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Controllability properties for some semilinear parabolic PDE with a quadratic gradient term ☆

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ABSTRACT

We study several controllability properties for some semilinear parabolic PDE with a quadratic gradient term. For internal distributed controls, it is shown that the system is approximately and null controllable. The proof relies on the Cole–Hopf transformation. The same approach is used to deal with initial controls.

1. Introduction

We study some controllability properties for the following semilinear parabolic boundary value problem:

$$\begin{cases} y_t(x,t) - \Delta y(x,t) = |\nabla y(x,t)|^2 \cdot \phi(y(x,t)) + u(x,t)\chi_\omega(x) & \text{in } Q_T, \\ y(x,t) = 0 & \text{on } \Sigma_T, \\ y(x,0) = y_0(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where $Q_T = \Omega \times (0,T)$ and $\Sigma_T = \Gamma \times (0,T)$, Ω being an open bounded domain of $\mathbb{R}^n$, with boundary Γ of class C^2 and T a positive (fixed) real number. Moreover, ω is a nonempty open subset of Ω (possibly small), χ_ω is the characteristic function of ω and ϕ is a real function. As usual, we will denote $y_t(x,t) = \frac{\partial y}{\partial t}(x,t)$, $\nabla y(x,t) = (\frac{\partial y}{\partial x_1}(x,t), \ldots, \frac{\partial y}{\partial x_n}(x,t))$ and $\Delta y = \sum_{i=1}^n \frac{\partial^2 y}{\partial x_i^2}(x,t)$.

Given $T > 0$ fixed, it is said that problem (1) is approximately controllable in $L^2(\Omega)$ at time T by using controls in $\mathcal{U}$ if, for each initial condition y_0 in a certain space, $y_d \in L^2(\Omega)$ and $\epsilon > 0$, there exist $u \in \mathcal{U}$ and a solution y of the problem (1) that satisfies $\|y(\cdot,T) - y_d\|_{L^2(\Omega)} < \epsilon$. Furthermore, it is said that problem (1) is null controllable at time T by using controls in $\mathcal{U}$ when for each initial condition y_0, there exist $u \in \mathcal{U}$ and a solution y of the problem (1) satisfying $y(x,T) = 0$ in Ω.

In the last few years, there has been a great interest in the controllability of parabolic systems with internal distributed control (see, among others, [1–7] and the references therein). Mainly, these works try to characterize the class of parabolic problems having the aforementioned controllability properties. It is now well known that the approximate controllability holds for most of the linear parabolic PDE. In the nonlinear case, many results are related to the system

$$\begin{cases} y_t(x,t) - \Delta y(x,t) + f(y(x,t), \nabla y(x,t)) = u(x,t)\chi_\omega(x) & \text{in } Q_T, \\ y(x,t) = 0 & \text{on } \Sigma_T, \\ y(x,0) = y_0(x) & \text{in } \Omega. \end{cases} \tag{2}$$

Roughly speaking, the situation can be described as follows:

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- for $n = 1$ and $f(y, y_x) = yy_x$ (i.e. Burgers' equation), it is known that the system (2) is not approximately controllable in $L^2(\Omega)$ at time T by using controls in $L^2(\omega \times (0, T))$; see [5, Theorem 6.3, Chapter I].
- For functions $f(y, \nabla y)$ globally Lipschitz with respect to $(y, \nabla y)$, the system (2) is approximately controllable in $H_0^\rho(\Omega)$ at time T for all $\rho \in [0, 1)$—see [4, Theorem 3.3]—and null controllable at time T (when $f(0, 0) = 0$)—see [6]—by using controls in $L^2(\omega \times (0, T))$.
- For functions $f(y, \nabla y)$ growing more slowly than

$$|y| \log^{3/2}(1 + |y| + |\nabla y|) + |\nabla y| \log^{1/2}(1 + |y| + |\nabla y|), \tag{3}$$

as $|(y, \nabla y)| \to +\infty$, the system (2) is null (when $f(0, 0) = 0$) and approximately controllable in $L^2(\Omega)$ at time T, by using controls in $L^\infty(\omega \times (0, T))$; see [2].

