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(0, 1)-Matrices with No Half–Half Submatrix of Ones

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We consider the minimum number of zeroes in a $2m \times 2n$ $(0, 1)$-matrix M that contains no $m \times n$ submatrix of ones. We show that this number, denoted by $f(m, n)$, is at least $2n + m + 1$ for $m \leq n$. We determine exactly when this bound is sharp and determine the extremal matrices in these cases. For any m, the bound is sharp for $n = m$ and for all but finitely many $n > m$. A general upper bound due to Gentry, $f(m, n) \leq 2m + 2n - \gcd(m, n) + 1$, is also derived. Our problem is a special case of the well-known Zarankiewicz problem.

1. INTRODUCTION

We consider rectangular matrices M with entries that are 0 or 1. The intersection of a rows and b columns of a matrix is called an $a \times b$ *submatrix*. Wunsch [10] asked: What is the least number of zeroes in a square $(0, 1)$-matrix M of even order, say $2n \times 2n$, such that M contains no $n \times n$ submatrix of ones? We solve his problem here and go on to propose an extension of it to general rectangular matrices. We say that a $2m \times 2n$ matrix M has *Property Z* if every $m \times n$ submatrix has at least one zero, i.e. M has no half–half all ones submatrix. An equivalent formulation of Property Z, that is typically more useful in our study, is to require that for every m rows of M at least $n+1$ columns contain a zero somewhere in those rows. We denote by $f(m, n)$ the minimum number of zeroes in such a matrix M with Property Z. For simplicity, we often assume that $m \leq n$, since we may switch to the transpose when $m > n$. One can ask equivalently for the maximum number of ones in such a matrix M, which is $4mn - f(m, n)$.

The graph-theoretic formulation is obtained by viewing M as the incidence matrix for a bipartite graph. Then $4mn - f(m, n)$ is the maximum number of edges of a bipartite graph (A, B) with part sizes $|A| = 2m$ and $|B| = 2n$, such that there is no complete bipartite subgraph $K_{m,n}$ with m vertices in A and n vertices in B.

We recognize this as a particular case of the famous problem of Zarankiewicz [11] from 1951, which is far from settled. The general problem asks for the maximum number of edges, denoted by $Z_{a,b}(k, l)$, of a bipartite graph (K, L) with $|K| = k$, $|L| = l$ that contains no subgraph $K_{a,b}$ with a vertices in K and b vertices in L. Our number f is then given by

$$f(m, n) = 4mn - Z_{m,n}(2m, 2n).$$

Most results in the literature on the Zarankiewicz problem assume that the dimensions a, b of the forbidden submatrix are fixed and consider large dimensions k, l. However, in our problem, a and b are growing along with k and l. We shall see that $Z_{m,n}(2m, 2n)$ is so close to $4mn$ that it is helpful to work instead with our new $f(m, n)$ notation, which grows only linearly with m and n.

We now briefly mention some of the results on the Zarankiewicz problem. Culik [3] has shown that if a, b and k are fixed, then

$$Z_{a,b}(k, l) = (a-1)l + (b-1)\binom{k}{a}, \quad \text{for} \quad l \geq (b-1)\binom{k}{a}.$$

Another result, due to Roman [9], covers our half–half case:

$$Z_{a,b}(k, l) \leq \frac{b-1}{\binom{p}{a-1}}\binom{k}{a} + \frac{(p+1)(a-1)}{a} l, \quad \text{for integers} \quad p \geq a-1.$$

Znam [12] has shown that

$$Z_{a,a}(k,k) \le (a-1)^{\frac{1}{a}} k\left(k - \frac{3}{8}(a-1)\right)^{\frac{a-1}{a}} + k\frac{(a-1)}{2}, \quad \text{for} \quad a \ge 2.$$

We obtain much better bounds than these for our half–half case.

Since we found it advantageous to work with $(0,1)$-matrices, rather than their associated bipartite graphs, we present our results in matrix terminology. However, the graph-theoretic applications are at least as important, and most of the related literature is in graph-theoretic language. A survey of work on the Zarankiewicz problem appears in Bollobás [2, Sec. VI.2]. Some of the more recent work includes the papers [1, 5, 8].

In Section 2 of this paper, we present constructions of some matrices with Property Z and few zeroes. In particular, we present constructions, which turn out to be optimal, for the cases

(1) m divides n (including $m = n$), and
(2) $m < n$ where $n = km + r$, $1 < r < m$, such that $k + r \ge m$. So the constructions apply, given m, for all but finitely many n.

