J. Differential Equations 245 (2008) 2522–2533

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Small amplitude limit cycles for the polynomial Liénard system ☆

Maciej Borodzik, Henryk Żołądek *

Institute of Mathematics, University of Warsaw, ul. Banacha 2, 02-097 Warsaw, Poland

ARTICLE INFO

Article history:
Received 1 October 2007
Revised 2 April 2008
Available online 15 August 2008

MSC:
primary 45C05
secondary 37G15

Keywords:
Liénard system
Limit cycles

ABSTRACT

We estimate for the maximal number of limit cycles bifurcating from a focus for the Liénard equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$, where f and g are polynomials of degree m and n respectively. These estimates are quadratic in m and n and improve the existing bounds. In the proof we use methods of complex algebraic geometry to bound the number of double points of a rational affine curve.

1. The results

Consider the Liénard vector field

$$\dot{x} = y - F(x), \qquad \dot{y} = -G'(x), \tag{1.1}$$

where F and G are polynomials of degree $m+1$ and $n+1$ respectively. It is related with the second order Liénard equation via the formulas $f(x) = F'(x)$, $g(x) = G'(x)$. The principal problem concerning the system (1.1) is to find a maximal number $H(m,n)$ of its limit cycles (a special case of the Hilbert's 16th problem). In this paper we study a weaker problem, we ask about the number of small limit cycles.

☆ Supported by Polish KBN Grant No. 1 P03A 015 29.
* Corresponding author.
E-mail addresses: mcboro@mimuw.edu.pl (M. Borodzik), zoladek@mimuw.edu.pl (H. Żołądek).

0022-0396/$ – see front matter
doi:10.1016/j.jde.2008.07.032

We assume that the origin $x = y = 0$ is a singular point of the center or focus type. Therefore

$$F(x) = a_1 x + \cdots + a_{m+1} x^{m+1}, \qquad G(x) = b_2 x^2 + \cdots + b_{n+1} x^{n+1}, \tag{1.2}$$

where $a_1^2 < 8b_2$. We can also assume that

$$b_2 = 1. \tag{1.3}$$

When we introduce the local analytic variable

$$u = \sqrt{G(x)} = x + \cdots,$$

then the system (1.1) becomes orbitally equivalent to

$$\dot{u} = y - \Phi(u), \qquad \dot{y} = -2u, \qquad \Phi = c_1 u + c_2 u^2 + \cdots. \tag{1.4}$$

Here the series

$$X = c_1 Y^{1/2} + c_2 Y + c_3 Y^{3/2} + \cdots \tag{1.5}$$

is the Puiseux expansion at the point $X = Y = 0$ of the curve

$$C: \quad X = F(x), \quad Y = G(x). \tag{1.6}$$

It is well known, see [3], that the system (1.1) (equivalently, the system (1.4)) has center at the origin if and only if $c_1 = c_3 = \cdots = 0$, i.e. $\Phi(u) = \widetilde{\Phi}(u^2)$ is an even function. From the algebraic point of view this means that the curve (1.6) is *multiply covered* (or *non-primitive*). By the Lüroth theorem (see [7]) we have $F(x) = \widetilde{F} \circ \omega(x)$, $G(x) = \widetilde{G} \circ \omega(x)$ for a polynomial $\omega(x) = x^2 + \cdots$. From the dynamical point of view this means that the system (1.4) is *time–reversible* and the system (1.1) is *rationally reversible*, i.e. it can be pushed forward via the map $(x, y) \to (\omega(x), y)$.

The coefficients $c_1, c_3, c_5, \ldots$ are called the *essential Puiseux quantities* of the singularity $X = Y = 0$ of the curve C (see [1]). They are related with the *Poincaré–Lyapunov quantities* $g_1, g_3, \ldots,$ which appear in the Taylor expansion of the Poincaré return map

$$r \to P(r) = r + g_1 r(1 + \cdots) + g_3 r^3 (1 + \cdots) + \cdots, \quad r \to 0^+, \tag{1.7}$$

from the section $\{(x, y) = (r, 0)\colon\ r \geqslant 0)\}$ to itself. Namely, g_j are proportional to c_j with coefficients depending only on j. We refer the reader to [5] for details.

Since the fixed points of the map (1.7) correspond to the limit cycles of the Liénard vector field, the essential Puiseux quantities of the curve C become responsible for the small amplitude limit cycles of the system (1.1).

The quantities c_j and g_j depend on the coefficients a_k and b_l in the polynomials F and G (see (1.2)). In fact, they are polynomials in $a = (a_1, \ldots, a_{m+1})$ and $b = (b_3, \ldots, b_{n+1})$, e.g. for $b_2 = 1$. So the expansion (1.6) varies with varying (a, b). This variation results in bifurcation of fixed points of the map $P(r)$ from the point $r = 0$ (the generalized Hopf bifurcation). For instance, when $g_{2\nu+1} \neq 0$ and the coefficients $g_1, g_3, \ldots, g_{2\nu-1}$ vary independently, then they can be chosen such that either

$$\begin{gathered} 0 < g_1 \ll -g_3 \ll g_5 \ll \cdots \ll \pm g_{2\nu+1}, \quad \text{or} \\ 0 < -g_1 \ll g_3 \ll -g_5 \ll \cdots \ll \mp g_{2\nu+1}. \end{gathered} \tag{1.8}$$

Thus one finds exactly ν limit cycles of small amplitude.

Since $g_j(a,b)$ are real polynomials, one cannot ensure free choice of signs, like in (1.8) (although the functions g_j may be independent).

