An algorithm for semistandardising homomorphisms

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A B S T R A C T

Suppose μ is a partition of n and λ a composition of n, and let S^μ, M^λ denote the Specht module and permutation module defined by Dipper and James for the Iwahori–Hecke algebra $\mathscr{H}_n$ of the symmetric group W_n. We give an explicit fast algorithm for expressing a tableau homomorphism $\hat{\phi}_A : S^\mu \to M^\lambda$ as a linear combination of semistandard homomorphisms. Along the way we provide a utility result related to removing rows from tableaux.

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1. Introduction

Throughout this paper, $\mathbb{F}$ is an arbitrary field and q is a fixed non-zero element of $\mathbb{F}$. Given a positive integer n, W_n denotes the symmetric group of degree n and $\mathscr{H}_n$ denotes the Iwahori–Hecke algebra of W_n: this is the unital associative $\mathbb{F}$-algebra with generators $T_1, \ldots, T_{n-1}$ and relations

$$(T_i - q)(T_i + 1) = 0 \quad \text{for } 1 \leqslant i \leqslant n - 1,$$

$$T_i T_j = T_j T_i \quad \text{for } 1 \leqslant i \leqslant j - 2 \leqslant n - 3,$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad \text{for } 1 \leqslant i \leqslant n - 2.$$

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The representation theory of $\mathscr{H}_n$ closely resembles (and informs) the modular representation theory of W_n. This theory was first explored in detail by Dipper and James, whose account [DJ1] we closely follow. In particular, they define a 'permutation module' M^λ for each composition λ of n, and a Specht module S^μ for each partition μ of n. Understanding the structure of Specht modules is an important goal in understanding the representation theory of $\mathscr{H}_n$, and one well-studied aspect of this is the examination of $\mathscr{H}_n$-homomorphisms between Specht modules.

The aim of this paper is to provide a 'straightening' result which yields (as long as $q \neq -1$) a fast algorithm to determine $\mathrm{Hom}_{\mathscr{H}_n}(S^\mu, S^\lambda)$. This result actually concerns the space of homomorphisms from S^μ to M^λ; an important family of such homomorphisms is labelled by μ-tableaux of type λ; when $q \neq -1$, this family spans the space $\mathrm{Hom}_{\mathscr{H}_n}(S^\mu, M^\lambda)$ [DJ2, Corollary 8.2]. Regardless of q, the space spanned by the tableau homomorphisms has a basis consisting of the homomorphisms labelled by *semistandard* tableaux. However, existing proofs of this result involve a loss of information which means that it is not easy to express a given tableau homomorphism explicitly in terms of semistandard homomorphisms. In this paper we prove a relation between tableau homomorphisms which allows us to do exactly this; this closely resembles the *Garnir relations* which are typically used to straighten elements of the Specht module.

In the next section, we very briefly recall the background information that we need, referring to [DJ1] for most of it. In Section 3 we prove our Garnir-like relation for tableaux with only two rows. In Section 4 we prove the necessary results concerning row removal to show that our Garnir relation holds generally. Finally, in Section 5 we give an algorithm for expressing a tableau homomorphism as a linear combination of semistandard homomorphisms.

2. Background

2.1. Basic definitions

We take almost all of our notation from [DJ1], with only very minor modifications. We provide a brief index here of most of the notation, referring the reader to [DJ1] for definitions. Homomorphisms are considered in more detail below. Note that we follow [DJ1] by considering right modules and having all functions act on the right.

W_n	the symmetric group on $\{1, \ldots, n\}$
l	the (Coxeter) length function on W_n
$\lambda \vDash n$	λ is a composition of n
$\lambda \vdash n$	λ is a partition of n
λ'	the conjugate partition to λ
W_λ	the standard Young subgroup of W_n defined by $\lambda \vDash n$
$\mathscr{D}_\lambda$	the set of minimal-length right coset representatives for W_λ in W_n
$\sim$	the row-equivalence relation on tableaux
$\mathfrak{t}^\lambda$	the λ-tableau which has $1, \ldots, n$ in order along successive rows, for $\lambda \vDash n$
$\mathfrak{t}_\lambda$	the λ-tableau which has $1, \ldots, n$ in order down successive columns, for $\lambda \vdash n$[1]
w_λ	the permutation such that $\mathfrak{t}^\lambda w_\lambda = \mathfrak{t}_\lambda$
$\mathscr{H}_n$	the Iwahori–Hecke algebra of W_n over $\mathbb{F}$ with parameter q
$T_1, \ldots, T_{n-1}$	the standard generators of $\mathscr{H}_n$
T_w	standard basis element of $\mathscr{H}_n$, for $w \in W_n$
x_λ	$\sum_{w \in W_\lambda} T_w$
y_λ	$\sum_{w \in W_\lambda} (-q)^{-l(w)} T_w$
M^λ	the 'permutation module' $x_\lambda \mathscr{H}_n$
S^λ	the Specht module $x_\lambda T_{w_\lambda} y_{\lambda'} \mathscr{H}_n$

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[1] N.b. this notation is not defined in [DJ1], although this tableau is used there. This notation is standard elsewhere.

2.2. Multisets

In this paper we frequently employ multisets of positive integers. If C is a multiset of positive integers, then we write C_i for the number of is in C. Given multisets C, D, we write $C \sqcup D$ for the multiset with $(C \sqcup D)_i = C_i + D_i$ for each i. If A is a tableau, we write A^i for the multiset of entries in row i of A.

2.3. Quantum binomial coefficients

Given our fixed q, we write $[n] = 1 + q + \cdots + q^{n-1}$ for any non-negative integer n, and $[n]! = \prod_{i=1}^{n}[i]$. These expressions are useful in the study of $\mathscr{H}_n$, since we have $[n]! = \sum_{w \in W_n} q^{l(w)}$. Furthermore, if $\lambda \vDash n$, then any $w \in W_n$ can be written uniquely as vd with $v \in W_\lambda$, $d \in \mathscr{D}_\lambda$ and $l(w) = l(v) + l(d)$, so we have

$$\sum_{w \in \mathscr{D}_\lambda} q^{l(w)} = \frac{[n]!}{\prod_i [\lambda_i]!}.$$

Note that the right-hand side of this expression is a polynomial in q, so makes sense even when $\prod_i [\lambda_i]!$ is zero.