A recursive construction then gives this result that yields Gentry's general upper bound formula:

THEOREM 1.1. *Let $1 \le m < n$, and put $n = km + r$, where $1 \le r \le m$. Then*

$$f(m,n) \le 2km + f(r,m).$$

COROLLARY 1.2 ([6]). *For general m, n, we have*

$$f(m,n) \le 2m + 2n - \gcd(m,n) + 1.$$

Here, $\gcd(m,n)$ is the greatest common divisor of m and n. Corollary 1.2 was discovered by Craig Gentry [6]. We first circulated our work, presenting the problem, our main result (Theorem 1.3 below, giving a sharp lower bound on f), and the constructions for Cases (1) and (2). Gentry, then a student in a summer research program, learned of this problem from his supervisor, Anant Godbole. He sent us this result (without proof), and we devised our proof via the recursive construction described by the stronger Theorem 1.1.

We now state our main result, a lower bound for $f(m,n)$ which is sharp in exactly the two cases given by the constructions of Section 2. It includes the solution to Wunsch's original problems.

THEOREM 1.3. *Assume that $m \le n$. Let $f(m,n)$ denote the least number of zeroes for a $2m \times 2n$ $(0,1)$-matrix having Property Z. Then*

$$f(m,n) \ge 2n + m + 1,$$

where the equality holds precisely when:

(1) n is a multiple of m, or
(2) $k + r \ge m$, where $n = km + r$, and $0 < r < m$.

Notice that if we fix m and let n increase, then condition (2) of Theorem 1.3 is true. So $f(m,n) = 2n + m + 1$ for fixed m and all sufficiently large n; specifically for $n > (m-1)^2$.

To prove the theorem, we associate to any $2m \times 2n$ $(0,1)$-matrix a certain graph on $2m$ vertices (which may have multiple edges, but need not be bipartite), which we call the *associated graph*. We prove in Section 3 a proposition about the existence of equipartitions of the vertex set of graphs such that there is at most one edge between the parts. So, even though we adopt the matrix interpretation of our problem, most of our arguments end up being graph-theoretic, concerning equipartitions.

We thus require some basic graph theory notation. If $G = (V, E)$ is a graph (or multigraph), then $\nu(G) = |V|$ and $\varepsilon(G) = |E|$. The disjoint union of two graphs G and H is denoted $G + H$, and the disjoint union of k copies of G is denoted kG. For a bipartite graph (A, B), we denote the number of its edges by $\varepsilon(A, B)$.

The proof of Theorem 1.3 follows in Section 4. In Section 5, we describe in Theorem 5.1 all extremal matrices for Theorem 1.1. They are unique up to permutations in Case (1), but not necessarily unique in Case (2). The statement of Theorem 5.1 depends on the discussion that precedes it, so we do not present it until then.

The determination of $f(m, n)$ remains open for general m, n. In Section 6, we conclude with some examples and suggested directions for future consideration of this problem.

2. Upper Bounds: Constructing Matrices with Property Z

We describe constructions of $2m \times 2n$ matrices, where $m \leq n$, with Property Z and few zeroes. We first suppose that m divides n, which is Case (1) in Theorem 3. Say we have $n = km$. For the first $m - 1$ rows, let each row have k zeroes so that no column has more than one zero. From the columns without zeroes, assign $k + 1$ zeroes to each of the remaining $m + 1$ rows so that each of these $m + 1$ rows has exactly one zero in common with the next of the $m + 1$ rows, ordered cyclically, and no column has more than two zeroes. For example, we have the following matrix when $m = 4$, $n = 12$:

$$\begin{pmatrix}
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&1&1\\
1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&0&0&0
\end{pmatrix}.$$

It is easily verified that Property Z holds and that the matrix ends up with $2n + m + 1$ zeroes.

There is a nice interpretation of the construction if we view the complement of the matrix above (replace zeroes by ones and vice versa) as the edge–vertex incidence matrix of a hypergraph on $2n$ vertices. The hypergraph has $m - 1$ disjoint edges of size (number of vertices) k, and it has $m + 1$ edges of size $k + 1$ on the remaining vertices such that the edges, denoted by $e_1, \ldots, e_{m+1}$, are disjoint from all others except that there exist vertices $v_1, \ldots, v_{m+1}$ with v_i belonging to both e_i and e_{i+1}, taking the subscripts mod $m + 1$. In Wunsch's original case, $m = n$, we just have the edge–vertex incidence matrix for the graph $C_{m+1} + (m-1)K_1$, where we treat each isolated K_1 as a loop.