In general, the proofs of these results are quite technical and involve Carleman estimates together with sharp parabolic regularity results.

The main contribution of this work is presented in Theorem 1, where it is shown that (1) is approximately controllable in $L^2(\Omega)$ and null controllable at time T by using controls in $L^\infty(\omega \times (0, T))$.

2. Internal distributed control

Let us begin by recalling some spaces that appear commonly in the framework of parabolic problems:

$$W_{2,0}^{1,1}(Q_T) = \{y \in L^2(0, T; H_0^1(\Omega)) : y_t \in L^2(Q_T)\},$$
$$W_{2,0}^{2,1}(Q_T) = \{y \in L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega)) : y_t \in L^2(Q_T)\}.$$

For when Γ is C^2, it is known that $W_{2,0}^{2,1}(Q_T) = \{y \in L^2(0, T; H_0^1(\Omega)) : y_t \in L^2(Q_T),\ \Delta y \in L^2(Q_T)\}$ (see [8, p. 109 and 113]). It is also well known that

$$W_{2,0}^{1,1}(Q_T) \subset C([0, T]; L^2(\Omega)), \qquad W_{2,0}^{2,1}(Q_T) \subset C([0, T]; H_0^1(\Omega)),$$

with continuous imbeddings.

A crucial assumption throughout this work is the following:

(H) $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function and there exists a real number $\alpha \le 0$ such that

$$\int_0^r \phi(s)ds \ge \alpha, \quad \forall r \in \mathbb{R}.$$

Now, let us introduce the function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$\varphi(r) = \int_0^r \exp\left(\int_0^v \phi(s)ds\right) dv, \quad \forall r \in \mathbb{R}. \tag{4}$$

Under condition **(H)**, it is straightforward to show that φ is a strictly increasing C^2 function with range equal to $\mathbb{R}$, thanks to

$$\varphi'(r) = \exp\left(\int_0^r \phi(s)ds\right) \ge \exp(\alpha) > 0, \quad \text{and} \quad \varphi''(r) = \phi(r)\varphi'(r), \quad \forall r \in \mathbb{R}. \tag{5}$$

We will consider the problem (1) with control $u \in L^\infty(Q_T)$ and initial condition $y_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$. A function $y \in W_{2,0}^{1,1}(Q_T) \cap L^\infty(Q_T)$ is said to be a solution of (1) if it verifies its PDE in the distribution sense and the initial condition in $L^2(\Omega)$.

Let us show that the problem (1) can be transformed into a semilinear one by using the Cole–Hopf transformation (4): given a solution y of problem (1), we define a new function given by

$$z(x, t) = \varphi(y(x, t)). \tag{6}$$

It is easy to show that $z \in W_{2,0}^{2,1}(Q_T) \cap L^\infty(Q_T)$, because $\varphi(0) = 0$ and

$$z_t(x, t) = \varphi'(y(x, t)) \cdot y_t(x, t), \qquad \nabla z(x, t) = \varphi'(y(x, t)) \cdot \nabla y(x, t). \tag{7}$$

Furthermore, taking into account (5), it follows that (in the distribution sense)

$$\begin{aligned} \Delta z(x, t) &= \varphi''(y(x, t))|\nabla y(x, t)|^2 + \varphi'(y(x, t))\Delta y(x, t) \\ &= \varphi'(y(x, t))(\phi(y(x, t))|\nabla y(x, t)|^2 + \Delta y(x, t)) \\ &= \varphi'(y(x, t))(y_t(x, t) - u(x, t)\chi_\omega(x)) = z_t(x, t) - u(x, t)\chi_\omega(x)\varphi'(y(x, t)). \end{aligned} \tag{8}$$