If q is an indeterminate and $n \geqslant r \geqslant 0$, then we can define the *quantum binomial coefficient*

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{[n]!}{[r]![n-r]!}.$$

In fact, this is a polynomial in q, so we can extend the definition to arbitrary q by defining $\begin{bmatrix} n \\ r \end{bmatrix}$ to be the specialisation of this polynomial.

2.4. Homomorphisms between permutation modules

We now recall the definitions from [DJ1, §3] on homomorphisms between permutation modules. Throughout this subsection we fix $\lambda, \mu \vDash n$.

Recall that a μ-tableau A of type λ is *row-standard* if its entries increase weakly along the rows. We write $\mathcal{T}_r(\mu, \lambda)$ for the set of row-standard μ-tableaux of type λ. If in addition $\mu \vdash n$, then we say that a μ-tableau of type λ is *semistandard* if its entries are weakly increasing along the rows and strictly increasing down the columns, and we write $\mathcal{T}_0(\mu, \lambda)$ for the set of semistandard μ-tableaux of type λ.

Given a μ-tableau A of type λ, we define the permutation 1_A by specifying that $t^\lambda 1_A$ is the row-standard λ-tableau (of type 1^n) in which i belongs to row r if the position occupied by i in t^μ is occupied by r in A. For example, if

$$A = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 1 & 2 \\ \hline \end{array}$$

then

$$t^\lambda 1_A = \begin{array}{|c|c|c|} \hline 1 & 4 & 5 \\ \hline 2 & 6 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}$$

so that $1_A = (2\ 4)(3\ 5\ 6)$.

The map $A \mapsto 1_A$ defines a bijection from $\mathcal{T}_r(\mu, \lambda)$ to the set $\mathcal{D}_\lambda \cap \mathcal{D}_\mu^{-1}$ of minimal-length (W_λ, W_μ) double coset representatives in W_n. For later use we record the simple fact that if $A \in \mathcal{T}_r(\mu, \lambda)$, then

$$l(1_A) = \sum_{g < h} \sum_{i < j} A_j^g A_i^h.$$

Now we consider homomorphisms. If $A \in \mathcal{T}_r(\mu, \lambda)$, then we define an $\mathcal{H}_n$-homomorphism $\phi_A : M^\mu \to M^\lambda$ by

$$x_\mu \phi_A = \sum_{w \in W_\lambda 1_A W_\mu} T_w. \tag{H1}$$

We call this a *tableau homomorphism*. In the case where μ is a partition, we write $\hat{\phi}_A$ for the restriction of ϕ_A to the Specht module S^μ. We remark that in more recent literature ϕ_A, $\hat{\phi}_A$ are written as Θ_A, $\hat{\Theta}_A$.

Now we have the following theorem.

Theorem 2.1. *(See [DJ1, Theorem 3.4], [DJ2, Corollary 8.7].)*

(1) *If $\mu, \lambda \vDash n$, then the set $\{\phi_A \mid A \in \mathcal{T}_r(\mu, \lambda)\}$ is a basis for $\mathrm{Hom}_{\mathcal{H}_n}(M^\mu, M^\lambda)$.*
(2) *If $\mu \vdash n$ and $\lambda \vDash n$, then the set $\{\hat{\phi}_A \mid A \in \mathcal{T}_0(\mu, \lambda)\}$ is a basis for the subspace of $\mathrm{Hom}_{\mathcal{H}_n}(S^\mu, M^\lambda)$ spanned by $\{\hat{\phi}_A \mid A \in \mathcal{T}_r(\mu, \lambda)\}$.*
(3) *If $\mu \vdash n$, $\lambda \vDash n$ and $q \neq -1$, then every $\mathcal{H}_n$-homomorphism $S^\mu \to M^\lambda$ can be extended to a homomorphism $M^\mu \to M^\lambda$. Hence $\{\hat{\phi}_A \mid A \in \mathcal{T}_0(\mu, \lambda)\}$ is a basis for $\mathrm{Hom}_{\mathcal{H}_n}(S^\mu, M^\lambda)$.*

The focus of this paper is part (2) of the above theorem; we provide an explicit algorithm for writing a given homomorphism ϕ_A as a linear combination of semistandard homomorphisms. Of course, the proof of Theorem 2.1(2) does yield an algorithm for 'semistandardising' a tableau homomorphism, but it is rather slow, even in the symmetric group case $q = 1$; one must consider the image of a polytabloid under the different homomorphisms ϕ_A, and keep track of the coefficients of various tabloids in these images. Our algorithm involves just manipulation of row-standard tableaux, forgetting the underlying homomorphisms. In simple computer experiments, our algorithm seems considerably quicker.

There are several other useful ways to write the image $x_\mu \phi_A$. From [DJ1, p. 30], we have

$$x_\mu \phi_A = \sum_{A^\diamond \sim A} x_\lambda T_{1_{A^\diamond}}. \tag{H2}$$

By considering the row permutations sending A to the various $A^\diamond$, we can also write this (following [DJ1, Theorem 3.4]) as

$$x_\mu \phi_A = x_\lambda T_{1_A} \sum_{w \in \mathcal{D}_\nu \cap W_\mu} T_w, \tag{H3}$$

where ν is the 'composition defined by reading A along rows', i.e. the composition

$$\left(A_1^1, A_2^1, A_3^1, \ldots, A_1^2, A_2^2, A_3^2, \ldots, A_1^3, A_2^3, \ldots \right).$$

We will often use the expression (H3), and write the term $\sum_{w \in \mathcal{D}_\nu \cap W_\mu} T_w$ as $|A\rangle$.

Symmetrically, we have

$$x_\mu \phi_A = \sum_{w \in \mathscr{D}_\pi^{-1} \cap W_\lambda} T_w T_{1_A} x_\mu, \tag{H4}$$

where π is the composition

$$\left(A_1^1, A_1^2, A_1^3, \ldots, A_2^1, A_2^2, A_2^3, \ldots, A_3^1, A_3^2, \ldots \right).$$

We shall write the factor $\sum_{w \in \mathscr{D}_\pi^{-1} \cap W_\lambda} T_w$ as $\langle A |$.