The construction for (1) is quickly verified to be optimal, as follows. Assume that a $2m \times 2n$ matrix M has Property Z. Since row permutations of a matrix preserve Property Z, we may assume that the number of zeroes r_i in row i is non-increasing. By Property Z, the bottom m rows of M have at least $n + 1 = km + 1 > km$ zeroes, so that $r_{m+1} \geq k + 1$. It follows that $r_i \geq k + 1$ for $1 \leq i \leq m$, and altogether M has at least $m(k + 1) + (n + 1) = 2n + m + 1$ zeroes, which is achieved by the construction.

Now we move on to Case (2) of Theorem 1.3, in which $n = km + r$, where $0 < r < m$ and $k + r - m \geq 0$. For the first $m - r - 1$ rows, give each row k zeroes, and for the next $m + r$ rows, give each row $(k + 1)$ zeroes, so that no column has more than one zero. So these $2m - 1$ rows have zeroes in $k \times (m - r - 1) + (k + 1) \times (m + r) = 2n - k - r + m$ columns. For the last row, we first fill the last $k + r - m$ positions with zeroes. Then we distribute

$(m+1)$ zeroes to the columns with indices greater than $(m-r-1)\times k$ while avoiding a 2×2 submatrix of zeroes. For example, here is the matrix when $m=4$ and $n=14$:

$$\begin{pmatrix}
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&1\\
1&1&1&1&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0
\end{pmatrix}.$$

We check Property Z by verifying that any m rows have zeroes in at least $n+1=km+r+1$ columns (we do two cases depending on whether or not the bottom row is used). Again, we have $2n+m+1$ zeroes altogether.

In this case, the incidence matrix interpretation gives a hypergraph on $2n$ vertices in which there are $m-r-1$ edges of size k and $r-1$ edges of size $k+1$, all disjoint. On the remaining vertices, there is an edge e of size $k+r+1$, which has $m+1$ pendant edges of size $k+1$. The pendant edges meet e in one distinct vertex each but are otherwise isolated.

We now give the recursive construction and resulting bound. This very simple recursion yields the construction for Case (1) above, m divides n, from the construction for the case $m=n$. We derive Gentry's general upper bound as a corollary.

THEOREM 1.1. *Let $1\le m<n$, and put $n=km+r$, where $1\le r\le m$. Then*

$$f(m,n)\le 2km+f(r,m).$$

PROOF. We build a $2m\times 2n$ matrix by taking k copies of the matrix $J-I$ of order $2m$, which is all ones except zeroes on the main diagonal. In the remaining $2m\times 2r$ space we insert a matrix achieving $f(m,r)$. For any m rows of this matrix, there are km zeroes in columns from the copies of $J-I$, and there are zeroes in at least $r+1$ columns of the last space, by Property Z, so we have zeroes in at least $km+r+1=n+1$ columns, and Property Z holds. This matrix has $k(2m)+f(m,r)=2km+f(r,m)$ zeroes. □

COROLLARY 1.2 ([6]). *For general m,n, we have*

$$f(m,n)\le 2m+2n-\gcd(m,n)+1.$$

PROOF. We prove this by induction on $\max(m,n)$. The bound is symmetric in m and n, so we may assume $m\le n$. For $m=n$, it holds by the construction for Case (1). For $m<n$, we apply Theorem 1.1 and obtain

$$f(m,n)\le 2km+f(r,m)\le 2km+2r+2m-\gcd(r,m)+1=2m+2n-\gcd(m,n)+1,$$

by induction, using $\gcd(m,n)=\gcd(r,m)$ (just like the Euclidean algorithm to compute $\gcd(m,n)$). □

Note that whenever we improve the bound on $f(r,m)$, Theorem 1.1 allows us to improve on the bound on $f(m,n)$ in Corollary 1.2, which is useful for improving bounds on particular values of $f(m,n)$.

3. Associated Graphs and Equipartitions of Graphs with Few Edges

Before proving Theorem 1.3, we introduce some definitions and a proposition that will be useful in the proof. Given a $k \times l$ $(0, 1)$-matrix M, we define the *associated graph* $G(M)$ to be the undirected loopless multigraph with vertex set $\{1, \ldots, k\}$ and edges as follows: For column j of M, $1 \leq j \leq l$, say the zeroes are located in rows $i_1 < i_2 < \cdots < i_s$. Then put the $s-1$ edges $\{i_1, i_2\}, \ldots, \{i_{s-1}, i_s\}$ into $G(M)$. For example, the matrix below gives a graph on $V = \{1, 2, 3, 4\}$ with edges $\{1, 2\}$, $\{1, 3\}$, $\{3, 4\}$, $\{3, 4\}$.

$$\begin{pmatrix} 0\,0\,0\,1\,1\,1\,1 \\ 1\,1\,0\,0\,1\,1\,1 \\ 0\,1\,1\,1\,0\,0\,1 \\ 0\,1\,1\,1\,0\,1\,0 \end{pmatrix}.$$

For the matrices obtained by the two constructions in the last section corresponding to the cases in Theorem 1.3, we see that the associated graphs are:

(1) $C_{m+1} + (m-1)\ K_1$;
(2) $K_{1,m+1} + (m-2)\ K_1$.