Therefore, z can be viewed as a solution in $W^{2,1}_{2,0}(Q_T) \cap L^\infty(Q_T)$ of the semilinear problem

$$\begin{cases} z_t(x,t) - \Delta z(x,t) = u(x,t)\chi_\omega(x) \cdot \varphi'(\varphi^{-1}(z(x,t))) & \text{in } Q_T, \\ z(x,t) = 0 & \text{on } \Sigma_T, \\ z(x,0) = \varphi(y_0(x)) & \text{in } \Omega. \end{cases} \tag{9}$$

Remark 1. The existence of a solution for problem (1) has been studied in many papers, even for less regular data u and y_0 (see [9, Theorem 5.6]), and for more general operators (see [10, Theorem 2.2]). Let us stress that we are dealing with solutions more regular than those obtained in these works. On the other hand, it was proved in [9] that the problem (1) admits infinitely many weak solutions. Clearly, this nonuniqueness result is not relevant from the controllability viewpoint, where only one solution satisfying the required conditions is needed.

Now, we can take advantage of the Cole–Hopf transformation in order to obtain the main result of this section.

Theorem 1. *Let us assume condition* (**H**) *and* $y_0 \in H^1_0(\Omega) \cap L^\infty(\Omega)$. *Then, the system* (1) *is approximately controllable in* $L^2(\Omega)$ *and null controllable at time* T *by using controls in* $L^\infty(\omega \times (0,T))$.

Proof. Let us begin by proving the approximate controllability property for (1). Due to the density of $L^\infty(\Omega)$ in $L^2(\Omega)$, it is enough to approximate any element $y_d \in L^\infty(\Omega)$. Given $\epsilon > 0$, by the approximate controllability for the heat equation (see [11, section 10, chapter III] and [12]), we can guarantee the existence of a control $v \in L^\infty(Q_T)$ such that the unique solution z in $W^{2,1}_{2,0}(Q_T) \cap L^\infty(Q_T)$ (see [8, p. 112] and [13, Theorem 7.1, p. 181 and Corollary 7.1, p. 186]) of the linear problem

$$\begin{cases} z_t(x,t) - \Delta z(x,t) = v(x,t)\chi_\omega(x) & \text{in } Q_T, \\ z(x,t) = 0 & \text{on } \Sigma_T, \\ z(x,0) = \varphi(y_0(x)) & \text{in } \Omega, \end{cases} \tag{10}$$

satisfies $\|z(\cdot,T) - \varphi(y_d)\|_{L^2(\Omega)} \leq \epsilon \cdot \exp(\alpha)$, where α is taken from **(H)**. Inspired by the argumentation developed at the beginning of this section (that we are reversing now), we define

$$y(x,t) = \varphi^{-1}(z(x,t)) \quad \text{in } Q_T. \tag{11}$$

Of course, y is well defined: $\varphi^{-1}(s)$ exists for all $s \in \mathbb{R}$, because φ is a strictly increasing function with range equal to $\mathbb{R}$. In fact, thanks to condition **(H)**, we know that $\varphi^{-1}(s)$ is a globally Lipschitz increasing C^2 function with

$$\frac{d\varphi^{-1}}{ds}(s) = \frac{1}{\varphi'(\varphi^{-1}(s))} \in (0, \exp(-\alpha)], \qquad \frac{d^2\varphi^{-1}}{ds^2}(s) = \frac{-\varphi''(\varphi^{-1}(s))}{(\varphi'(\varphi^{-1}(s)))^3}. \tag{12}$$

Furthermore, it is easy to show that $y \in W^{1,1}_{2,0}(Q_T) \cap L^\infty(Q_T)$, combining that $z \in W^{2,1}_{2,0}(Q_T) \cap L^\infty(Q_T)$, $\varphi(0) = 0$ and

$$y_t(x,t) = \frac{d\varphi^{-1}}{ds}(z(x,t)) \cdot z_t(x,t), \qquad \nabla y(x,t) = \frac{d\varphi^{-1}}{ds}(z(x,t)) \cdot \nabla z(x,t). \tag{13}$$