Our main interest is in understanding linear combinations $\sum_{i=1}^r c_i \phi_{A_i}$ (where $A_1, \ldots, A_r \in \mathcal{T}_r(\mu, \lambda)$ and $c_1, \ldots, c_r \in \mathbb{F}$) for which $\sum_{i=1}^r c_i \hat{\phi}_{A_i} = 0$. Since S^μ is generated by $x_\mu T_{w_\mu} y_{\mu'}$, this condition amounts to saying that $\sum_{i=1}^r c_i x_\mu \phi_{A_i}$ is annihilated by $T_{w_\mu} y_{\mu'}$.

3. A Garnir-type relation for tableaux with two rows

Now we come to our main result. We begin with tableaux having only two rows.

Theorem 3.1. *Suppose $\mu = (m, n - m)$ is a partition, and R, S, T are multisets of positive integers with $|R| + |S| + |T| = n$ and $|S| > m$. Let $\mathcal{S}$ be the set of all pairs (U, V) such that $S = U \sqcup V$ and $|U| = m - |R|$. For each $(U, V) \in \mathcal{S}$, let $A[U, V]$ denote the row-standard μ-tableau with*

$$A[U, V]^1 = R \sqcup U, \qquad A[U, V]^2 = T \sqcup V.$$

Then

$$\sum_{(U, V) \in \mathcal{S}} \prod_i \begin{bmatrix} R_i + U_i \\ R_i \end{bmatrix} \begin{bmatrix} T_i + V_i \\ T_i \end{bmatrix} \prod_{i < j} q^{R_j U_i + T_i V_j} \hat{\phi}_{A[U, V]} = 0.$$

An example of this theorem is given in Section 5. The way we prove the theorem will be to express the given homomorphism (extended to M^μ) as a composition $\psi \chi$ such that $\psi |_{S^\mu} = 0$. To do this, we need a few simple results concerning composition of tableau homomorphisms. Some of these are probably known, but we prove everything in order to keep our account self-contained.

Our first result shows how to use a tableau of the form $\boxed{1}\boxed{1}\ \cdots\ \boxed{1}\boxed{2}\ \cdots\ \boxed{2}$ to merge two rows of a tableau.

Proposition 3.2. *Suppose $\xi = (r, m - r)$ is a two-part composition of m, and C is a row-standard ξ-tableau of type $\alpha \vDash m$. Let B be the unique row-standard (m)-tableau of type ξ, and A the unique row-standard (m)-tableau of type α. Then*

$$\phi_B \phi_C = \prod_i \begin{bmatrix} A_i^1 \\ C_i^1 \end{bmatrix} \prod_{i < j} q^{C_j^1 C_i^2} \phi_A.$$

Proof. Eq. (H1) shows that for a row-standard (m)-tableau D of any type, we have

$$x_{(m)} \phi_D = x_{(m)}.$$

So we just need to show that $x_{(m)}\phi_B\phi_C$ is the appropriate scalar multiple of $x_{(m)}$. We have

$$x_{(m)}\phi_B = x_\xi \sum_{w\in\mathscr{D}_\xi} T_w$$

by (H3), while (H4) gives

$$x_\xi\phi_C = \sum_{w\in\mathscr{D}_\pi^{-1}\cap W_\alpha} T_w T_{1_C} x_\xi,$$

where π is the composition $(C_1^1, C_1^2, C_2^1, C_2^2, C_3^1, C_3^2, \ldots)$. So

$$x_{(m)}\phi_B\phi_C = \sum_{w\in\mathscr{D}_\pi^{-1}\cap W_\alpha} T_w T_{1_C} x_\xi \sum_{w\in\mathscr{D}_\xi} T_w = \sum_{w\in\mathscr{D}_\pi^{-1}\cap W_\alpha} T_w T_{1_C} x_{(m)}.$$

To evaluate this expression, we use the fact that $T_w x_{(m)} = q^{l(w)} x_{(m)}$ for $w \in W_n$. The length of 1_C is $\sum_{i<j} C_j^1 C_i^2$, and since $W_\pi \leqslant W_\alpha$, we have

$$\sum_{w\in\mathscr{D}_\pi^{-1}\cap W_\alpha} q^{l(w)} = \frac{\sum_{s\in W_\alpha} q^{l(s)}}{\sum_{s\in W_\pi} q^{l(s)}} = \frac{\prod_i [\alpha_i]!}{\prod_i [\pi_i]!} = \prod_i \begin{bmatrix} A_i^1 \\ C_i^1 \end{bmatrix}.$$

The result follows. □

The next result is a more complicated version of the result above.

Proposition 3.3. *Suppose (r, u, v, t) is a composition of n. Let C be a row-standard (r, u, v, t)-tableau of type $\lambda \vDash n$, and let B be the row-standard $(r + u, v + t)$-tableau in which there are r 1s and u 2s in the first row, and v 3s and t 4s in the second. Let A be the row-standard $(r + u, v + t)$-tableau with $A^1 = C^1 \sqcup C^2$ and $A^2 = C^3 \sqcup C^4$. Then*

$$\phi_B\phi_C = \prod_i \begin{bmatrix} A_i^1 \\ C_i^1 \end{bmatrix} \begin{bmatrix} A_i^2 \\ C_i^3 \end{bmatrix} \prod_{i<j} q^{C_j^1 C_i^2 + C_j^3 C_i^4} \phi_A.$$

Proof. Since $1_B = 1$, (H3) gives

$$x_{(r+u,v+t)}\phi_B = x_{(r,u,v,t)} \sum_{w\in\mathscr{D}_{(r,u,v,t)}\cap W_{(r+u,v+t)}} T_w$$

and (H4) gives

$$x_{(r,u,v,t)}\phi_C = \sum_{w\in\mathscr{D}_\pi^{-1}\cap W_\lambda} T_w T_{1_C} x_{(r,u,v,t)},$$

where π is the composition $(C_1^1, C_1^2, C_1^3, C_1^4, C_2^1, C_2^2, C_2^3, C_2^4, C_3^1, C_3^2, C_3^3, C_3^4, \ldots)$. So

$$x_{(r,u,v,t)}\phi_B\phi_C = \sum_{w\in\mathscr{D}_\pi^{-1}\cap W_\lambda} T_w T_{1_C} x_{(r,u,v,t)} \sum_{w\in\mathscr{D}_{(r,u,v,t)}\cap W_{(r+u,v+t)}} T_w$$