For a multigraph $G = (V, E)$ with $\nu(G)$ even, we define an *equipartition* $[A, B]$ of G to be the bipartite subgraph of G obtained by partitioning V into two equally sized subsets, A and B, and taking all edges between A and B. Equipartitions will prove to be valuable in studying our matrices.

Proposition 1.2. *Let M be a $2m \times 2n$ matrix with Property Z. Then, for any equipartition $[A, B]$ of the associated graph $G(M)$, $\varepsilon(A, B) \geq 2$.*

Proof. For any set of m rows, indexed by set A, there are at least $n+1$ columns containing a zero, and the same is true for the set B of remaining m rows. By the pigeonhole principle, M has at least two columns that have zeroes in rows indexed by A and by B both. In each such column, there will be some A row and some B row, each with a zero, such that no row in between has a zero. Hence, for any equipartition $[A, B]$ of the associated graph $G(M)$, $\varepsilon(A, B) \geq 2$. □

By contrast, we shall now prove a proposition giving sufficient conditions for a graph to have an equipartition with at most one edge. This will be the main tool for proving Theorem 1.3.

Proposition 1.2. *Let $G = (V, E)$ be a multigraph, with $\nu(G) = 2m$, $m \geq 1$.*

(I) If $\varepsilon(G) \leq m$, there exists an equipartition $[A, B]$ with $\varepsilon(A, B) = 0$, unless:

(i) $G = T + (m-1)K_1$, where T is a tree on $m+1$ vertices, or
(ii) $G = m\,K_2$, where m is odd.

(II) If $\varepsilon(G) = m+1$, there exists an equipartition $[A, B]$ with $\varepsilon(A, B) \leq 1$, unless:

(iii) $m = 1$ (two vertices with a double edge);
(iv) $G = C_{m+1} + (m-1)K_1$; or
(v) $G = K_{1,m+1} + (m-2)K_1$.

Proof (of (I)). We do this by induction on m. Consider G having $2m$ vertices and at most m edges. When $m \leq 2$, it is true trivially, so we may assume $m \geq 3$. Let C denote a component of G with $\nu(C)$ as large as possible. If $\nu(C) \leq 2$, we can easily verify the claim. (This includes Case (ii).)

So we assume that $\nu(C) \geq 3$. Then $\varepsilon(C) \geq \nu(C) - 1$. Let $G' = G - C$. The number of isolated vertices in G' is at least $\nu(G') - 2\varepsilon(G')$, since each edge absorbs at most two of them.

We have $\nu(G') = 2m - \nu(C)$ and $\varepsilon(G') \le m - \varepsilon(C) \le m - \nu(C) + 1$. So there are at least $\nu(G') - 2\varepsilon(G') \ge \nu(C) - 2$ isolated vertices.

Let $H = G' - (\nu(C) - 2)K_1$. Then $\nu(H) = 2(m - \nu(C) + 1)$ and $\varepsilon(H) \le m - \nu(C) + 1$. If H is the empty graph, G is in (i). Otherwise, we apply induction on H, so H must be one of situations in (I). In case $H = T + (m - \nu(C))K_1$, where T is a tree on $m - \nu(C) + 2$ vertices, we have an equipartition $[A, B]$ of G with $\varepsilon(A, B) = 0$ by putting $C + (m - \nu(C))K_1$ in A and $T + (\nu(C) - 2)K_1$ in B. In case $H = (m - \nu(C) + 1)K_2$, where $m - \nu(C) + 1$ is odd, we put C in A, $(\nu(C) - 2)K_1$ in B, and distribute remaining K_2's to balance A and B, so that again $\varepsilon(A, B) = 0$. In the case of the existence of an equipartition $[A', B']$ of H with $\varepsilon(A', B') = 0$, let D be a smallest component of H, which is either K_1 or K_2. Say D is in A'. Put $C + A' - D$ in A, $B' + D$ in B, and distribute remaining K_1's to balance A and B, so that $\varepsilon(A, B) = 0$.

