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Further Discussion of a Time-Continuous Gaussian Channel*

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The purpose of this note is to simplify the proof of the coding theorem given by the author (Ash, 1963) for a time-continuous Gaussian channel, and to correct an error in the proof of the weak converse. In addition, further comment will be made concerning the difference between the channel model considered by the author (see also Fortet, 1961 and Bethoux, 1962) and the model which is implicit in the Shannon theory.

A SHORT PROOF OF THE DIRECT HALF OF THE CODING THEOREM AND OF THE EXPONENTIAL BOUND

Using the terminology of the previous paper (Ash, 1963) we wish to prove that the capacity C of the given time-continuous Gaussian channel is at least $K/2$, and in addition, given any $R < C$, there are positive constants A and B (depending on C and R) such that for each T there is a code $(T, M, \beta(T))$ with $\beta(T) \leqq Ae^{-BT}$.

From the beginning of the previous paper until Eq. (11) the discussion is as before. We begin at this point.

Equivalently,

$$\left(\frac{y_1}{\sqrt{\lambda_1}}, \cdots, \frac{y_n}{\sqrt{\lambda_n}}\right) = \left(\frac{x_1}{\sqrt{\lambda_1}}, \cdots, \frac{x_n}{\sqrt{\lambda_n}}\right) + \left(\frac{z_1}{\sqrt{\lambda_1}}, \cdots, \frac{z_n}{\sqrt{\lambda_n}}\right) \quad \text{(B1)}$$

where the random variables $(z_i/\sqrt{\lambda_i})$ are normal (0, 1) and the vectors $(x_1', \cdots, x_n') = ((x_1/\sqrt{\lambda_1}), \cdots, (x_n/\sqrt{\lambda_n}))$ satisfy the constraint

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$$\frac{1}{n}\sum_{i=i}^{n}(x_i')^2 \leqq \frac{KT}{n}. \tag{B2}$$

First assume T is an integer. Since for any T we are free to choose n, we may take $n = \alpha T$, where α is a fixed positive integer. We therefore have a memoryless time-discrete Gaussian channel with noise variance unity and an input "average power" limitation of $(KT/n) = (K/\alpha)$. By the coding theorem for such a channel (Thomasian, 1960, Theorem 1A), if R_0 is a positive number $< D_\alpha = \frac{1}{2}\log(1 + K/\alpha)$, there is a code $(n, [e^{nR_0}], \beta_0(n))$, i.e., a code consisting of $[e^{nR_0}]$ vectors of dimension n with a probability of error $\leqq \beta_0(n)$, such that

$$\beta_0(n) \leqq 3\exp\left\{-\frac{n}{4}\left[\left(1 + 0.64(D_\alpha - R_0)^2\left(1 + \frac{\alpha}{K}\right)\right)^{1/2} - 1\right]\right\}. \tag{B3}$$

Since $e^{nR_0} = e^{(\alpha R_0)T}$, we have the following result:

If $R < C_\alpha = (\alpha/2)\log(1 + K/\alpha)$, there is a code $(T, [e^{RT}], \beta(T))$ for the original channel such that

$$\beta(T) \leqq 3\exp\left\{-\frac{\alpha T}{4}\left[\left(1 + .64\left(\frac{C_\alpha - R}{\alpha}\right)^2\left(1 + \frac{\alpha}{K}\right)\right)^{1/2} - 1\right]\right\} = 3e^{-\gamma T}. \tag{B4}$$

If T is not an integer, then by the above procedure we can construct a code $([T], [e^{R[T]}], \beta'(T))$ where $\beta'(T) \leqq 3e^{-\gamma[T]} \leqq 3e^{\gamma}e^{-\gamma T}$. This code immediately gives a code $(T, [e^{R[T]}], \beta(T))$ where $\beta(T) = \beta'(T)$, as follows:

If $s_{[T]}(t)$ is a code word of the code $([T], [e^{R[T]}], \beta'(T))$, then we let $s_T(t)$ be a code word of the code $(T, [e^{R[T]}], \beta(T))$. We decode by observing the output only over the interval $[-[T], [T]]$ and then using the decision procedure dictated by the code $([T], [e^{R[T]}], \beta'(T)$. Since $e^{R[T]} \geqq e^{R(T-1)} = e^{(R-R/T)T} \geqq e^{(R-\varepsilon)T}$ for any fixed $\varepsilon > 0$ and sufficiently large T, we conclude that for the purpose of proving that $C \geqq K/2$ and establishing the exponential approach of the probability of error to zero, we may assume without loss of generality that T is an integer. Any explicit bound on $\beta(T)$ derived for this case may have to be modified before it will apply in general.