$$= \sum_{w\in\mathscr{D}_\pi^{-1}\cap W_\lambda} T_w T_{1_C} x_{(r+u,v+t)}.$$

T_{1_C} factorises as $T_{1_A} T_d$, where $d \in W_{(r+u,v+t)}$, so the above expression becomes

$$q^{l(d)} \sum_{w\in\mathscr{D}_\pi^{-1}\cap W_\lambda} T_w T_{1_A} x_{(r+u,v+t)} = q^{l(d)} \sum_{w\in\mathscr{D}_\rho^{-1}\cap W_\lambda} T_w \sum_{w\in\mathscr{D}_\pi^{-1}\cap W_\rho} T_w T_{1_A} x_{(r+u,v+t)}$$

where ρ is the composition $(A_1^1, A_1^2, A_2^1, A_2^2, \ldots)$, and $l(d) = \sum_{i<j}(C_j^1 C_i^2 + C_j^3 C_i^4)$. Now 1_A conjugates the pair (W_π, W_ρ) to the pair (W_σ, W_τ), where

$$\sigma = (C_1^1, C_1^2, C_2^1, C_2^2, C_3^1, C_3^2, \ldots, C_1^3, C_1^4, C_2^3, C_2^4, C_3^3, C_3^4, \ldots),$$
$$\tau = (A_1^1, A_2^1, A_3^1, \ldots, A_1^2, A_2^2, A_3^2, \ldots),$$

and this gives

$$\sum_{w\in\mathscr{D}_\pi^{-1}\cap W_\rho} T_w T_{1_A} = T_{1_A} \sum_{w\in\mathscr{D}_\sigma^{-1}\cap W_\tau} T_w.$$

W_τ is contained in $W_{(r+u,v+t)}$, so

$$\sum_{w\in\mathscr{D}_\sigma^{-1}\cap W_\tau} T_w x_{(r+u,v+t)} = \sum_{w\in\mathscr{D}_\sigma^{-1}\cap W_\tau} q^{l(w)} x_{(r+u,v+t)} = \prod_i \begin{bmatrix} A_i^1 \\ C_i^1 \end{bmatrix} \begin{bmatrix} A_i^2 \\ C_i^3 \end{bmatrix} x_{(r+u,v+t)}.$$

So we have

$$x_{(r,u,v,t)}\phi_B\phi_C = \prod_i \begin{bmatrix} A_i^1 \\ C_i^1 \end{bmatrix} \begin{bmatrix} A_i^2 \\ C_i^3 \end{bmatrix} \prod_{i<j} q^{C_j^1 C_i^2 + C_j^3 C_i^4} \sum_{w\in\mathscr{D}_\rho^{-1}\cap W_\lambda} T_w T_{1_A} x_{(r+u,v+t)}$$

$$= \prod_i \begin{bmatrix} A_i^1 \\ C_i^1 \end{bmatrix} \begin{bmatrix} A_i^2 \\ C_i^3 \end{bmatrix} \prod_{i<j} q^{C_j^1 C_i^2 + C_j^3 C_i^4} x_{(r,u,v,t)}\phi_A. \qquad \square$$

The next result effectively enables us to split a row of a tableau in two, using a tableau of the form

1	1	$\cdots$	1	1
2	2	$\cdots$		2	2
2	2	$\cdots$	2	2
3	3	$\cdots$	3	3

.

Lemma 3.4. *Suppose (r, u, v, t) is a composition of n, and let D be the unique (r, u, v, t)-tableau in which the entries in row 1 are equal to 1, the entries in rows 2 and 3 are all equal to 2, and the entries in row 4 are all equal to 3. Let E be a row-standard $(r, u + v, t)$-tableau of any type, and let $\mathcal{C}$ be the set of all row-standard*

(r, u, v, t)-*tableaux* C *such that*

$$C^1 = E^1, \qquad C^2 \sqcup C^3 = E^2, \qquad C^4 = E^3.$$

Then

$$\phi_D \phi_E = \sum_{C \in \mathcal{C}} \phi_C.$$

Proof. We have $x_{(r,u,v,t)} \phi_D = x_{(r,u+v,t)}$, so by (H2)

$$x_{(r,u,v,t)} \phi_D \phi_E = x_\lambda \sum_{E^\diamond \sim E} T_{1_{E^\diamond}},$$

where λ is the type of E. But it is easy to see that

$$\{ 1_{E^\diamond} \mid E^\diamond \sim E \} = \bigsqcup_{C \in \mathcal{C}} \{ 1_{C^\diamond} \mid C^\diamond \sim C \}$$

so

$$x_{(r,u,v,t)} \phi_D \phi_E = x_\lambda \sum_{C \in \mathcal{C}} \sum_{C^\diamond \sim C} T_{1_{C^\diamond}} = \sum_{C \in \mathcal{C}} x_\lambda \sum_{C^\diamond \sim C} T_{1_{C^\diamond}} = \sum_{C \in \mathcal{C}} x_\lambda \phi_C. \qquad \square$$

The preceding results allow us to express the homomorphism of Theorem 3.1 as a composition of homomorphisms indexed by row-standard tableaux.

Proposition 3.5. *Suppose* $R, T, S, \mathcal{S}$ *are as in Theorem 3.1, and let* $r = |R|$, $s = |S|$, $t = |T|$, $u = m - r$, $v = n - m - t$. *Let* E *be the row-standard* (r, s, t)-*tableau which has* $(E^1, E^2, E^3) = (R, S, T)$, *let* B *be the* $(r + u, v + t)$-*tableau from Proposition 3.3, and let* D *be the* (r, u, v, t)-*tableau from Lemma 3.4. Then*

$$\phi_B \phi_D \phi_E = \sum_{(U,V) \in \mathcal{S}} \prod_i \begin{bmatrix} R_i + U_i \\ R_i \end{bmatrix} \begin{bmatrix} T_i + V_i \\ T_i \end{bmatrix} \prod_{i<j} q^{R_j U_i + T_i V_j} \phi_{A[U,V]}.$$

Proof. From Lemma 3.4, we have

$$\phi_D \phi_E = \sum_{C \in \mathcal{C}} \phi_C;$$

each tableau $C \in \mathcal{C}$ satisfies $(C^1, C^2, C^3, C^4) = (R, U, V, T)$ for some pair $(U, V) \in \mathcal{S}$. When we compose ϕ_B with ϕ_C using Proposition 3.3, the tableau A we obtain is precisely $A[U, V]$, and the coefficient of ϕ_A is the coefficient of $\phi_{A[U,V]}$ in the proposition. $\square$

