Oscillation Criteria for
Second-Order Neutral Equations
with Distributed Deviating Arguments

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Abstract—In this paper, we establish several sufficient conditions for the oscillation of solutions of second-order neutral functional differential equations with distributed deviating arguments, which generalize and improve some known results. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

In the last decades, the oscillatory behavior of solutions of second-order neutral equations are of both theoretical and practical interest. Some applicable examples can be found in the monograph of Hale [1]. We should also note that the behavior of solutions of neutral-type equations may be quite different than that of nonneutral equations, refer to the monograph of Bainov and Mishev [2]. There have been some results on the oscillatory and asymptotic behavior of second-order neutral equations, refer to the monographs of Bainov and Mishev [2] and Erbe, Kong and Zhang [3]. We note that the main concern of the past research has been discrete delay. However, in many areas of their actual application, models describing these problems are often effected by such factors as seasonal changes. Therefore, it is necessary, either theoretically or practically, to study a type of equations in a more general sense—differential equations with distributed deviating arguments. Several papers concerning neutral functional differential equations with distributed deviating arguments have appeared recently, refer to [4–7] and their references cited therein.

In this paper, we consider the oscillation behavior of solutions of the second-order neutral differential equations of the form

\[ \left[ a(t) \left( x(t) + c(t) x(t - \tau(t)) \right) \right]' + \int_{a}^{b} p(t, \xi) x \left[ g(t, \xi) \right] d\sigma(\xi) = 0, \quad t \geq t_{0}, \]

where \( \tau > 0 \) is a constant.

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We restrict our attention to solutions \( x(t) \) of equation (1) which exist on some half-line and nontrivial for all large \( t \). It is tacitly assumed that such solutions exist.

As is customary, a solution \( x(t) \) of equation (1) is called oscillatory if it has arbitrarily large zeros. Otherwise, it is nonoscillatory; that is, if it is eventually positive or eventually negative.

The objective of this paper is to obtain some general oscillatory criteria of solutions of equation (1). In oscillatory criteria obtained, there are a general class of function \( H(t, s) \) as the parameter function. By choosing different functions \( H(t, s) \), we are able to derive some useful corollaries. The corollaries generalize and improve some known results. For example, in [4], the authors discussed the following neutral equations of the form

\[
|x(t) + c(t)x(t-\tau)|'' + \int_a^b p(t, \xi) x[g(t, \xi)] \, d\sigma(\xi) = 0, \quad t \geq t_0, \tag{2}
\]

and obtained the following results.

**Theorem A.** If

\[
\int_{t_0}^\infty \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} \, d\sigma(\xi) \, ds = \infty, \tag{3}
\]

then every solution of equation (2) is oscillatory.

**Theorem B.** If there exist \( \varphi(t) \in C([t_0, \infty), (0, \infty)) \) such that

\[
\int_{t_0}^\infty \left[ \varphi(s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} \, d\sigma(\xi) - \frac{\varphi^2(s)}{4\varphi(s)g'(s, a)} \right] ds = \infty, \tag{4}
\]

then every solution of equation (2) is oscillatory.

We assume throughout this paper that the following conditions hold:

\begin{enumerate}
\item [(H1)] \( a(t), c(t) \in C([t_0, \infty), R_+) \), and \( c(t) \leq 1, \int_{t_0}^\infty (1/a(s)) \, ds = \infty, R_+ = [0, \infty) \);
\item [(H2)] \( p(t, \xi) \in C([t_0, \infty) \times [a, b], R_+) \), and \( p(t, \xi) \) is not eventually zero on any half-line \([t_\mu, \infty) \times [a, b] \);
\item [(H3)] \( g(t, \xi) \in C([t_0, \infty) \times [a, b], R), g(t, \xi) \leq t, \xi \in [a, b], g(t, \xi) \) is nondecreasing with respect to \( t \) and \( \xi \), respectively, and \( \liminf_{t \to \infty, \xi \in [a, b]} \{g(t, \xi)\} = \infty \);
\item [(H4)] \( \sigma(\xi) \in ([a, b], R) \) is nondecreasing, and the integral of equation (1) is a Stieltjes one.
\end{enumerate}

To obtain the oscillatory criteria of solutions of equation (1), we first give the following lemma.