Now (B4) implies, by definition of C, that $C \geqq C_\alpha$ *for any* α; allowing α to approach infinity we obtain $C \geqq K/2$, proving the direct half of the coding theorem. The exponential bound follows if we note that, given $R < K/2$, there is an $\alpha = \alpha_R$ such that $R < C_{\alpha_R} < K/2$. Hence for each positive integer T, there is a code $(T, [e^{RT}], \beta(T))$ such that $\beta(T) \leqq$

$A_0e^{-B_0T}$ where A_0 and B_0 are as in Eq. (B4) with $\alpha = \alpha_R$. The proof is complete.

CORRECTION TO THE PROOF OF THE WEAK CONVERSE

We wish to prove that any code (T, M, β) with $\beta < \frac{1}{2}$ must satisfy $\log M < [(KT/2 + \log 2)/(1 - 2\beta)]$. We follow the proof given in the previous paper until Eq. (36). The statement after this equation that "for each A_i^* there is measurable cylinder $B_i C A_i^*$ such that $P\{(s_{i1}, s_{i2}, \cdots) + (z_1, z_2, \cdots) \in B_i\} \geqq 1 - 2\beta$" is not correct. However, we may reason as follows:

By a standard approximation theorem (e.g., Halmos, 1950, p. 56, Theorem D), given any positive number ε there exists, for each set A_i^*, a measurable cylinder B_i^* such that

$$P\{(s_{i1}, s_{i2}, \cdots) + (z_i, z_2, \cdots) \in A_i^* \Delta B_i^*\} < \frac{\varepsilon}{2M} \tag{B5}$$

where $A_i^* \Delta B_i^* = (A_i^* - B_i^*) \cup (B_i^* - A_i^*)$ is the symmetric difference between A_i^* and B_i^* (If A and B are sets, A-B will denote the set of elements which belong to A but not to B). The sets B_i^* may not be disjoint, but if we define sets B_i by

$$B_i = B_i^* - (\bigcup_{j \neq i} B_j^*), \qquad i = 1, 2, \cdots, M \tag{B6}$$

then the B_i are a disjoint collection of measurable cylinders. If we let $\mathbf{y}_i = (s_{i1}, s_{i2}, \cdots) + (z_1, z_2, \cdots)$, we have

$$\begin{aligned} P\{\mathbf{y}_i \in B_i\} &= P\{\mathbf{y}_i \in B_i^*\} \\ &\quad - P\{\mathbf{y}_i \in (\bigcup_{j \neq i} B_j^*) \cap B_i^*\} \geqq P\{\mathbf{y}_i \in B_i^*\} \\ &\quad - \sum_{j \neq i} P\{\mathbf{y}_i \in B_j^* \cap B_i^*\}. \end{aligned} \tag{B7}$$

Since the A_i^* are disjoint, $B_i^* \cap B_j^* \subset (A_i^* \Delta B_i^*) \cup (A_j^* \Delta B_j^*)$ for $i \neq j$. Hence $P\{\mathbf{y}_i \in B_i\} \geqq P\{\mathbf{y}_i \in A_i^*\} - (\varepsilon/2M) - (M-1)(\varepsilon/M) > P\{\mathbf{y}_i \in A_i^*\} - \varepsilon$. Since $P\{\mathbf{y}_i \in A_i^*\} \geqq 1 - \beta$, if $\beta > 0$ we may take $\varepsilon = \beta$ to obtain $P\{\mathbf{y}_i \in B_i\} \geqq 1 - 2\beta$, $i = 1, 2, \cdots, M$, just as in the second line below Eq. (36). In fact we may assume without loss of generality that $\beta > 0$ since any code (T, M, β) is a code (T, M, β') for any $\beta' \geqq \beta$ by our definition of a code. From this point on, the proof proceeds exactly as before; the B_i need not be subsets of A_i^*.

A FURTHER CORRECTION

After Eq. (31) of the previous paper, the text should read:

For each T we are free to choose n. For simplicity assume T is an integer and let $n = \delta T$, where δ is a positive integer large enough so that Eq. (19) is satisfied. An examination of Eq. (31) shows that $\beta(T) \leqq A_0 e^{-B_0 T}$, where

$$A_0 = 3$$

and

$$B_0 = \min\left(\frac{\epsilon}{2}, \frac{\delta}{4}\left(\sqrt{1 + \epsilon^2/16\delta^2} - 1\right), \frac{\delta}{2}\log b(K, \epsilon)\right]. \tag{32}$$

A similar but more cumbersome bound can be developed when T is not an integer (see discussion after Eq. B4). In either case $\beta(T) \to 0$ exponentially as $T \to \infty$, proving Theorem 2.