C. Christopher and S. Lynch in [5] introduced the following quantities:

$\widehat{H}(m,n)$—the maximal number of limit cycles which can bifurcate from the origin;
$H^*(m,n)$—the maximal cyclicity of the focus at $x=y=0$, i.e. $\max\{\nu\colon\ c_1=c_3=\cdots=c_{2\nu-1}=0\neq c_{2\nu+1}\}$;
$\widehat{H}_{\mathbb{C}}(m,n)$—the maximal number of limit cycles bifurcating from the origin in the complex sense, i.e. $\frac{1}{2}\times$ maximal number of zeroes $r_i\neq 0$ of the function $P(r)-r$ for $r\in(\mathbb{C},0)$ (counted with multiplicities);
$H^*_{\mathbb{C}}(m,n)$—the maximal cyclicity of $x=y=0$ in the complex sense.

In the definitions of $\widehat{H}_{\mathbb{C}}(m,n)$ and $H^*_{\mathbb{C}}(m,n)$ one assumes complex coefficients a_i, b_j and considers the complex foliation defined by (1.1) in $(\mathbb{C}^2,(0,0))$.

We have the following simple relations

$$\widehat{H}(m,n)\leqslant H^*(m,n)\leqslant H^*_{\mathbb{C}}(m,n)=\widehat{H}_{\mathbb{C}}(m,n). \tag{1.9}$$

In [4] Cristopher and Lloyd proved the general inequality

$$H^*_{\mathbb{C}}(m,n)\leqslant\frac{1}{2}mn. \tag{1.10}$$

In the proof they used the Bezout theorem for estimating the number of finite solutions of the following system

$$\frac{F(x)-F(x')}{x-x'}=\frac{G(x)-G(x')}{x-x'}=0. \tag{1.11}$$

In [5] Christopher and Lynch stated several conjectures concerning the above quantities. To formulate them we introduce the space $\mathcal{X}$ of curves of the form (1.6) with F, G like in (1.2), thus $\mathcal{X}\simeq\mathbb{C}^{m+n+1}$. This space is acted on by a group $\mathcal{G}$ of equivalences of curves, generated by:

- rescalings $x\to\alpha x$, $X\to\beta X$, $Y\to\gamma Y$;
- elementary Cremona transformations $X\to X+\mathrm{const}\cdot Y^j$, $1\leqslant j\leqslant[(m+1)/(n+1)]$, if $n\leqslant m$; or of the form $Y\to Y+\mathrm{const}\cdot X^j$ if $m<n$.

These changes have no influence on the property of vanishing of successive coefficients c_{2j-1}. Therefore the equations $c_1=c_3=\cdots=c_{2\nu-1}=0$ can be regarded as equations on the quotient space $\mathcal{X}/\mathcal{G}$. They define varieties in $\mathcal{X}$ composed of whole orbits of the action of $\mathcal{G}$ on $\mathcal{X}$.

If $n\leqslant m$ and $\frac{m+1}{n+1}\notin\mathbb{Z}$ then there exists one (exceptional) orbit, which contains the quasi-homogeneous curve $F(x)=x^{m+1}$, $G(x)=x^{n+1}$, of dimension $2+[\frac{m+1}{n+1}]$; other orbits have dimension $3+[\frac{m+1}{n+1}]$. If $(m+1)=k(n+1)$ then the orbit of the curve $F(x)=x^{m+1}$, $G(x)=x^{n+1}$ has dimension $1+k$ and other orbits have dimension $3+k$. Also for $m<n$ there is such division.

Since we assume $b_2\neq 0$, the first case for $n\leqslant m$ (i.e. with quasi-homogeneous curve) occurs when $n=1$ (and $G(x)=x^2$). But here $c_j=a_j$ and the problem is elementary: we have $\widehat{H}(m,1)=\widehat{H}_{\mathbb{C}}(m,n)=[\frac{m}{2}]$, where $[\cdot]$ denotes the integer part. When $m,n\geqslant 2$ we have the following

Conjecture 1. *(See [5].)*

1. $\widehat{H}_{\mathbb{C}}(m,n)=\widehat{H}_{\mathbb{C}}(n,m)=m+n-2-[\frac{m+1}{n+1}]$ *for* $2\leqslant n\leqslant m$;
2. $\widehat{H}(m,n)=\widehat{H}(n,m)$;
3. $H^*(m,n)=H^*(n,m)$.

Remark 1. In [5] one finds the following conjectured bounds $\widehat{H}_{\mathbb{C}}(m,n)=[\frac{n(m+2)}{n+1}]+n-3$ for $2\leqslant n<m$ (which agrees with the above) and $\widehat{H}_{\mathbb{C}}(n,n)=2n-4+[\frac{2}{n}]$ (which is stronger than above).

Christopher and Lynch proved the formula $\widehat{H}(m,2)=[\frac{2m+1}{3}]=m-[\frac{m+1}{3}]$, using some Petrov's [15] ideas. They also proved that $\widehat{H}(m,3)=2[\frac{2(m+2)}{8}]$ when $3\leqslant m\leqslant 50$ and $\widehat{H}_{\mathbb{C}}(m,3)=[\frac{3(m+2)}{4}]$ when $6\leqslant m\leqslant 50$. They found examples where $\widehat{H}_{\mathbb{C}}(m,3)>\widehat{H}(m,3)$ (e.g. $\widehat{H}_{\mathbb{C}}(7,3)=7$ and $\widehat{H}(7,3)=6$).