Proof of Theorem 3.1. Using Proposition 3.5, we must show that the composition $\phi_B \phi_D \phi_E$ kills S^μ. In fact, we will show that $\phi_B \phi_D$ kills S^μ. The codomain of $\phi_B \phi_D$ is the permutation module $M^{(r,s,t)}$, while the Specht module S^μ is generated by $x_\mu T_{w_\mu} y_{\mu'}$, so it suffices to show that $M^{(r,s,t)} y_{\mu'} = 0$. But this is straightforward: $M^{(r,s,t)}$ is spanned by elements $x_{(r,s,t)} T_d$ which correspond to (r, s, t)-tabloids (this correspondence is introduced on [DJ1, p. 31]). If u is an (r, s, t)-tabloid, then because $s > m$ there must be some pair $(i, i+1)$ which lie in the same column of t_μ and both lie in the second row of u. $y_{\mu'}$ may be factorised as $(1 - q^{-1} T_i) h$, and the fact that $i, i+1$ both lie in the same row of u means that $u T_i = q u$. Hence $u y_{\mu'} = u(1 - q^{-1} T_i) h = 0$. $\square$

Remark. The argument used here to show that $M^{(r,s,t)} y_{\mu'} = 0$ is a special case of [DJ1, Lemma 4.1] which shows that $M^\nu y_{\mu'} = 0$ whenever $\nu' \not\trianglerighteq \mu'$.

4. Row removal

In this section, we prove a result which allows us to apply our Garnir-type relation to tableaux with more than two rows. Using the results of this section, we will be able to prove a q-analogue of [FM, Lemma 4], which essentially says that given a linear relation between homomorphisms $\hat{\phi}_A$, we can add a fixed combination of rows to each of the tableaux and preserve the relation.

We remark that row-removal for tableau homomorphisms has been considered at length, by the author and Lyle [FL1] for the symmetric group, and by Lyle and Mathas [LM] for the Hecke algebra. However, the result we prove here seems not to have appeared before.

We need to introduce some more notation. Given $m \leqslant n$ and $0 \leqslant r \leqslant n - m$, we define the homomorphism $\Gamma_r : \mathscr{H}_m \to \mathscr{H}_n$ by $T_i \mapsto T_{i+r}$. So Γ_0 is the usual embedding of $\mathscr{H}_m$ in $\mathscr{H}_n$. Clearly we have $\Gamma_r \Gamma_s = \Gamma_{r+s}$ for any r, s. We will also occasionally use Γ_r to denote the corresponding map $W_m \to W_n$.

4.1. Adding a row at the top

The main result we want to prove here is the following.

Proposition 4.1. *Suppose $\mu \vdash n$ and $\lambda \vDash n$, and $A_1, \ldots, A_r$ are row-standard μ-tableaux of type λ which are identical in row 1. Write $\tilde{\mu} = (\mu_2, \mu_3, \ldots)$, and let $\tilde{A}_1, \ldots, \tilde{A}_r$ be the $\tilde{\mu}$-tableaux obtained by deleting the first row from each of $A_1, \ldots, A_r$. If $c_1, \ldots, c_r \in \mathbb{F}$ are such that $\sum_{i=1}^r c_i \hat{\phi}_{\tilde{A}_i} = 0$, then $\sum_{i=1}^r c_i \hat{\phi}_{A_i} = 0$.*

In order to prove Proposition 4.1, we make two comparisons:

- we compare $x_\mu \phi_A$ with $x_{\tilde{\mu}} \phi_{\tilde{A}}$, for a row-standard μ-tableau A;
- we compare the term $T_{w_\mu} y_{\mu'}$ which defines the Specht module with $T_{w_{\tilde{\mu}}} y_{\tilde{\mu}'}$.

Proposition 4.2. *Suppose A is a row-standard μ-tableau of type λ. Write $\tilde{\mu} = (\mu_2, \mu_3, \ldots)$, and let $\tilde{A}$ be the $\tilde{\mu}$-tableau obtained by deleting the first row of A. Let $\tilde{\lambda}$ be the type of $\tilde{A}$. Then we have*

$$x_\mu \phi_A = h \Gamma_{\mu_1}(x_{\tilde{\mu}} \phi_{\tilde{A}})$$

for some $h \in \mathscr{H}_n$ depending only on $\mu, \lambda, \tilde{\lambda}$.

The way we prove Proposition 4.2 is to express ϕ_A as a composition of tableau homomorphisms. Let B be the μ-tableau which is obtained from A by replacing every entry in the first row with a 1 and adding 1 to every entry below the first row, and let C be the tableau whose first row is the same as the first row of A, and whose ith row consists of $\tilde{\lambda}_{i-1}$ $(i-1)$s, for $i \geqslant 2$. Note that both the type of B and the shape of C equal the composition $\kappa = (\mu_1, \tilde{\lambda}_1, \tilde{\lambda}_2, \ldots)$.

Example. Suppose $\mu = (4, 4, 2, 1)$, $\lambda = (3, 4, 2, 1, 1)$ and

$$A = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 3 \\ \hline 1 & 2 & 4 & 5 \\ \hline 2 & 2 \\ \cline{1-2} 1 \\ \cline{1-1} \end{array}.$$

Then $\tilde{\lambda} = (2, 3, 0, 1, 1)$, and

$$B = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\\hline 2 & 3 & 5 & 6 \\\hline 3 & 3 \\\cline{1-2} 2 \\\cline{1-1}\end{array}, \qquad C = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 3 \\\hline 1 & 1 \\\cline{1-2} 2 & 2 & 2 \\\cline{1-3} \\\hline 4 \\\cline{1-1} 5 \\\cline{1-1}\end{array} \quad .$$

Lemma 4.3. *With the notation above, we have* $\phi_B \phi_C = \phi_A$.