WHAT IS A TIME-CONTINUOUS CHANNEL?

In the model which we have been using, the code words are truncated versions $s_T(t)$ of signals $s(t)$ satisfying the constraint

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|S(\omega)|^2}{N(\omega)}\, d\omega \leqq KT$$

The output $s_T(t) + n_T(t)$ is observed over the interval $[-T, T]$ and then a decision is made as to the identity of the input signal. The model ignores the effect of the "tail" of $s(t)$ on the transmission of code words in the intervals $[T, 3T]$, $[3T, 5T]$, $[-3T, -T]$, etc. Shannon's formulation (Shannon, 1948) does take the interference problem into account, but in the Shannon model a capacity has not yet been established. To fix ideas, let us consider band-limited noise, i.e., let $N(\omega) = N/2$, $-2\pi W \leqq \omega \leqq 2\pi W$; $N(\omega) = 0$ elsewhere, and let $K = 2P/N$. The "allowable" signals $s(t)$ are those which are limited to the same frequency band as the noise, and which satisfy the input constraint

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |S(\omega)|^2\, dw \leqq PT,$$

i.e., the classical "average power" limitation.

We may formulate Shannon's problem as follows: *A code* (T, M, β) is a set $\{(s_1(t), A_1), \cdots, (s_M(t), A_M)\}$ where each $s_i(t)$ is an allowable function $s(t)$, truncated to $[-T, T]$, and the A_j are disjoint Borel sets in the space of real functions with domain $[-T, T]$ such that the following condition holds:

Let $\cdots, s^{(-1)}(t), s^{(0)}(t), s^{(1)}(t), \cdots$ be *any* sequence of allowable signals such that for each i, the truncated version $s_T^{(i)}(t)$ is a code word, i.e., $(s_T^{(i)}(t), A^{(i)})$ is an element of the code for some Borel set $A^{(i)}$. Then

$$P\left\{\left(\sum_{j=-\infty}^{\infty} s^{(j)}(t-2jT)\right)_T + n_T(t) \in A^{(0)}\right\} \geqq 1-\beta \qquad \text{(B8)}$$

the term $(\sum_{j=-\infty}^{\infty} s^{(j)}(t-2jT))_T$, the truncation of $\sum_{j=-\infty}^{\infty} s^{(j)}(t-2jT)$, is the sum of a signal transmitted during the interval $[-T, T]$ plus the total contribution of the interference in that interval, for a particular sequence of code words. The condition (B8) states that the probability of error should not exceed β for any such sequence.

As before, a number R is called *permissible rate of transmission* if for each T there is a code $(T, [e^{RT}], \beta(T))$ such that $\beta(T) \to 0$ as $T \to \infty$. The *channel capacity* C^* is the supremum of all premissible transmission rates. Shannon (1948) proved, using the Sampling Theorem, that $C^* \geqq W \log [1 + (P/NW)]$. Since Shannon assumed a specific decoding procedure and a specific method of choosing code words, namely the observation of the output at discrete sampling instants and the restriction of the class of allowable signals to those with only a finite number of non-zero samples, it is not possible to conclude from his results that $C^* = W \log [1 + (P/NW)]$; the evaluation of C^* is still on open problem. All we can do at this time is to observe that $C^* \leqq C = P/N$.

Note that if the bandwidth is infinite, we may solve the interference problem by using time-limited code words; then Shannon's problem is the same as ours and $C^* = C = P/N$.

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References

Ash, R. B. (1963), Capacity and error bounds for a time-continuous gaussian channel. *Inform. and Control*, **6**, 14–27.

Bethoux, P. (1962), Test et estimations concernant certaines fonctions aléatoires en particulier Laplaciennes, *Ann. Inst. Henri Poincaré*, **27**, 255–322.

FORTET, R. (1961), Hypothesis testing and estimation for Laplacian functions. *In* "Fourth Berkeley Symposium on Mathematical Statistics and Probability," Vol. I, pp. 289–305. Univ. of California Press.

HALMOS, P. (1950), "Measure Theory." Van Nostrand, Princeton, New Jersey.

SHANNON, C. E. (1948), A mathematical theory of communication. *Bell System Tech. J.* **27**, 379–423, 623–656.

THOMASIAN, A. J. (1960), Error bounds for continuous channels. *In* "Fourth London Symposium on Information Theory," pp. 46–60. Butterworths, London.