Also other computer calculations confirm the above conjecture.

We do not prove the Christopher–Lynch conjecture in this paper (although initially we aimed at it). We are able to show the following quadratic bounds for $H^*_{\mathbb{C}}(m,n)$. Introduce the number

$$\delta_{\max}=\delta_{\max}(m,n)=\frac{1}{2}\big(mn-\gcd(m+1,n+1)+1\big); \tag{1.12}$$

in the next section we interpret $\delta_{\max}$ as the maximal number of double points of a curve of the form (1.6). The following result slightly improves the Christopher–Lloyd bound and is proved in the next section.

Theorem 1. *If $m,n\geqslant 2$ then $H^*_{\mathbb{C}}\leqslant\delta_{\max}-1$.*

In the next result we replace the factor $\frac{1}{2}$ in (1.10) and (1.12) with $\frac{1}{4}$. We prove it in Section 3.

Theorem 2. *If $m,n\geqslant 2$ and the curve C has one-branch singularity at $X=Y=0$ then*

$$H^*_{\mathbb{C}}\leqslant\frac{1}{4}(mn+3m+3n+1).$$

Remark 2. The bound from Theorem 2 still holds true when more than one branches of C go through $X=Y=0$. However the proof is more involved. Namely, the proof of Lemma 7 below becomes much more technical.

2. Double points of a curve via a Hamiltonian vector field

If $A\subset(\mathbb{C}^2,0)$ is a germ of holomorphic curve defined by $H(X,Y)=0$ then the (complex) Hamiltonian vector field

$$V_H=H'_Y\partial_X-H'_X\partial_Y$$

is tangent to A. Below we shall regard V_H as a real vector field in $\mathbb{R}^4$ (i.e. with real time). One can check that the real field V_H is also Hamiltonian with $\operatorname{Re}H$ as the Hamilton function, but with respect to the symplectic structure given by $d\operatorname{Re}X\wedge d\operatorname{Re}Y-d\operatorname{Im}X\wedge d\operatorname{Im}Y$.

We denote $W:=V_H|_A$. If 0 is an isolated singular point of A, then we consider the normalization $N:\widetilde{A}\to A$; thus each topological component $\widetilde{A}_j$, $j=1,\dots,r$ of $\widetilde{A}$ (preimage of an analytic component A_j of A) is a disc. The pull-back $\widetilde{W}:=N^*W=(N_*)^{-1}W\circ N$ of the vector field W is a vector field on the smooth manifold $\widetilde{A}$ with isolated equilibrium points $p_j\in N^{-1}(0)$, $j=1,\dots,r$. Thus one can define the indices $i_{p_j}\widetilde{W}$.

We call the quantity

$$\delta_0=\delta_0(A):=\frac{1}{2}\sum_j i_{p_j}\widetilde{W} \tag{2.1}$$

the *number of double points of A hidden at* 0. In the literature δ_0 is sometimes called the *δ-invariant* of the singularity or the *virtual number of double points*. The next lemma justifies this definition.

Lemma 1. *The number δ_0 equals to the number of simple double points of a typical perturbation N' of the normalization map $N : \widetilde{A}_1 \sqcup \cdots \sqcup \widetilde{A}_r \to \mathbb{C}^2$.*

Proof. If, after perturbation, in the disc $\widetilde{A_j}$ there remain only preimages of simple double points then the number of such preimages equals to the sum of indices of the vector field $\widetilde{W}'|_{\widetilde{A}_j} = (N')^* V_{H'}|_{\widetilde{A}_j}$, where H' defines the perturbed curve. But this is exactly the index of the field $\widetilde{W}'$ along ∂A_j. The latter index equals the index of the field $\widetilde{W}|_{\widetilde{A}_j}$ at p_j.

Summing-up all this over j we get twice the number of double points of the perturbation. $\square$

Lemma 2. *We have*

$$\delta_0(A) = \sum_j \delta_0(A_j) + \sum_{i<j} (A_i \cdot A_j)_0, \tag{2.2}$$

where $(A_i \cdot A_j)_0$ is the intersection number at 0 of the components A_i and A_j. In particular,

$$\delta_0(A) \geqslant \frac{1}{2} r(r-1) \geqslant r-1. \tag{2.3}$$

Proof. Let $N : (\mathbb{C}, 0) \to (A_j, 0)$, $z \to (X(z), Y(z))$ be the local parametrization (normalization) of A_j. Assume also that the coordinates X, Y are such that A_j does not lie in the line $X = 0$. Then we get $\dot{z} = (dX/dz)^{-1}(\partial H/\partial Y)|_{A_j}$ and $i_{p_j}\widetilde{W} = \operatorname{ord}_{z=0}(dX/dz)^{-1}(\partial H/\partial Y)|_{A_j}$. If $H = H_1 \cdots H_r$, where H_j defines A_j, then $\operatorname{ord}_{z=0}(dX/dz)^{-1}(\partial H/\partial Y)|_{A_j}$ equals

$$\operatorname{ord}_{z=0}(dX/dz)^{-1}(\partial H_j/\partial Y)|_{A_j} + \sum_{i \neq j} \operatorname{ord}_{z=0} H_i|_{A_j} = 2\delta_0(A_j) + \sum_{i \neq j} (A_i \cdot A_j)_0.$$

This gives Eq. (2.2). $\square$

Consider now the curve C of the form (1.6), where we assume that $a_{m+1} b_2 b_{n+1} \neq 0$.