Proof. Using (H3), we get

$$x_\mu \phi_B \phi_C = x_\lambda T_{1_C} |C\rangle T_{1_B} |B\rangle.$$

Since the entries in the ith row of C are constant for $i \geqslant 2$, we have $|C\rangle \in \mathscr{H}_{\mu_1}$. On the other hand, since the entries in the first row of B all equal 1, we have

$$T_{1_B} = \Gamma_{\mu_1}(T_{1_{\tilde{A}}}) \in \Gamma_{\mu_1}(\mathscr{H}_{n-\mu_1}).$$

So $|C\rangle$ and T_{1_B} commute. It is not too hard to see that $T_{1_C} T_{1_B} = T_{1_A}$, and it is immediate from the definition of $|A\rangle$ that $|C\rangle|B\rangle = |A\rangle$, so

$$x_\mu \phi_B \phi_C = x_\lambda T_{1_A} |A\rangle = x_\mu \phi_A. \qquad \square$$

Proof of Proposition 4.2. We can write $x_\kappa \phi_C$ as $\langle C| T_{1_C} x_\kappa$, so

$$x_\mu \phi_A = x_\mu \phi_B \phi_C = \langle C| T_{1_C} x_\kappa T_{1_B} |B\rangle.$$

We have $x_\kappa = x_{(\mu_1)} \Gamma_{\mu_1}(x_{\tilde{\lambda}})$; as noted in the proof of Lemma 4.3, T_{1_B} is just $\Gamma_{\mu_1}(T_{1_{\tilde{A}}})$, and it is immediate that $|B\rangle = \Gamma_{\mu_1}(|\tilde{A}\rangle)$. Putting this together, we get

$$x_\mu \phi_A = \langle C| T_{1_C} x_{(\mu_1)} \Gamma_{\mu_1}\big(x_{\tilde{\lambda}} T_{1_{\tilde{A}}} |\tilde{A}\rangle\big);$$

the tableau C depends only on $\mu, \lambda, \tilde{\lambda}$, so the result follows. $\square$

The next thing we need in order to prove Proposition 4.1 is the following.

Proposition 4.4. *Suppose* μ *is a partition of* n, *and write* $\tilde{\mu} = (\mu_2, \mu_3, \ldots)$. *Then*

$$T_{w_\mu} y_{\mu'} \in \Gamma_{\mu_1}(T_{w_{\tilde{\mu}}} y_{\tilde{\mu}'}) \mathscr{H}_n.$$

To prove this, it will be helpful to generalise to *skew partitions*. For us, a skew partition is an ordered pair of partitions (written μ/λ) where $\mu_i \geqslant \lambda_i$ for each i. We define the Young diagram $[\mu/\lambda]$ to be the set difference $[\mu] \setminus [\lambda]$. We identify two skew-partitions which have the same Young diagram, so that for example $(3, 2^2)/(3, 1^1) = (2^3, 1)/(2, 1^3)$. If μ/λ is a skew-partition, then we define its conjugate $(\mu/\lambda)'$ to be μ'/λ'. We say that a skew partition $\tilde{\mu}$ is obtained from a skew partition μ *by translation* if for some i, j we have $[\tilde{\mu}] = \{(r+i, c+j) \mid (r, c) \in [\mu]\}$. For most purposes $\tilde{\mu}$ and μ can be treated as equal in this case.

If μ is a skew partition, then the tableaux $\mathfrak{t}^\mu$, $\mathfrak{t}_\mu$, and hence the permutation w_μ, can be defined in the obvious way, and so can $y_{\mu'}$. Note that if $\tilde{\mu}$ is a skew-partition obtained from μ by translation, then $w_{\tilde{\mu}} = w_\mu$ and $y_{\tilde{\mu}'} = y_{\mu'}$.

Now Proposition 4.4 follows by applying the next lemma μ_1 times.

Lemma 4.5. *Suppose μ is a skew partition of $n > 0$, and let ν be the skew partition obtained by removing the first node from the first non-empty row of $[\mu]$. Then $T_{w_\mu} y_{\mu'} \in \Gamma_1(T_{w_\nu} y_{\nu'}) \mathcal{H}_n$.*

Proof. Suppose the node removed from $[\mu]$ to obtain $[\nu]$ is (r, c). Then $\mathfrak{t}^\mu \Gamma_1(w_\nu)$ is the standard tableau in which the number 1 appears in position (r, c), while the numbers $2, \ldots, n$ appear in order down successive columns. So, if the entry at the bottom of column $c - 1$ is m, then $w_\mu = \Gamma_1(w_\nu) w$, where w is the cycle $(m\ m - 1\ \ldots\ 1)$. Moreover, we have $l(w_\mu) = l(w_\nu) + l(w)$, so $T_{w_\mu} = \Gamma_1(T_{w_\nu}) T_w = \Gamma_1(T_{w_\nu}) T_1 T_2 \ldots T_{m-1}$.

Now $y_{\mu'}$ is the commuting product of factors y_k, where $y_k = \sum_{s \in W(k)} (-q)^{-l(s)} T_s$ (with $W(k)$ being the group of permutations of the entries in column k of $\mathfrak{t}_\mu$). $y_{\nu'}$ is the product of factors $\bar{y}_k$ defined similarly. For $k > c$ we have $y_k = \Gamma_1(\bar{y}_k)$, and y_k commutes with T_w. For $k = c$, we can factorise y_k as $\Gamma_1(\bar{y}_k) y$ using right coset representatives, and $\Gamma_1(\bar{y}_k)$ again commutes with T_w. For $k < c$ we have $y_k = \bar{y}_k$ and $T_w y_k = \Gamma_1(y_k) T_w$. And so

$$T_{w_\mu} y_{\mu'} = \Gamma_1(T_{w_\nu}) T_w \prod_k y_k = \Gamma_1(T_{w_\nu}) \prod_k \Gamma_1(\bar{y}_k) T_w y \in \Gamma_1(T_{w_\nu} y_{\nu'}) \mathcal{H}_n. \qquad \square$$

Example. Let $\mu = (3^2, 2)/(1)$, so that $\nu = (3^2, 2)/(2)$. Then $w_\nu = (1\ 5\ 2)(4\ 6)$, and

$$\mathfrak{t}^\mu \Gamma_1(w_\nu) = \begin{array}{|c|c|c|} \hline & 1 & 2 \\ \hline 3 & 4 & 5 \\ \hline 6 & 7 \\ \cline{1-2} \end{array} (2\ 6\ 3)(5\ 7) = \begin{array}{|c|c|c|} \hline & 1 & 6 \\ \hline 2 & 4 & 7 \\ \hline 3 & 5 \\ \cline{1-2} \end{array}, \qquad w_\mu = \Gamma_1(w_\nu)(1\ 3\ 2).$$

