length by inserting repetitions if necessary. Using commutativity we obtain a zigzag of the form (1) over S^n with value u, where

$$\begin{aligned} x_r &= x_r{}^1 x_r{}^2 \cdots x_r{}^n & (r &= 1, 2,..., m). \\ y_r &= y_r{}^1 y_r{}^2 \cdots y_r{}^n & (r &= 1, 2,..., m). \\ a_r &= a_r{}^1 a_r{}^2 \cdots a_r{}^n & (r &= 0, 1,..., 2m). \end{aligned}$$

LEMMA 3.3.2. *If S is embedded epimorphically in U, then $U \backslash S \subseteq U^n$.*

Proof. Let d be an element of $U \backslash S$ and let (1) be a zigzag of minimum length with value d. Then $d = x_1 a_1 y_1$, where $x_1, y_1 \in U \backslash S$. Thus $d \in U^3$. The same argument can now be applied to x_1 (or y_1); clearly in this way we show that $d \in U^n$ for any positive integer n.

A further simple consequence of the lemmas is

COROLLARY 3.4. *If S^n is saturated for some positive integer n, then S is saturated.*

For if S were epimorphically embedded in U, then S^n would be epimorphically embedded in U^n. Thus $U \backslash S \subseteq U^n = S^n \subseteq S$ and so $S = U$.

Theorem 3.1 gives a sufficient condition for a commutative semigroup to be saturated. The next result gives a necessary condition. First, let us call a commutative semigroup *inverse closed* if an element a in S has an inverse (in the usual semigroup sense) whenever a^2 is a potential divisor of a.

THEOREM 3.5. *Commutative saturated semigroups are inverse closed.*

Proof. Let S be a saturated semigroup and suppose that there exists a in S such that a^2 is a potential divisor of a, but a has no inverse. Then a^2 is not an actual divisor of a, for $a = a^2x$ would imply that x^2a was an inverse of a. By a result of Šutov [5] (see also [3], Theorem 3.9) one can embed S in a semigroup T containing an element x such that $a^2x = a$. The element xa of T does not belong to S, for if it did we should have that a^2 was a diviso of a in S. Hence the subsemigroup U generated in T by S and xax properl contains S. However, there is a zigzag

$$xax = a(xax)^2, \qquad a = (xax)^2 a^3, \qquad a^3(xax)^2 = a, \qquad (xax)^2 a = xa$$

in U over S with value xax and so the dominion of S in U (being a subsemi group of U) must coincide with U. That is, S is embedded epimorphicall in U, in contradiction to our supposition that S is saturated.

A converse to Theorem 3.5 can be stated for finitely generated semigroups

THEOREM 3.6. *A finitely generated, commutative, inverse closed semigrou is saturated.*

Proof. We show that a semigroup S satisfying these conditions must als satisfy the minimum condition on principal ideals, which is sufficient b Theorem 3.2.

First, by a result of Rédei [4], all congruences on S are finitely generated that is, every congruence is generated by a finite subset of $S \times S$. It follow that from every subset of $S \times S$ generating a given congruence ρ on S w can extract a finite subset which still generates ρ. There cannot exist a infinite ascending chain of congruences on S, for if $\rho_1 \subset \rho_2 \subset \rho_3 \subset \cdots$ wer such a chain, we could choose a finite set of generators $\mathfrak{R}_i$ for each ρ_i such tha $\mathfrak{R}_1 \subset \mathfrak{R}_2 \subset \mathfrak{R}_3 \subset \cdots$. Then $\bigcup \rho_i$ would be a congruence on S having an infinit set $\bigcup \mathfrak{R}_i$ of generators no finite subset of which would suffice to generate it.

In S, by virtue of commutativity, the relation $\rho(a)$ defined by

$$(x, y) \in \rho(a) \qquad \text{if} \qquad xa = ya$$

is a congruence (for any element a in S). Moreover,

$$\rho(a) \subseteq \rho(a^2) \subseteq \rho(a^3) \subseteq \cdots$$

and so for some n we must have that

$$\rho(a^{n-1}) = \rho(a^n) = \rho(a^{n+1}) = \cdots = \rho(a^{2n}) = \cdots.$$

It follows that a^{2n} is a potential divisor of a^n: the only case that is not immediately obvious is where $a^{2n} = a^{2n}y$ for some y in S, in which case we observe that $(a, ay) \in \rho(a^{2n-1}) = \rho(a^{n-1})$, so that $a^n = a^ny$ as required. Since S is inverse closed, there must therefore exist b in S such that $a^{2n}b = a^n$.

Writing e for the idempotent a^nb, we note that a^n belongs to the maximal subgroup H_e. Also

$$a^{n+1}e = a^{2n+1}b = a \cdot a^n = a^{n+1} \qquad \text{and} \qquad a^{n+1} \cdot a^{n-1}b^2 = a^{2n}b^2 = e^2 = e;$$

hence $a^{n+1} \in H_e$ also. Thus a^{n-1} divides a^n.

Applying this argument to each member of any finite set $\{g_1, g_2, \ldots, g_m\}$ of generators of S, we find that there is (for $j = 1, 2, \ldots, m$) a positive exponent $e(j)$ such that $g_j^{e(j)+1}$ divides $g_j^{e(j)}$.

Now suppose that elements $s_1, s_2, s_3, \ldots$ of S generate an infinite descending sequence of principal ideals, where

$$s_i = g_1^{p_{i1}} g_2^{p_{i2}} \cdots g_m^{p_{im}} \qquad (i = 1, 2, 3, \ldots),$$

with $p_{ij} \geqslant 0$. We can suppose that for $j = 1, 2, \ldots, m$,

$$p_{1j} \leqslant p_{2j} \leqslant p_{3j} \leqslant \cdots.$$

Now for each j, either p_{ij} stops increasing, so that $p_{ij} \leqslant q_j$ (say) for every i, or else p_{ij} increases indefinitely. We can assume that the first possibility occurs for $j = 1, 2, \ldots, r$ and the second for $j = r + 1,\ r + 2, \ldots, m$. For each j between 1 and r, let $i(j)$ be the smallest i for which $p_{ij} = q_j$; let

$$q = \max \{i(j) : j = 1, 2, \ldots, r\}.$$

Thus in every member of the sequence

$$s_q, s_{q+1}, s_{q+2}, \ldots,$$

g_j $(1 \leqslant j \leqslant r)$ occurs with exponent q_j. For $j = r + 1,\ r + 2, \ldots, m$, let l_j be the smallest $i > q$ for which $p_{ij} \geqslant e(j)$; let

$$l = \max \{l_j : j = r + 1, r + 2, \ldots, m\}.$$

Then s_i divides s_l if $i > l$, a contradiction to our assumption that the elements $s_1, s_2, s_3, \ldots$ generate an infinite descending sequence of principal ideals. This completes the proof.

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