Lemma 3. *The quantity ν for the curve* (1.6) *such that $c_1 = c_3 = \cdots = c_{2\nu-1} = 0 \neq c_{2\nu+1}$* (*i.e. the codimension of the singularity $x = 0$ of a parametrized curve*) *equals δ_0, the number of double points at the singularity $X = Y = 0$ of C* (*which is of the type $\mathbf{A}_{2\nu}$*).

Proof. The $\mathbf{A}_{2\nu}$ singular curve $H = X^2 - Y^{2\nu+1}$ admits parametrization $X = z^{2\nu+1}$, $Y = z^2$. Then the Hamiltonian vector field restricted to this curve takes the form $\dot{z} = -H'_X/2z = -z^{2\nu}$. $\square$

Remark 3. In [1,10,11] it is proved that the number $2\delta_0$ for a *cuspidal singularity*, i.e. with only one branch ($r = 1$), equals the Milnor number of this singularity.

Denote by $\xi = (F, G) : \mathbb{C} \to C$ the parametrization of the curve C and let $H(X, Y) = 0$ be the equation for C. The extension of the map ξ to a map from $\mathbb{CP}^1$ is the normalization of the closure $\overline{C} = C \cup p_\infty \subset \mathbb{CP}^2$ of the curve C. We define a (real) vector field W on $\overline{C}$, or $\widetilde{W}$ on $\mathbb{CP}^1$, by the formula

$$\widetilde{W}(x) = \chi(x) \cdot (\xi^* V_H)(x), \quad x \in \mathbb{CP}^1 \setminus \infty.$$

Here $\chi(x) > 0$ is a smooth function tending to 0 as $x \to \infty$ in a way that $\widetilde{W}$ becomes smooth at ∞. Namely, in the variable $z = 1/x$ the pull-back vector field $\xi^* V_H$ usually has pole, $\xi^* V_H = z^{-\alpha}(c + \cdots)\frac{d}{dz}$ for $c \neq 0$. Then we put $\chi(x) = |z|^{2\alpha}$ near $z = 0$. We find that

$$i_\infty \widetilde{Y} = -\alpha. \tag{2.4}$$

Lemma 4. *If C has only simple double points as singularities, then their number equals*

$$\delta := 1 - \frac{1}{2} i_\infty \widetilde{W}.$$

For general curve C the number $\delta = 1 - \frac{1}{2} i_\infty \widetilde{W}$ equals the sum of the numbers of double points hidden at the (finite) singular points of C.

Proof. It follows from the Poincaré–Hopf formula, Eq. (2.4) and $\chi(\mathbb{CP}^1) = 2$. □

Let us calculate the number $i_\infty \widetilde{W}$ in terms of the Puiseux expansion of the curve C at infinity:

$$\begin{aligned} Y &= X^{\frac{u_1}{p_1}}(d_1 + \cdots) + X^{\frac{u_2}{p_1 p_2}}(d_1 + \cdots) + \cdots + X^{\frac{u_r}{p_1 \cdots p_r}}(d_1 + \cdots) \\ &= X^{\frac{v_1}{m+1}}(d_1 + \cdots) + X^{\frac{v_2}{m+1}}(d_2 + \cdots) + \cdots + X^{\frac{v_r}{m+1}}(d_r + \cdots). \end{aligned} \tag{2.5}$$

Here $p_j > 1$ for $j \geqslant 2$, $\deg F = m + 1 = p_1 \cdots p_r$, $\gcd(u_j, p_j) = 1$ and $v_1 > v_2 > \cdots > v_r$. The coefficients $d_j \neq 0$ and the dots denote power series in $X^{1/p_1 \cdots p_j}$ in the jth summand. Moreover, $v_1 = \deg G = n + 1$. The pairs $(p_1, u_1), (p_2, u_2), \ldots, (p_r, u_r)$ are often called the *characteristic pairs* (at infinity). We call the expansion (2.5) the *topologically arranged Puiseux expansion.*

Lemma 5. *The number $i_\infty \widetilde{W}$ equals*

$$2 - \big\{(v_1 - 1)(p_1 - 1)p_2 \cdots p_r + (v_2 - 1)(p_2 - 1)p_3 \cdots p_r + \cdots + (v_r - 1)(p_r - 1)\big\}.$$

In particular, the number of double points of C equals

$$\delta = \frac{1}{2} \sum (v_j - 1)(p_j - 1)p_{j+1} \cdots p_r. \tag{2.6}$$

Proof. Formula (2.6) is well known in the literature. It is the same formula as the formula for the Milnor number of a cuspidal singularity via the characteristic pars given in [11]. Using the Hamiltonian differential equation on C, i.e. $\dot{X} = H'_Y$, it can be proved as follows.

In the local variable $z = 1/x$ we get $\dot{z} = z^{-m} H'_Y(c + \cdots)$ for some constant $c \neq 0$. So we have to calculate the order of $H'_Y|_C$ at $z = 0$.

Formula (2.5) gives one branch $Y = f_{\zeta^*}(x)$ of the multi-valued solution to the equation $H(X, Y) = 0$. All branches $Y = f_\zeta(X)$ of this solution take the form

$$\zeta_1\big\{d_1 X^{\frac{v_1}{m+1}} + \cdots + \zeta_r\big\{d_r X^{\frac{v_1}{m+1}} + \cdots\big\}\cdots\big\},$$

where the coefficient ζ_1 takes p_1 values, ζ_2 takes p_2 values, etc. We have $\zeta^* = (1, \ldots, 1)$.