**Lemma 1.** If \( x(t) \) is an eventually positive solution of equation (1), and

\[
y(t) = x(t) + c(t)x(t-\tau), \tag{5}
\]

then there exists a \( t_1 \geq t_0 \) such that

\[
y(t) > 0, \quad [a(t)y'(t)]' \leq 0, \quad \text{and} \quad y'(t) \geq 0, \quad t \geq t_1. \tag{6}
\]

**Proof.** Let \( x(t) \) be an eventually positive solution of equation (1), and from (H3),

\[
\liminf_{t \to \infty, \xi \in [a, b]} \{g(t, \xi)\} = \infty,
\]

there exists a \( t_1 \geq t_0 \) such that

\[
x(t) > 0, \quad x(t-\tau) > 0, \quad \text{and} \quad x[g(t, \xi)] > 0, \quad t \geq t_1, \quad \xi \in [a, b].
\]

From (5), we have \( y(t) > 0, t \geq t_1 \), and from (1), we have \([a(t)y'(t)]' \leq 0, t \geq t_1\).
Next, we prove \( y'(t) \geq 0, \ t \geq t_1 \). In fact, if there is a \( t_2 \geq t_1 \) with \( y'(t_2) < 0 \), then by \([a(t)y'(t)]' \leq 0\), integrating from \( t_2 \) to \( t \), we have \( a(t)y'(t) \leq a(t_2)y'(t_2) < 0 \), \( t \geq t_2 \), then

\[
y'(t) \leq \frac{a(t_2)y'(t_2)}{a(t)}, \quad t \geq t_2.
\]

Integrating the above inequality from \( t_3 \) to \( t, \ t > t_3 \geq t_2 \), we have

\[
y(t) \leq y(t_3) + a(t_2)y(t_2) \int_{t_3}^{t} \frac{1}{a(s)} ds, \quad t \geq t_3,
\]

therefore, from the condition of (H_3), we have \( \lim_{t \to \infty} y(t) = -\infty \), this contradicts \( y(t) > 0, \ t \geq t_1 \). This completes the proof of Lemma 1.

### 2. MAIN RESULTS

Now, we give the main results of this paper.

Let \( D_0 = \{(t, s) \mid t > s \geq t_0 \} \), \( D = \{(t, s) \mid t \geq s \geq t_0 \} \).

**Theorem 1.** Assume that there exist \( \frac{d}{dt}g(t, a) \) and functions \( H(t, s) \in C'(D; \mathbb{R}), \ h(t, s) \in C(D_0; \mathbb{R}) \), and \( \rho(t) \in C'([t_0, \infty), (0, \infty)) \), such that

(I) \( H(t, t) = 0, \ H(t, s) > 0; \)

(II) \( H_s'(t, s) \leq 0, \) and \(-H_s'(t, s) - H(t, s)(/\rho'(s)/\rho(s)) = h(t, s)\sqrt{H(t, s)}.

If

\[
\limsup_{t \to \infty} \frac{1}{H(t, t_0)} \times \int_{t_0}^{t} H(t, s) \rho(s) \int_{a}^{b} p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) - \frac{a[g(s, a)] \rho(s) h^2(t, s)}{4g'(s, a)} ds = \infty,
\]

then every solution of equation (1) is oscillatory.

**Proof.** Assume that there exists a nonoscillatory solution \( x(t) \) of equation (1) on \([t_0, \infty)\), such that \( x(t) \not= 0 \) on \([t_0, \infty)\). Without loss of generality, assume that \( x(t) > 0, \ t \geq t_0 \). Then, proceeding as in the proof of Lemma 1, there exists a \( T_0 \geq t_0 \) such that

\[
x(t) > 0, \quad x(t - \tau) > 0, \quad \text{and} \quad x[g(t, \xi)] > 0, \quad t \geq T_0, \quad \xi \in [a, b].
\]

Thus, from (1) and (5), we have

\[
0 = [a(t)y'(t)]' + \int_{a}^{b} p(t, \xi) x[g(t, \xi)] d\sigma(\xi)
\]

\[
= [a(t)y'(t)]' + \int_{a}^{b} p(t, \xi) \{y[g(t, \xi)] - c[g(t, \xi)] x[g(t, \xi) - \tau]\} d\sigma(\xi).
\]

From Lemma 1, and using \( y(t) \geq x(t), \ t \geq t_1 \), we get \( y[g(t, \xi)] \geq x[g(t, \xi) - \tau] \geq x[g(t, \xi) - \tau], \) thus,

\[
[a(t)y'(t)]' + \int_{a}^{b} p(t, \xi) \{1 - c[g(t, \xi)]\} y[g(t, \xi)] d\sigma(\xi) \leq 0.
\]

Furthermore, using \( g(t, \xi) \) is nondecreasing with respect to \( \xi \), we have \( y[g(t, a)] \leq y[g(t, \xi)] \), thus,