Taking into account (5) and (10)–(13), the next equalities follow in the distribution sense:

$$\begin{aligned} \Delta y(x,t) &= \frac{d^2\varphi^{-1}}{ds^2}(z(x,t))|\nabla z(x,t)|^2 + \frac{d\varphi^{-1}}{ds}(z(x,t))\Delta z(x,t) \\ &= \frac{-\varphi''(y(x,t))}{(\varphi'(y(x,t)))^3}|\nabla z(x,t)|^2 + \frac{\Delta z(x,t)}{\varphi'(y(x,t))} = -\phi(y(x,t))\frac{|\nabla z(x,t)|^2}{(\varphi'(y(x,t)))^2} + \frac{z_t(x,t) - v(x,t)\chi_\omega(x)}{\varphi'(y(x,t))} \\ &= -\phi(y(x,t))|\nabla y(x,t)|^2 + y_t(x,t) - u(x,t)\chi_\omega(x), \end{aligned}$$

where we have selected the control

$$u(x,t) = \frac{v(x,t)}{\varphi'(y(x,t))}. \tag{14}$$

Obviously, u belongs to $L^\infty(Q_T)$, thanks to (5):

$$\|u\|_{L^\infty(Q_T)} \leq \|v\|_{L^\infty(Q_T)} \cdot \exp(-\alpha).$$

This means that y is a solution of problem (1). Finally, combining with the Mean Value Theorem, (5) and (12), we obtain

$$\begin{aligned} \|y(\cdot,T) - y_d\|_{L^2(\Omega)} &= \|\varphi^{-1}(z(\cdot,T)) - \varphi^{-1}(\varphi(y_d))\|_{L^2(\Omega)} \\ &= \left(\int_\Omega \left| \frac{d\varphi^{-1}}{ds}(\theta(x)) \right|^2 |z(x,T) - \varphi(y_d(x))|^2 dx \right)^{1/2} \leq \exp(-\alpha)\|z(\cdot,T) - \varphi(y_d)\|_{L^2(\Omega)} \leq \epsilon, \end{aligned}$$

where $\theta(x)$ denotes some intermediate value between $z(x,T)$ and $\varphi(y_d(x))$. This is exactly what we were looking for.

The proof of the null controllability property can be seen as a particular case of the previous argumentation, taking $\epsilon = 0$, $y_d = 0$ and applying the null controllability result for the heat equation with bounded controls; see [2, Theorem 3.1] and also [14]. □

Of course, there exist many continuous functions ϕ verifying condition **(H)**. Typical examples are $\phi(y) = \exp(y)$ (with $\alpha = -1$) and $\phi(y) = y^{2k+1}$ for any natural number k (with $\alpha = 0$). More generally, $\phi(y)$ can be any polynomial with highest term of odd order and positive main coefficient. It is also clear that some other usual functions do not verify condition **(H)**, like $\phi(y) = y^{2k}$ for any natural number k. These cases deserve a specific treatment: for instance, the case $\phi(y) = 1$ was studied in [15].

From Theorem 1 it follows that the hypotheses assumed in [2] (see (3)) are far from being necessary for deriving the controllability properties, because clearly they are not satisfied in our framework.

3. Initial control

Previous argumentation can be also applied when the control is acting through the initial condition, like in the problem

$$\begin{cases} y_t(x,t) - \Delta y(x,t) = |\nabla y(x,t)|^2 \cdot \phi(y(x,t)) & \text{in } Q_T, \\ y(x,t) = 0 & \text{on } \Sigma_T, \\ y(x,0) = u(x) & \text{in } \Omega. \end{cases} \tag{15}$$

As in the previous case, it is said that problem (15) is approximately controllable in $L^2(\Omega)$ at time T by using initial controls u in a certain space $\mathcal{U}$ if, for each $y_d \in L^2(\Omega)$ and $\epsilon > 0$, there exist $u \in \mathcal{U}$ and a solution y of the problem (15) that satisfies $\|y(\cdot, T) - y_d\|_{L^2(\Omega)} < \epsilon$.

The following result can be proved in this context.