The polynomial H can be represented in the form $H = \prod_\zeta (Y - f_\zeta(X))$ near infinity and $H'_Y|_C = \prod_{\zeta \neq \zeta^*}(Y - f_\zeta(X))$. In the latter product we have $(p_1 - 1)p_2 \cdots p_r$ factors with $\zeta_1 \neq 1$ and of order $X^{\frac{v_1}{m+1}} \sim z^{-v_1}$ each, we have $(p_2 - 1)p_3 \cdots p_r$ factors of order z^{-v_2}, etc. We find $\mathrm{ord}_{z=0} H'_Y = -\sum v_j(p_j - 1)p_{j+1} \cdots p_r$.

Together with $(m + 1) - 1 = \sum (p_j - 1)p_{j+1} \cdots p_r$, this gives the thesis of the lemma. □

Note that when $m + 1 = l(n + 1)$, then $p_1 = 1$, $v_1 = l$ and the first term in the sum in (2.6) gives zero contribution to δ.

Lemma 6. *The number δ is maximal when either*:

(i) *$m + 1$ and $n + 1$ are relatively prime (here $\delta = \delta_{\max} = \frac{1}{2} mn$), or*

(ii) *when there are exactly two essential terms in the expansion* (2.5): $d_1 X^{(n+1)/(m+1)} + d_2 X^{n/(m+1)}$ (*here* $\delta_{\max} = \frac{1}{2}(mn - p_2 + 1)$ *where* $p_2 = \gcd(m+1, n+1)$).

These numbers agree with (1.12). We obtain the following bound, which is weaker than in Theorem 1.

Corollary 1. $H^*_{\mathbb{C}}(m, n) \leqslant \delta_{\max}$.

Remark 4. The latter bound can be obtained also by estimating the number of finite solutions to the system (1.11) using the Bezout theorem and taking into account solutions at infinity.

In order to improve this estimate we use the following theorem of M. Zaidenberg and V. Lin.

Theorem 3. *(See [17].) If an algebraic curve of the form* (1.6) *has only one singular point which is cuspidal then it is equivalent to a quasi-homogeneous curve.*

Proof of Theorem 1. If the curve C is not equivalent to a quasi-homogeneous curve then the Zaidenberg–Lin theorem says that it must have another double point (simple or hidden at another singularity). Hence the number of double points hidden at the point $X = Y = 0$ does not exceed $\delta_{\max} - 1$.

So it remains to consider the possibilities when C could be reduced to a quasi-homogeneous curve using Cremona transformations, like $X \to X + aY^j$.

This cannot occur when none of the ratios $\frac{m+1}{n+1}$ and $\frac{n+1}{m+1}$ is integer and $m, n \geqslant 1$. Indeed, by the condition $b_2 \neq 0$ the curve should be equivalent to $X = x^{2\nu+1}$, $Y = x^2$. So the Cremona transformations should result in decreasing the bi-degree $(m+1, n+1)$. Our assumption about the ratios forbids it.

Suppose, for instance, that $(m+1) = k(n+1)$ and a change $X \to X + aX^k$ gives a curve with $X = F_1(x)$, $\deg F_1 = m_1 + 1 < m + 1$. Then like in Lemma 6 one checks that $\delta_{\max}(m_1, n) \leqslant \delta_{\max}(m, n) = \frac{1}{2}kn(n+1)$ and the equality takes place only when $m_1 = m - 1$. In the latter case the curve cannot be reduced to a quasi-homogeneous one. □

3. The Bogomolov–Miyaoka–Yau inequality

This section could be not easy for specialists in ODEs. It uses methods of complex geometry of open algebraic surfaces. This theory was developed mainly by Japanese geometers (see [6]) and one of its main results is the Bogomolov–Miyaoka–Yau (BMY) inequality which we present below. We will apply it using as few algebro-geometric language as possible.

Recall that a *divisor* on a smooth compact complex 2-dimensional manifold V is a linear combination of closed complex curves with integer coefficients. Such a divisor (modulo some equivalence) can be viewed either as an element in $H^2(V, \mathbb{Z})$ or as an element in the homology group $H_2(V, \mathbb{Z})$. The latter point of view allows to define intersection product of divisors.

If we have a line bundle (i.e. with the fiber $\mathbb{C}$) over V then we can associate with it a divisor of zeroes and poles of a meromorphic section of this bundle, the corresponding element in $H_2(V, \mathbb{Z})$ is dual to the *first Chern class* of the bundle. Conversely with any divisor we can associate a line bundle whose first Chern class equals this divisor. A special case of such a bundle is the canonical bundle $K = K_V$ associated with the sheaf Ω^2_V of holomorphic 2-forms on V. The corresponding (class of) divisor is called the *canonical divisor*. We refer the reader to [7] and [8] for more informations on divisors and Chern classes.

Let V be a complex projective surface, K its canonical divisor and D a divisor of the form $D = \sum D_i$, where D_i are smooth projective curves on V and each D_i intersects other D_j's transversally. We say that D is a *normal crossings divisor*.

Definition 1. We say that the *logarithmic Kodaira dimension* of (V, D) equals $-\infty$ if none of the line bundles associated with the divisors $n(K + D)$, $n = 1, 2, \dots$, has a nontrivial holomorphic section.[1]

Definition 2. We say that the pair (V, D) is *relatively minimal* if none of the components D_i can be blown down. In other words, either $D_i^2 \neq -1$, or D_i is rational with $D_i^2 = -1$ but D_i intersects the divisor $D - D_i$ in at least three points.