PROOF (OF (II)). Suppose the loopless multigraph G has $\nu(G) = 2m$ and $\varepsilon(G) = m + 1$. We may assume $m \ge 2$, or else (iii) applies. If the largest component C of G has $m + 2$ vertices, then C must be a tree T. If T is not $K_{1,m+1}$, the diameter of T is at least 3, so there exists an edge e the deletion of which gives two components T_1, T_2, where each $\nu(T_i) \ge 2$. Since $\nu(C) = m + 2$ vertices, each $\nu(T_i) \le m$. We construct an equipartition $[A, B]$ of G as follows: Put T_1 in A, T_2 in B, and then distribute the K_1's to obtain an equipartition with $\varepsilon(A, B) = 1$.

Next assume $\nu(C) = m + 1$ vertices. If C is not a cycle C_{m+1}, it must have an end vertex, say v. We obtain an equipartition $[A, B]$ by taking $A = C - v$. Then $\varepsilon(A, B) = 1$.

If $\nu(C) = m$, we just take $A = C$ for an equipartition $[A, B]$.

Finally, suppose that $\nu(C) \le m - 1$. Consider $G' = (V', E')$ with $V' = V$ and $E' = E - \{e\}$, where e is some edge. In case $G' = mK_2$, an equipartition $[A, B]$ with $\varepsilon(A, B) \le 1$ can be obtained easily. Otherwise apply part (I) to obtain an equipartition $[A, B]$ with $\varepsilon(A, B) = 0$ with respect to G'. That means $\varepsilon(A, B) \le 1$ with respect to G. $\square$

4. LOWER BOUND: MAIN RESULT

We can now prove our sharp lower bound, which we again include for the convenience of the reader.

THEOREM 1.2. *Assume that $m \le n$. Let $f(m, n)$ denote the least number of zeroes for a $2m \times 2n$ $(0, 1)$-matrix having Property Z. Then*

$$f(m, n) \ge 2n + m + 1,$$

where the equality holds precisely when:

(1) n is a multiple of m; or
(2) $k + r \ge m$, where $n = km + r$, and $0 < r < m$.

PROOF. If $m = 1$, we easily check that $f(m, n) = 2n + 2 = 2n + m + 1$, and of course, m divides n in this case.

Assume for the remainder that $2 \le m \le n$. Notice that an associated graph $G(M)$ of a matrix M has the following property: If every column of M has a zero, then the number of zeroes in M is the number of the columns plus $\varepsilon(G(M))$.

To prove the lower bound in the theorem, assume for contradiction that there is a $2m \times 2n$ matrix M with Property Z and at most $2n + m$ zeroes. In fact, we may assume that M has exactly $2n + m$ zeroes; or else we could just replace some ones by zeroes and still have Property Z.

If every column of M has a zero, it follows that $\varepsilon(G(M)) \le m$. Then, by Proposition 3.2, $G(M)$ has an equipartition $[A, B]$ with $\varepsilon(A, B) \le 1$. But this contradicts Proposition 3.1.

Therefore, M has some column j without zeroes. Now M has $2n+m \geq 2n$ zeroes, so there is some column i with zeroes at distinct rows k and row l. We then modify M by switching the zero at entry (k, i) with the one at entry (k, j). The new matrix has the same number of zeroes as before and still has Property Z. Also, its associated graph has fewer edges than before. Repeating this process, we eventually obtain a matrix with Property Z and just $2n+m$ zeroes such that every column has a zero, which we saw was impossible.

So if M has Property Z, it must have at least $2n+m+1$ zeroes. This proves the lower bound in the theorem. The constructions in Section 2 achieve the lower bound in Case (1) m divides n, and in Case (2) $k+r \geq m$, $r > 0$.

To complete the proof, it must be shown that $f(m, n) = 2n+m+1$ only in these two cases. Suppose that matrix M has Property Z and just $2n+m+1$ zeroes. Arguing as before, we may perform switches until every column of M has a zero. Assuming this, we deduce that $\varepsilon(G(M)) = m+1$. Since, by Proposition 3.1, we have $\varepsilon(A, B) \geq 2$ for any equipartition $[A, B]$ of $G(M)$, we learn from Proposition 3.2 that $G(M)$ is given by (iv) or (v). The two lemmas below treat these respective cases.