We have $y_{\mu'} = y_1 y_2 y_3$ and $y_{\nu'} = \bar{y}_1 \bar{y}_2 \bar{y}_3$, where

$$\bar{y}_1 = y_1 = 1 - q^{-1} T_1,$$

$$\bar{y}_2 = 1 - q^{-1} T_3, \qquad y_2 = \bar{y}_2 \left(1 - q^{-1} T_4 + q^{-2} T_4 T_3 \right),$$

$$\bar{y}_3 = 1 - q^{-1} T_5, \qquad y_3 = 1 - q^{-1} T_6 = \Gamma_1(\bar{y}_3).$$

So

$$T_{w_\mu} y_{\mu'} = \Gamma_1(T_{w_\nu} y_{\nu'}) T_1 T_2 \left(1 - q^{-1} T_4 + q^{-2} T_4 T_3 \right).$$

Proof of Proposition 4.1. S^μ is generated by $x_\mu T_{w_\mu} y_{\mu'}$, so we need to show that

$$\left(\sum_{i=1}^r c_i x_\mu \phi_{A_i} \right) T_{w_\mu} y_{\mu'} = 0.$$

Using Propositions 4.2 and 4.4, this equals

$$\left(\sum_{i=1}^r h c_i \Gamma_{\mu_1}(x_{\tilde{\mu}} \phi_{\tilde{A}_i}) \right) \Gamma_{\mu_1}(T_{w_{\tilde{\mu}}} y_{\tilde{\mu}'}) k,$$

where $h, k \in \mathscr{H}_n$. Note that h is independent of i: because $A_1, \ldots, A_r$ all have the same first row, $\tilde{A}_1, \ldots, \tilde{A}_r$ all have the same type. So we have

$$h \Gamma_{\mu_1} \left(x_{\tilde{\mu}} \sum_{i=1}^r c_i \phi_{\tilde{A}_i} T_{w_{\tilde{\mu}}} y_{\tilde{\mu}'} \right) k.$$

The term in the middle is $(x_{\tilde{\mu}} T_{w_{\tilde{\mu}}} y_{\tilde{\mu}'}) \sum_{i=1}^r c_i \phi_{\tilde{A}_i}$, which is zero by assumption. $\square$

4.2. Adding a row at the bottom

We now consider a counterpart to Proposition 4.1 in which we remove a row from the bottom of a tableau.

Proposition 4.6. *Suppose $\mu \vdash n$ and $\lambda \vDash n$, and let l be maximal such that $\mu_l > 0$. Suppose $A_1, \ldots, A_r$ are row-standard μ-tableaux of type λ which are identical in row l. Write $\tilde{\mu} = (\mu_1, \ldots, \mu_{l-1}, 0, 0, \ldots)$, and let $\tilde{A}_1, \ldots, \tilde{A}_r$ be the $\tilde{\mu}$-tableaux obtained by deleting the lth row from each of $A_1, \ldots, A_r$. If $c_1, \ldots, c_r \in \mathbb{F}$ are such that $\sum_{i=1}^r c_i \hat{\phi}_{\tilde{A}_i} = 0$, then $\sum_{i=1}^r c_i \hat{\phi}_{A_i} = 0$.*

This is proved in a very similar way to Proposition 4.1; the proof is in fact slightly simpler, because we do not need the functions Γ_s, and we can more easily avoid skew partitions. So we just give the two main propositions which are analogous to Propositions 4.2 and 4.4, and leave the reader to fill in the remaining details.

Proposition 4.7. *Suppose A is a row-standard μ-tableau of type λ, and let l be maximal such that $\mu_l > 0$. Write $\tilde{\mu} = (\mu_1, \ldots, \mu_{l-1}, 0, 0, \ldots)$, and let $\tilde{A}$ be the $\tilde{\mu}$-tableau obtained by deleting the last row of A. Let $\tilde{\lambda}$ be the type of $\tilde{A}$. Then we have*

$$x_\mu \phi_A = h x_{\tilde{\mu}} \phi_{\tilde{A}}$$

where $h \in \mathscr{H}_n$ depends only on μ, λ, $\tilde{\lambda}$.

Proposition 4.8. *Suppose μ is a partition of n, and write $\tilde{\mu} = (\mu_1, \ldots, \mu_l, 0, 0, \ldots)$, where l is maximal such that $\mu_l > 0$. Then*

$$T_{w_\mu} y_{\mu'} \in T_{w_{\tilde{\mu}}} y_{\tilde{\mu}'} \mathscr{H}_n.$$

5. Expressing a homomorphism as a linear combination of semistandard homomorphisms

5.1. Two-part partitions

We now explain how to use Theorem 3.1 to write a row-standard homomorphism $\hat{\phi}_A$ as a linear combination of semistandard homomorphisms. We begin by considering two-part partitions.

Suppose $\mu = (m, n - m)$ is a two-part partition, and A is a row-standard μ-tableau of arbitrary type. If A is not semistandard, then there is some $i \leqslant n - m$ such that the $(1, i)$-entry of A is at least as large as the $(2, i)$-entry. Suppose the $(1, i)$-entry equals j. Now define the following multisets:

$$R = \{k \in A^1 \mid k < j\};$$
$$S = \{k \in A^1 \mid k \geqslant j\} \sqcup \{k \in A^2 \mid k \leqslant j\};$$
$$T = \{k \in A^2 \mid k > j\}.$$

Then $|R| < i$ and $|T| \leqslant n - m - i$, so R, S, T satisfy the conditions of Theorem 3.1. Moreover, the tableau A arises as $A[U_0, V_0]$, where

$$U_0 = \{ k \in A^1 \mid k \geqslant j \}, \qquad V_0 = \{ k \in A^2 \mid k \leqslant j \};$$

note in particular that U_0 consists of the $|U_0|$ largest elements of S. Note also that all the elements of R are strictly less than all the elements of U_0, and the same is true for V_0 and T, so the coefficient of $\hat{\phi}_A$ in Theorem 3.1 is 1.