Theorem 4. *(BMY inequality,[2] see [9].) Let V be a complex projective surface, K its canonical divisor and $D = \sum D_i$ is a normal crossings divisor in V. Assume moreover that the logarithmic Kodaira dimension of (V, D) is not $-\infty$ and the pair (V, D) is relatively minimal.*

Then

$$(K + D)^2 \leqslant 3\chi(V \setminus D), \tag{3.1}$$

where χ denotes the classical, topological Euler characteristic.

Let us consider the curve C from Section 1 in $\mathbb{CP}^2$. More precisely, we consider the closure of this curve, which we still denote by C. Recall that C: $X = F(x)$, $Y = G(x)$, where F and G are polynomials of degree $m + 1$ and $n + 1$ respectively. Recall also that C has $\mathbf{A}_{2\nu}$ type singularity at $X = Y = 0$.

C is a singular rational curve. Let $z_1, \dots, z_k$ be singular points of C with $z_1 = (0, 0)$ and z_k the only point at infinity (when it is singular). Let r_i be the number of branches of C around the singular point z_i.

Then, as C is rational, its Euler characteristics is equal to

$$\chi(C) = 2 - \sum_{i=1}^{k} (r_i - 1).$$

In fact, C is topologically a sphere with sets of r_i-tuples of points (for $i = 1, \dots, k$) glued together. We denote

$$R = \sum_{i=1}^{k} (r_i - 1). \tag{3.2}$$

Let us *resolve* the singularities of C. This means that we construct a map

$$(V, \tilde{C}) \stackrel{\pi}{\mapsto} \left(\mathbb{CP}^2, C\right)$$

such that $\tilde{C}$ is smooth, π is a composition of blow-downs and $\pi^{-1}(C) = \tilde{C} + \sum a_i E_i$ as divisors, where $\sum a_i E_i$ is the sum of *exceptional divisors*. Assume that the resolution is *minimal* in the sense that there is no other triple $(V', \tilde{C}', \pi')$ sharing the same properties as $(V, \tilde{C}, \pi)$ such that π' is a composition of fewer blow-downs than π.

Define

$$D = \tilde{C} + \sum E_i,$$

which from the algebro-geometric point of view can be regarded as taking a reduced inverse image of C. Here $\tilde{C} = \pi^*(C)$ is the *strict transform* of C.

[1] The logarithmic Kodaira dimension is defined via growth of dimensions of the spaces $H^0(V, n(K + D))$ as $n \to \infty$ (see [6]).

[2] The BMY inequality for compact complex surfaces states that $c_1^2 \leqslant 3c_2$ where c_1 and c_2 are the Chern classes of the surface. Here c_2 is the Euler class and its integral equals the Euler characteristic.

Lemma 7. *If the curve C has a self-intersection, i.e. $R \geqslant 1$ in* (3.2), *then the logarithmic Kodaira dimension of* (V, D) *is not* $-\infty$ *and the pair* (V, D) *is relatively minimal.*

Proof. The first statement, i.e. about the logarithmic Kodaira dimension, follows from a theorem of I. Wakabayashi [16].

In order to show the relative minimality of the pair (V, D) we have to show that no component of D is a rational curve D_i such that $D_i^2 = -1$ and $D_i(D - D_i) \leqslant 2$ (see [2,6] and Definition 2). The components of D are the exceptional divisors E_i and the strict transform $\tilde{C}$ itself. By the minimality of resolution for each E_i either $E_i^2 < -1$ or $E_i(D - E_i) \geqslant 3$; otherwise we could contract the curve E_i and obtain a resolution with smaller number of blowing ups. On the other hand, if $R \geqslant 1$ the strict transform $\tilde{C}$ intersects other components of D in at least three points. □

Recall that by the Zaidenberg–Lin Theorem 3 and by the proof of Theorem 1 the curve C without self-intersections can be reduced to a quasi-homogeneous curve via Cremona transformations which reduce the bi-degree of C. It is easy to show that in this case the δ-invariant $\delta_1 \leqslant \frac{1}{2}\max(m+1, n+1)$. So in the sequel we assume that C is not simply connected and we can use the BMY inequality (3.1).

In order to make use of the BMY inequality we have to identify both of its sides. Let us begin with the right-hand side.

Observe that by definition $V \setminus D$ is isomorphic to $\mathbb{CP}^2 \setminus C$. In particular, the Euler characteristics of both spaces coincide. Therefore

$$\chi(V \setminus D) = \chi\left(\mathbb{CP}^2 \setminus C\right) = 1 + R. \tag{3.3}$$

The latter equality results from the additivity of the Euler characteristics.

Let us now deal with the left-hand side of (3.1). We will explain how this is done in this particular case. The general case is done in [2].

Let W_l be the subspace of $H_2(V, \mathbb{Q})$ spanned by components of $\pi^{-1}(z_l)$. Let W_0 be the subspace of $H_2(V, \mathbb{Q})$ spanned by the class H of a strict transform of a line in $\mathbb{CP}^2$ not passing by any of the points z_j.