LEMMA 1.2. *If matrix M has Property Z, every column has a zero, and its associated graph is $G(M) = C_{m+1} + (m-1)K_1$, then n is a multiple of m.*

PROOF. We claim that all rows in such M corresponding to vertices in C_{m+1} have an equal number of zeroes. Consider an equipartition $[A, B]$ in which A contains all the $m-1$ isolated vertices and one vertex from C_{m+1}. Since $\varepsilon(A, B) = 2$ and Z holds, there are exactly $n+1$ columns having zeroes from the m rows associated with A, and similarly for the m rows associated with B. Since the vertex in A from C_{m+1} is chosen arbitrarily, all the rows corresponding to vertices in C_{m+1} have an equal number of zeroes; let us denote it by t. Then the total number of columns with zeroes in the rows associated with subset B is $t + (m-1)(t-1) = n+1$, so that $n = km$, where $k = t-1$. □

LEMMA 1.2. *If matrix M has Property Z, every column has a zero, and its associated graph is $G(M) = K_{1,m+1} + (m-2)K_1$, then $k+r \geq m$, where $n = km+r$ and $0 < r < m$.*

PROOF. Let the vertex in $K_{1,m+1}$ which is not an end-vertex be v, and let p be the number of zeroes in its row. We have $p \geq m+1$, since v is adjacent to $m+1$ independent vertices. We consider an equipartition $[A, B]$, where A contains v and any $m-1$ end-vertices. By the same argument as above, the numbers of zeroes in all the rows that correspond to end-vertices of $K_{1,m+1}$ are the same, say t.

Next consider an equipartition $[A', B']$ where A' contains any m end-vertices in $K_{1,m+1}$. The rows for A' have zeroes in tm distinct columns, so $tm \geq n+1$ by Property Z. Letting $n = km+r$, $0 \leq r < m$, this gives $t \geq k+1$. Also, since $\varepsilon(A', B') = m$ and since B' has zeroes in at least $n+1$ columns, we obtain $tm \leq n+m < (k+2)m$. Thus, $t = k+1$.

The number of columns having zeroes from the rows associated with A is

$$p + (m-1)k = n+1.$$

Thus,

$$mk + r = n = mk + p - k - 1,$$

so

$$k + r = p - 1 \geq m.$$

The rows for the $m+1$ end-vertices contain zeroes in $(m+1)k$ columns that have a unique zero. Then we find that B' contains zeroes in the other columns and in the k columns with a unique zero from the row of the end-vertex in B'. Altogether, this gives zeroes in this number of columns:

$$2n - (m+1)k + k = 2n - mk = n + r,$$

which by Property Z forces $r > 0$, and we are in Case (2).

This completes the proof of our main result. $\square$

5. On the Extremal Matrices for Theorem 1.3

When the bound $2n + m + 1$ of Theorem 1.3 is sharp, in Cases (1) and (2), the matrices constructed in Section 2 are sometimes, but not always, the only ones achieving the bound up to row and column permutations. For example, when $m = 4$ and $n = 14$, here is a different extremal matrix from the one we presented earlier. Note that it has a column without zeroes:

$$\begin{pmatrix}
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0\\
1&1&1&1&1&1&1&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&1\\
1&1&1&1&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0
\end{pmatrix}.$$

We determine the extremal matrices here and summarize the discussion in Theorem 5.1, which we postpone to the end of the section since it depends on the discussion.

We first discuss Case (1), say $n = km$, and show that the construction we gave is in fact the unique extremal matrix. We then go into Case (2), say $n = km + r$, $0 < r < m$, $k + r - m \geq 0$, and show that our construction gives the unique extremal matrix with no column of ones, and all other extremal matrices are easily derived from it.

Case (1). Consider an extremal matrix M for Case (1) of Theorem 1.3. Arguing as in Section 4, we may shift zeroes until every column of M has a zero. The discussion preceding Lemma 4.2 tells us that we may assume that (iv) or (v) of Proposition 3.2 applies. Indeed, since n is a multiple of m, only (iv) is possible, in view of Lemma 4.3, which means that $G(M) = C_{m+1} + (m-1)K_1$. Continuing the proof of Lemma 4.2, define a new equipartition $[A', B']$, where A' contains $m-1$ consecutive vertices from C_{m+1} and one of the isolated vertices. The vertices from C_{m+1} have zeroes in $m + (m-1)(k-1) = n + 1 - k$ columns, so by Property Z, the isolated vertex has at least k zeroes. Since A in Lemma 4.2 has zeroes in

$$\begin{aligned} n + 1 &= mk + 1 \\ &= k + 1 + (m-1)k \end{aligned}$$

columns, the $m-1$ isolated vertices have zeroes in $(m-1)k$ columns. Thus, each has exactly k zeroes, and M comes from the construction for Case (1) in Section 2, up to row and column permutations. However, we may have obtained M by shifting zeroes.