Theorem 2. *Let us assume condition* (**H**). *Then, the system* (15) *is approximately controllable in $L^2(\Omega)$ at time T by using controls $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$.*

Proof. Given any element $y_d \in L^\infty(\Omega)$ and $\epsilon > 0$, by the approximate controllability proved in [11, Th. 11.2, p. 215], we can guarantee the existence of a control $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that the unique solution z in $W_{2,0}^{2,1}(Q_T) \cap L^\infty(Q_T)$ of the linear problem

$$\begin{cases} z_t(x,t) - \Delta z(x,t) = 0 & \text{in } Q_T, \\ z(x,t) = 0 & \text{on } \Sigma_T, \\ z(x,0) = v(x) & \text{in } \Omega, \end{cases} \tag{16}$$

satisfies $\|z(\cdot, T) - \varphi(y_d)\|_{L^2(\Omega)} \le \epsilon \cdot \exp(\alpha)$. We finish the proof as in Theorem 1, by selecting the control

$$u(x) = \varphi^{-1}(v(x)), \tag{17}$$

that belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$, thanks to the properties of φ. □

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References

[1] V. Barbu, Exact controllability of the superlinear heat equation, Appl. Math. Optim. 42 (1) (2000) 73–89.
[2] A. Doubova, E. Fernández-Cara, M. González-Burgos, E. Zuazua, On the controllability of parabolic systems with a nonlinear term involving the state and the gradient, SIAM J. Control Optim. 41 (3) (2002) 798–819.
[3] C. Fabre, J.P. Puel, E. Zuazua, Approximate controllability of the semilinear heat equation, Proc. Roy. Soc. Edinburgh 125A (1995) 31–61.
[4] L.A. Fernández, E. Zuazua, Approximate controllability for the semilinear heat equation involving gradient terms, J. Optim. Theory Appl. 101 (2) (1999) 307–328.
[5] A.V. Fursikov, O.Y. Imanuvilov, Controllability of evolution equations. Research Institute of Mathematics, Seoul National University, Korea, 1996.
[6] O.Y. Imanuvilov, M. Yamamoto, Carleman inequalities for parabolic equations in Sobolev spaces of negative order and exact controllability for semilinear parabolic equations, Publ. Res. Inst. Math. Sci. 39 (2) (2003) 227–274.
[7] A.Y. Khapalov, Exact null-controllability for the semilinear heat equation with superlinear term and mobile internal controls, Nonlinear Anal. 43 (6) (2001) 785–801.
[8] O.A. Ladyzhenskaya, The Boundary Value Problems of Mathematical Physics, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1985.
[9] B. Abdellaoui, A. Dall'Aglio, I. Peral, Regularity and nonuniqueness results for parabolic problems arising in some physical models, having natural growth in the gradient, J. Math. Pures Appl. 90 (3) (2008) 242–269.
[10] A. Dall'Aglio, D. Giachetti, S. Segura de León, Nonlinear parabolic problems with a very general quadratic gradient term, Differential Integral Equations 20 (4) (2007) 361–396.
[11] J.L. Lions, Optimal Control of Systems Governed by Partial Differential Equations, Springer-Verlag, New York, Heidelberg, Berlin, 1971.
[12] J.L. Lions, Remarques sur la contrôlabilité approchée, in: Jornadas Hispano-Francesas sobre Control de Sistemas Distribuidos, University of Málaga, Spain, 1991, pp. 77–87.
[13] O.A. Ladyzhenskaya, V.A. Solonnikov, N.N. Ural'tseva, Linear and Quasilinear Equations of Parabolic Type, Am. Math. Soc., Providence, RI, 1968.
[14] G. Lebeau, L. Robbiano, Contrôle exact de l'équation de la chaleur, Comm. Partial Differential Equations 20 (1–2) (1995) 335–356.
[15] P. Lei, Y. Li, P. Lin, Null controllability for a semilinear parabolic equation with gradient quadratic growth, Nonlinear Anal. 68 (1) (2008) 73–82.