Lemma 8. *(See [13].) We have the orthogonal* (*with respect to the intersection form*) *decomposition*

$$H_2(V, \mathbb{Q}) = \bigoplus_{i=0}^{k} W_i.$$

We denote by K_i and D_i the orthogonal projections of K and D onto the space W_i respectively. We have then

$$\begin{aligned}(K + D)^2 = {} & D(K + D) + K_0(K_0 + D_0) + K_1(K_1 + D_1) \\ & + \sum_{i=2}^{k-1} K_i(K_i + D_i) + K_k(K_k + D_k).\end{aligned} \tag{3.4}$$

It is easy to identify the first term in the right-hand side of Eq. (3.4). By the genus formula (see [7]) we have

$$D(K + D) = 2g_a - 2, \tag{3.5}$$

where g_a is the *arithmetic genus* of the reducible curve D.[3]

[3] The arithmetic genus g_a (or p_a) of an algebraic curve D on an algebraic surface V is defined as $\frac{1}{2}\chi(\mathcal{O}_D) + 1 = \frac{1}{2}(h^0(\mathcal{O}_D) - h^1(\mathcal{O}_D)) + 1$ and equals $\frac{1}{2}D(K_V + D) + 1$. If D is smooth then $K_D = K_V + D$ restricted to D (it is the adjunction formula or

Lemma 9. *The arithmetic genus $g_a(D)$ is equal to R defined in* (3.2).

Proof. The curve C is rational. Thus its geometric genus is zero. Now each r_i-tuple point increases the arithmetic genus by $r_i - 1$. $\square$

Also it is rather easy to deal with the term $K_0(K_0 + D_0)$.

Lemma 10. $K_0(K_0 + D_0) = -3\deg C + 9$.

Proof. Assume $n \leqslant m$; then the degree of C is equal to $m+1$. Therefore C intersects a generic line H in $\mathbb{CP}^2$ at $m+1$ points. If these points differ from $z_0, \dots, z_k$ the strict transform of H intersects D still at $m+1$ points. Thus $D_0 = (m+1)H$. Now the canonical divisor on $\mathbb{CP}^2$ can be represented as the class $-3H$ in $H_2(\mathbb{CP}^2, \mathbb{Z})$ (see [7]). By similar arguments as above we get $K_0 = -3H$. The lemma follows from the fact that $H \cdot H = 1$. $\square$

Now we pass to the other parts of Eq. (3.4). We have to interpret and estimate each intersection number $K_i(K_i + D_i)$ associated with a singular point z_i of the curve C.

Probably S. Orevkov [12] was the first who interpreted it as the codimension of the singularity in the cuspidal case, i.e. when $r_i = 1$; he calls it the *rough M-number*. If, in some local analytic coordinates x, y near z_i, we have the topologically arranged Puiseux expansion

$$x = \tau^p, \qquad y = x^{\frac{q_1}{p_1}}(d_1 + \cdots) + \cdots + x^{\frac{q_s}{p_1 \cdots p_s}}(d_s + \cdots) \tag{3.6}$$

(compare Eq. (2.5)) with the *multiplicity* $p = p_1 \cdots p_s$ and $1 < \frac{q_1}{p_1} < \cdots < \frac{q_1}{p_1 \cdots p_s}$ then the *codimension* of this singularity equals $p-2$ plus the number of vanishing essential terms in this Puiseux expansion. Explicitly we have

$$K_i(K_i + D_i) = (p-2) + \sum_{j=1}^{s}\left(q_j - 1 - \left[\frac{q_j - 1}{p_j}\right]\right). \tag{3.7}$$

Here $p-2$ is the number of conditions that $d^i x/d\tau^i = 0$, $i = 1, \dots, p-1$, at some point, $q_1 - 1 - [(q_1 - 1)/p_1]$ is the number of terms x^{j/p_1}, j/p_1 not integer, which are absent in (3.6), etc.

The proof uses a subtle analysis of the dual graph of the singularity which encodes the intersection matrix in the space W_i from Lemma 8. We refer the reader to the papers [13] and [14] by Orevkov and Zaidenberg. In [2] we generalize the interpretation of $K_i(K_i + D_i)$ as a codimension of singularity to the non-cuspidal case.

In particular, formula (35) of [14] directly implies the following

Lemma 11. *For $1 \leqslant i \leqslant k-1$ the intersection number $K_i(K_i + D_i)$ is non-negative. For $i = 1$, where the singularity at z_1 is of type $\mathbf{A}_{2\nu}$, it is equal to ν.*

For the rough M-number $K_k(K_k + D_k)$ associated with the cuspidal singularity at infinity we have a more subtle bound.

an algebro-geometric variant of the Gelfand–Leray form) and $D(K_V + D)$ is the degree of K_D. If $\tilde{D}$ is the normalization of D then $g_a(D) = g_a(\tilde{D}) + \sum \delta_z$, where δ_z are the numbers of double points of D at the singular points z. In particular, if D is a connected union of m rational curves with r simple double points as the only singularities (of D) then $g_a(D) = 1 - m + r$. All this can be found in the Hartshorne's book [8].

Lemma 12. *For $n < m$ we have*

$$K_k(K_k + D_k) \geqslant (m-n-1) + m - \left[\frac{m}{m-n}\right].$$

Proof. The singularity at infinity of the curve C can be locally parametrized by $s = 1/x$ as follows: $Y/X = s^{m-n} + \cdots$, $1/X = s^{m+1} + \cdots$. Then the first two terms in the right-hand side of Eq. (3.7) give our inequality. On the other hand, this inequality follows directly from Proposition 3 in [14] (see also [2]). □

From Theorem 4, Eqs. (3.3), (3.4) and (3.5) and Lemmas 9, 10, 11 and 12 we get that

$$\nu + 2R - m - n - \left[\frac{m}{m-n}\right] + 3 \leqslant (K+D)^2, \quad n < m.$$

Thus BMY inequality implies that

$$\nu \leqslant m + n + \left[\frac{m}{m-n}\right] + R.$$

To make the inequality more transparent, we bound $[\frac{m}{m-n}]$ by $\frac{1}{2}(m+n)$, getting finally

$$\nu \leqslant \frac{3}{2}(m+n) + R + \frac{1}{2}. \tag{3.8}$$

In case $m < n$ we have to switch from $\frac{m}{m-n}$ to $\frac{n}{n-m}$. In case $m = n$ we apply a Cremona change $Y \to Y + \text{const} \cdot X$, so n becomes smaller. Anyway, in all cases inequality (3.8) still holds true.