If we obtained M by shifting zeroes, the last shift would have moved a zero into a column that had been all ones. But looking at this in reverse, it can be checked that for the construction for Case (1), shifting any zero from a column with a single zero to another column destroys Property Z. So there can be no such predecessor: The construction for Case (1) gives the unique extremal matrix.

Case (2). Again begin by letting M be an extremal matrix such that every column has a zero. We show that M must be the matrix from the construction for Case (2) in Section 2, up to row or column permutations. By Lemma 4.2 and the discussion preceding it, we have that $G(M) = K_{1,m+1} + (m-2)K_1$. Recalling the proof of Lemma 4.3, it must be that each end-vertex has zeroes in precisely $k+1$ columns.

Now, as before, we take an equipartition $[A, B]$ where A contains vertex v and $m-1$ end-vertices of the star $K_{1,m-1}$. We showed that the row for v in M has

$$p = k + r + 1 \tag{5.1}$$

zeroes, and thus this row has $k + r - m$ zeroes in columns by themselves.

The $m-2$ isolated vertices have

$$2n - (p + (m+1)k) = k(m-2) + (r-1) \tag{5.2}$$

columns with a zero. Then B has zeroes in

$$2(k+1) + k(m-2) + (r-1) = n + 1 \tag{5.3}$$

columns.

Let the number of zeroes in the rows for isolated vertices be denoted $z_1 \le z_2 \le \cdots \le z_{m-2}$. Applying Property Z to the $m-r-1$ isolated vertices with the lowest numbers of zeroes together with $r+1$ end-vertices, we have

$$z_1 + \cdots + z_{m-r-1} + (r+1)(k+1) \ge n + 1 = mk + r + 1,$$

so that

$$z_1 + \cdots + z_{m-r-1} \ge (m-r-1)k. \tag{5.4}$$

Similarly, by taking the $m-r$ isolated vertices with the lowest z_i together with r end-vertices, we have

$$z_1 + \cdots + z_{m-r} \ge (m-r)k + 1,$$

which implies that

$$z_{m-2} \ge \cdots \ge z_{m-r} \ge k + 1. \tag{5.5}$$

Then, considering all rows for isolated vertices, and applying (5.2), (5.4) and (5.5), we obtain

$$\begin{aligned} k(m-2) + (r-1) &= z_1 + \cdots + z_{m-2} \\ &\ge (m-r-1)k + (r-1)(k+1) \\ &= k(m-2) + r - 1, \end{aligned}$$

which forces equalities in (5.4) and (5.5). If $r < m-1$, next take the rows for v, $m-2$ end-vertices, and the isolated vertex for z_1. By Property Z, these m rows have zeroes in

$$(k + r + 1) + (m-2)k + z_1 \ge n + 1 = mk + r + 1$$

columns. It follows that $z_1 \ge k$, and so

$$z_1 = \cdots = z_{m-r-1} = k, \quad z_{m-r} = \cdots = z_{m-2} = k + 1.$$

Thus, the matrix M is that of the construction for Case (2).

It remains to consider extremal matrices for Case (2) with a column of ones. Any such matrix can be converted to the construction for this case, call it M_0, by successively shifting zeroes into columns of ones. Let us consider the last matrix, call it M_1, before reaching M_0. So M_1 is extremal, has a single column of ones, and is formed by shifting a zero, which is in a column by itself, over in its row into a column that has a zero. How is this possible?

In M_0, the set A containing v and some $m-1$ end-vertices has zeroes in just $n+1$ columns, so we cannot do the shift in an A row to a column where A has a zero, without violating Property Z. Similarly, we can consider the complementary set B of two end-vertices and all

$m-2$ isolated vertices and the set A'' with v, $m-2$ end-vertices, and some isolated vertex with k zeroes in its row.

The only possibility is that we shift a zero from an isolated vertex with $k+1$ zeroes in its row into a column containing just one zero in the row for v (or vice versa). That is, we combine the zeroes in two such columns. This matrix M_1 has Property Z: For a set C of m vertices, there are zeroes in at least $n+1$ rows as before, if $v \notin C$, while if $v \in C$, we hit $k+r+1$ columns with v and at least k new columns with every other element of S, for a total of at least $k+r+1+(m-1)k = n+1$ columns.