So Theorem 3.1 allows us to express $\hat{\phi}_A$ as a linear combination of homomorphisms $\hat{\phi}_B$, for row-standard tableaux B satisfying

$$\sum_{k \in B^1} k < \sum_{k \in A^1} k.$$

We can repeat the process for each B which is not semistandard, and eventually express $\hat{\phi}_A$ as a linear combination of semistandard homomorphisms. The above inequality guarantees that this process will terminate.

Example. For this example, we abuse notation and write a homomorphism $\hat{\phi}_B$ just as B. Suppose

$$A = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 2 & 3 & 4 \\ \hline 1 & 3 & 3 & 3 \\ \cline{1-4} \end{array}.$$

Applying Theorem 3.1 with

$$R = \emptyset, \qquad S = \{1, 1, 2, 2, 3, 4\}, \qquad T = \{3, 3, 3\},$$

we get

$$\hat{\phi}_A = -[4] \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 4 \\ \hline 3 & 3 & 3 & 3 \\ \cline{1-4} \end{array} - \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 3 & 4 \\ \hline 2 & 3 & 3 & 3 \\ \cline{1-4} \end{array} - q^3 \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 3 \\ \hline 3 & 3 & 3 & 4 \\ \cline{1-4} \end{array}.$$

Applying Theorem 3.1 again, we have

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 3 & 4 \\ \hline 2 & 3 & 3 & 3 \\ \cline{1-4} \end{array} = -[2] \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 3 \\ \hline 3 & 3 & 3 & 4 \\ \cline{1-4} \end{array} - [2] \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 4 \\ \hline 3 & 3 & 3 & 3 \\ \cline{1-4} \end{array} - \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 3 & 3 \\ \hline 2 & 3 & 3 & 4 \\ \cline{1-4} \end{array}$$

so that

$$\hat{\phi}_A = -(q^2 + q^3) \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 4 \\ \hline 3 & 3 & 3 & 3 \\ \cline{1-4} \end{array} + (1 + q - q^3) \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 3 \\ \hline 3 & 3 & 3 & 4 \\ \cline{1-4} \end{array} + \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 3 & 3 \\ \hline 2 & 3 & 3 & 4 \\ \cline{1-4} \end{array}.$$

5.2. Arbitrary partitions

Now we consider generalising this to arbitrary partitions, using the results of Section 4. By applying Propositions 4.1 and 4.6 repeatedly, we obtain the following theorem.

Theorem 5.1. *Suppose $\mu \vdash n$ and $\lambda \vDash n$, and $1 \leqslant l < m$. Let $\tilde{\mu}$ be the partition $(\mu_l, \mu_{l+1}, \ldots, \mu_m, 0, 0, \ldots)$. Suppose $A_1, \ldots, A_r$ are row-standard μ-tableaux of type λ which are identical outside rows $l, l+1, \ldots, m$, and let $\tilde{A}_1, \ldots, \tilde{A}_r$ be the μ-tableaux obtained by deleting rows $1, \ldots, l-1$ and $m+1, m+2, \ldots$ of $A_1, \ldots, A_r$. If $c_1, \ldots, c_r \in \mathbb{F}$ are such that $\sum_{i=1}^{r} c_i \hat{\phi}_{\tilde{A}_i} = 0$, then $\sum_{i=1}^{r} c_i \hat{\phi}_{A_i} = 0$.*

Now suppose A is a row-standard μ-tableau. If A is not semistandard, then there must be some l such that the tableau $\tilde{A}$ consisting of rows l, $l+1$ of A is not semistandard. Using the algorithm described above, we can express $\hat{\phi}_{\tilde{A}}$ as a linear combination of homomorphisms of the form $\hat{\phi}_{\tilde{B}}$ where each $\tilde{B}$ satisfies $\sum_{k\in\tilde{B}^1} k < \sum_{k\in\tilde{A}^1} k$. Applying Theorem 5.1, we can express $\hat{\phi}_A$ as a linear combination of homomorphisms $\hat{\phi}_B$ where each B satisfies

$$\sum_{i\geqslant 1}\sum_{k\in B^i} ik > \sum_{i\geqslant 1}\sum_{k\in A^i} ik.$$

So by induction we can express ϕ_A as a linear combination of semistandard homomorphisms.

5.3. Computing the space of homomorphisms between two Specht modules

We now explain briefly how to compute the space of $\mathscr{H}_n$-homomorphisms between two Specht modules when $q \neq -1$, which is the main motivation for proving the results in this paper. A more detailed discussion of this topic can be found in the paper [L2] by Lyle.

Suppose $\mu, \lambda \vdash n$. Theorem 2.1(2) says that the set

$$\{\hat{\phi}_A \mid A \text{ is a semistandard } \mu\text{-tableau of type } \lambda\}$$

is a basis for $\mathrm{Hom}_{\mathscr{H}_n}(S^\mu, M^\lambda)$ as long as $q \neq -1$. Since $S^\lambda \subseteq M^\lambda$, an $\mathscr{H}_n$-homomorphism from S^μ to S^λ is simply a linear combination of semistandard homomorphisms whose image lies inside the Specht module S^λ. Fortunately, there is a straightforward way to check this condition: S^λ may be characterised [DJ1, Theorem 7.5] as the intersection of the kernels of certain tableau homomorphisms $\psi_{i,j}$ from M^λ to other permutation modules M^ν. Given a tableau homomorphism $\phi_A : M^\mu \to M^\lambda$, it is known how to compose ϕ_A with $\psi_{i,j}$; this result is [L1, Proposition 2.14], generalising the result [FM, Lemma 5] proved for the symmetric group. However, this composition is given as a linear combination of tableau homomorphisms which are not always semistandard (even if A is semistandard). Since the set $\{\hat{\phi}_A \mid A \in \mathcal{T}_r(\mu, \lambda)\}$ is linearly dependent in general, this can make it difficult to tell whether a given linear combination $\sum_A c_A \hat{\phi}_A \psi_{i,j}$ is non-zero. But using the results in this paper, one can quickly express this homomorphism as a linear combination of semistandard homomorphisms, and hence discover whether it is non-zero.

So one has a reasonably fast algorithm for computing the space of $\mathscr{H}_n$-homomorphisms between two Specht modules when $q \neq -1$. The author has implemented this in a small package for the GAP4 system [GAP2008]. This package has already yielded new results on homomorphisms between Specht modules [FL2,L3,DF].

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