We need another inequality relating R and ν. That one will be the consequence of computing the indices of the Hamiltonian vector field introduced in Section 2. Namely, observe that by Lemma 6 (in Section 2) we get $\nu + \sum_{i=2}^{k-1} \delta_k \leqslant \delta_{\max}$, where $\delta_{\max} = \frac{1}{2}(mn - \gcd(n+1, m+1) + 1)$ and δ_i is the δ-invariant of the ith singular point. For the sake of transparency we estimate $\gcd(n+1, m+1)$ by 1.

Now by Lemma 3 (see (2.3)) $\delta_i \geqslant r_i - 1$. So the above inequalities yield

$$R \leqslant \frac{1}{2}mn - \nu.$$

Then substituting R into (3.8) yields

$$\nu \leqslant \frac{1}{2}(mn + 3n + 3m + 1) - \nu.$$

This concludes the proof of Theorem 2.

Remark 5. This proof allows some improvement at the cost of legibility. We can consider the resolution not of the curve C but the curve $C+$ (line at infinity) and study carefully the cases when the line at infinity becomes a (-1)-curve. We can show that $(K+D)^2 \geqslant \nu + 2R - n - m + [\frac{m+1}{n+1}] + 1$ (if $m < n$). The whole procedure is explained in [2] and is technically quite complicated. Anyway, the bounds we obtain have the same leading term (i.e. $\nu \leqslant \sim \frac{1}{4}mn$) as the one we have written here.

Acknowledgments

We would like to thank the referee of JDE for pointing out the reference [4] and for convincing us to give the algebro-geometric proof of the $\frac{1}{4}mn$ bound.

References

[1] M. Borodzik, H. Żołądek, Complex algebraic curves via Poincaré–Hopf formula. I. Parametric lines, Pacific J. Math. 229 (2) (2007) 307–338.

[2] M. Borodzik, H. Żołądek, Complex algebraic plane curves via Poincaré–Hopf formula. III. Codimension bounds, preprint, University of Warsaw, 2007; available at: http://www.mimuw.edu.pl/~mcboro/pliki/artykuly/c4.pdf.

[3] L.A. Cherkas, On the conditions for a center for certain equations of the form $yy' = P(x) + Q(x)y + R(x)y^2$, Differ. Equ. 8 (8) (1974) 1104–1107; in Russian: Differ. Uravn. 8 (8) (1972) 1435–1439.

[4] C. Christopher, N.G. Lloyd, Small-amplitude limit cycles in polynomial Liénard systems, Nonlinear Differ. Equ. Appl. 3 (1996) 183–190.

[5] C. Christopher, S. Lynch, Small-amplitude limit cycle bifurcations for Liénard systems with quadratic damping or restoring forces, Nonlinearity 12 (1999) 1099–1112.

[6] T. Fujita, On the topology of non-complete algebraic surfaces, J. Fac. Sci. Univ. Tokyo (Ser. 1A) 29 (1982) 503–566.

[7] P. Griffiths, J. Harris, Principles of Algebraic Geometry, J. Wiley & Sons, New York, 1978.

[8] R. Hartshorne, Algebraic Geometry, Springer-Verlag, New York, 1993.

[9] R. Kobayashi, S. Nakamura, F. Sakai, A numerical characterization of ball quotients for normal surfaces with branch loci, Proc. Japan Acad. Ser. A Math. Sci. 65 (1989) 238–241.

[10] A. Lins-Neto, Algebraic solutions of polynomial differential equations and foliations in dimension two, in: Holomorphic Dynamics, Mexico, 1986, in: Lecture Notes in Math., vol. 1345, Springer-Verlag, Berlin–Heidelberg–New York, 1988, pp. 193–232.

[11] J. Milnor, Singular Points of Complex Hypersurfaces, Ann. of Math. Stud., vol. 61, Princeton University Press, Princeton, 1968.

[12] S.Yu. Orevkov, On rational cuspidal curves. I. Sharp estimate for degree via multiplicities, Math. Ann. 324 (2002) 657–673.

[13] S.Yu. Orevkov, M.G. Zaidenberg, Some estimates for plane cuspidal curves, in: Seminaire d'Algébre et Geometrie, Grenoble, 1993, pp. 1–13.

[14] S.Yu. Orevkov, M.G. Zaidenberg, On the number of singular points of plane curves, in: Algebraic Geometry, Saithana, 1995.

[15] G.S. Petrov, Number of zeroes of complete elliptic integrals, Funct. Anal. Appl. 18 (1984) 148–149; in Russian: Funktsional. Anal. i Prilozhen. 18 (2) (1984) 73–74.

[16] I. Wakabayashi, On the logarithmic Kodaira dimension of the complement of a curve in $\mathbf{P}^2$, Proc. Japan Acad. Ser. A Math. Sci. 54 (1978) 157–162.

[17] M.G. Zaidenberg, V.Ya. Lin, An irreducible, simply connected algebraic curve in C^2 is equivalent to a quasi-homogeneous curve, Dokl. Akad. Nauk SSSR 271 (1983) 1048–1052 (in Russian).