We can repeat the argument above, and show that from M_1 the only thing we can do is to combine a zero from another row of an isolated vertex with $k+1$ zeroes together with another column of v with just one zero. This gives a matrix M_2. We can continue this process of combination, so that after s shifts, we obtain a matrix M_s with s columns of ones such that

$$G(M_s) = K_{1,m+1-s} + (m-s-2)K_1,$$

where the v row has $k+r+1$ zeroes, $m-r-1$ isolated vertices have k zeroes, $r-s-1$ isolated vertices have $k+1$ zeroes, and the end-vertices have k zeroes each in columns by themselves.

Each M_s is the unique extremal matrix with s columns of ones. We can do this until we either run out of isolated vertices with $k+1$ zeroes or out of columns for v with a single zero, which means M_s is at most $\min(r-1, k+r-m)$.

The example we gave earlier in this section is M_1 for $m=4$, $n=14$. We now summarize our description of all extremal matrices for Theorem 1.3:

THEOREM 1.2. *Assume that $m \le n$. Suppose that M is a $2m \times 2n$ $(0,1)$-matrix having Property Z and just $2n+m+1$ zeroes. Then up to a permutation of rows or columns, in case*

(1) n is a multiple of m: The matrix M is the one constructed in Section 2; or
(2) $k+r \ge m$, where $n = km + r$, and $0 < r < m$: The matrix M is M_s above for some s, where

$$0 \le s \le \min(r-1, k+r-m).$$

6. FURTHER REMARKS AND PROBLEMS

The remaining undetermined values of $f(m,n)$ should be studied. All have $n \le (m-1)^2$. In one interesting open case, we have for $m \ge 2$ that

$$3m + 4 \le f(m, m+1) \le 4m + 2.$$

We should at least determine $\lim_{m\to\infty} f(m, m+1)/m$.

We have computed some particular values, and interesting things happen. Increasing m or n may actually decrease f! For example,

$$f(7,8) = 26 > f(8,8) = 25.$$

The lower bound $f(7,8) \ge 26$ was verified by tedious analysis of several cases, while $f(8,8)$ is given by Theorem 1.3. Chih-Chang Ho [7] has produced another such example:

$$47 = f(7,18) > f(7,19) = 46.$$

While the value of $f(7,18)$ required analysis of many cases, the value of $f(7,19)$ comes from Theorem 1.3. Ho has worked out various other values of f and interesting refinements and extensions of results presented here.

What are the bounds, if any, on how much f changes (up or down) when m or n changes by one? For instance, is there some g, perhaps $g = 2$, such that

$$|f(m, n+1) - f(m, n)| \leq g?$$

When is the upper bound of Corollary 1.2 sharp? It may not be hard to answer this.

Natural extensions of our problem are suggested. A more general setting would be to forbid all submatrices of ones with dimension $1/k$ by $1/l$ times the whole, or equivalently, to study $Z_{m,n}(km, ln)$.

Another direction, suggested to us by Paul Erdős [4], is the 'k-by-half' problem in which only one dimension of the forbidden submatrix grows: Specifically, we fix a value k, and (in the square case of even order $2n$) consider the least number of zeroes that prevent any $k \times n$ submatrix of ones. This number grows quadratically in n, not merely linearly, since any k rows must contain zeroes in at least $n+1$ columns. It is equivalent to investigate $Z_{k,n}(2n, 2n)$ for fixed k.

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REFERENCES

1. A. E. Andreev On a family of Boolean matrices *Vestnik Mosk. Univ. Ser. 1*, **41** (1986), 97–100 (in Russian); English translation, *Moscow Univ. Math. Bull.*, **41** (1986), 79–82.
2. B. Bollobás, *Extremal Graph Theory*, Academic Press, New York, 1978.
3. K. Culik, Teilweise Losung eines verallgeneinerten Problem von K. Zarankiewicz *Ann. Soc. Polon. Math.*, **3** (1968), 165–168.
4. P. Erdős, Personal communication, 1994.
5. Z. Füredi, New asymptotics for bipartite Turán numbers, *J. Combin. Theory Ser. A*, **75** (1996), 141–144.
6. C. Gentry, Personal communication, 1994.
7. C.-C. Ho, In preparation, 1996.
8. J. Kollar, L. Ronyai and T. Szabo, Norm-graphs and bipartite Turán numbers, *Combinatorica*, to appear.
9. S. Roman, A problem of Zarankiewicz, *J. Combin. Theory Ser. A*, **18** (1975), 187–198.
10. J. Wunsch, Personal communication, 1993.
11. K. Zarankiewicz, Problem P101, *Colloq. Math.*, **2** (1951), 301.
12. Z. Znam, Two improvements of a result concerning a problem of K. Zarankiewicz, *Colloq. Math.*, **13** (1965), 255–258.

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