\[
[a(t)y'(t)]' + y[g(t, a)] \int_{a}^{b} p(t, \xi) \{1 - c[g(t, \xi)]\} d\sigma(\xi) \leq 0, \quad t \geq t_1.
\]
Let
\[ z(t) = \rho(t) \frac{a(t) y'(t)}{y[g(t, a)]}, \]
then \( z(t) \geq 0 \), and according to the fact that there exists a \( \frac{d}{dt} g(t, a) \), we obtain \( y'[g(t, a)] = \frac{dy}{da} \frac{d}{dt} g(t, a) \), and noting that \( g(t, \xi) \) is nondecreasing with respect to \( \xi \), \( g(t, \xi) \leq t, \xi \in [a, b] \), and \( [a(t)y'(t)]' \leq 0 \), we obtain \( a(t)y'(t) \leq a[g(t, a)]y'[g(t, a)] \). Thus,
\[
\begin{align*}
 z'(t) &= \frac{\rho(t)[a(t)y'(t)] + \rho(t)[a(t)y'(t)]'}{y[g(t, a)]} - \frac{\rho(t)[a(t)y'(t)]}{y^2[g(t, a)]} g'(t, a) \\
 &\leq \frac{\rho(t)}{\rho(t)} z(t) - \rho(t) \int_{a}^{b} p(t, \xi) \left\{ 1 - c[g(t, \xi)] \right\} d\sigma(\xi) - \frac{g'(t, a)}{\rho(t)} \frac{\rho(t)}{a[g(t, a)]} z^2(t),
\end{align*}
\]
that is,
\[
\rho(t) \int_{a}^{b} p(t, \xi) \left\{ 1 - c[g(t, \xi)] \right\} d\sigma(\xi) \leq -z'(t) + \frac{\rho(t)}{\rho(t)} z(t) - \frac{g'(t, a)}{a[g(t, a)]} \frac{\rho(t)}{\rho(t)} z^2(t).
\]
Integrating by parts for any \( t > T > t_1 \), and using Properties (I) and (II), we have
\[
\int_{T}^{t} H(t, s) \rho(s) \int_{a}^{b} p(s, \xi) \left\{ 1 - c[g(s, \xi)] \right\} d\sigma(\xi) ds \\
\leq -\int_{T}^{t} H(t, s) z'(s) ds + \int_{T}^{t} H(t, s) \frac{\rho'(s)}{\rho(s)} z(s) ds - \int_{T}^{t} H(t, s) \frac{g'(s, a)}{a[g(s, a)]} \frac{\rho(s)}{\rho(s)} z^2(s) ds \\
= -\int_{T}^{t} H(t, s) dz(s) + \int_{T}^{t} H(t, s) \frac{\rho'(s)}{\rho(s)} z(s) ds - \int_{T}^{t} H(t, s) \frac{g'(s, a)}{a[g(s, a)]} \frac{\rho(s)}{\rho(s)} z^2(s) ds \\
= H(t, T) z(T) + \int_{T}^{t} H(t, s) \frac{\partial H}{\partial s} + \frac{\rho'(s)}{\rho(s)} z(s) ds - \int_{T}^{t} H(t, s) \frac{g'(s, a)}{a[g(s, a)]} \frac{\rho(s)}{\rho(s)} z^2(s) ds \\
= H(t, T) z(T) - \int_{T}^{t} H(t, s) \frac{g'(s, a)}{a[g(s, a)]} \frac{\rho(s)}{\rho(s)} z^2(s) ds \\
= H(t, T) z(T) - \int_{T}^{t} \left[ \frac{H(t, s) g'(s, a)}{a[g(s, a)]} \frac{\rho(s)}{\rho(s)} z(s) + \frac{\sqrt{a[g(s, a)]} \rho(s) h(t, s)}{2 \sqrt{g'(s, a)}} \right]^2 ds \\
+ \int_{T}^{t} \frac{a[g(s, a)]}{4g'(s, a)} \frac{\rho(s)}{\rho(s)} h^2(t, s) ds.
\]
Furthermore, we have
\[
\int_{T}^{t} \left[ H(t, s) \rho(s) \int_{a}^{b} p(s, \xi) \left\{ 1 - c[g(s, \xi)] \right\} d\sigma(\xi) - \frac{a[g(s, a)]}{4g'(s, a)} \rho(s) h^2(t, s) \right] ds \\
\leq H(t, T) z(T) - \int_{T}^{t} \left[ \frac{H(t, s) g'(s, a)}{a[g(s, a)]} \frac{\rho(s)}{\rho(s)} z(s) + \frac{\sqrt{a[g(s, a)]} h(t, s)}{2 \sqrt{g'(s, a)}} \right]^2 ds.
\]
From (II), \( H_t'(t, s) \leq 0 \), for \( t_1 \geq t_0 \), we have \( H(t, t_1) \leq H(t, t_0) \), and instead of \( T \) in (15) with \( t_1 \), we have
\[
\int_{t_1}^{t} \left[ H(t, s) \rho(s) \int_{a}^{b} p(s, \xi) \left\{ 1 - c[g(s, \xi)] \right\} d\sigma(\xi) - \frac{a[g(s, a)]}{4g'(s, a)} \rho(s) h^2(t, s) \right] ds \\
\leq H(t, t_1) z(t_1) - \int_{t_1}^{t} \left[ \frac{H(t, s) g'(s, a)}{a[g(s, a)]} \frac{\rho(s)}{\rho(s)} z(s) + \frac{\sqrt{a[g(s, a)]} h(t, s)}{2 \sqrt{g'(s, a)}} \right]^2 ds \\
\leq H(t, t_1) z(t_1) \leq H(t, t_0) z(t_1),
\]
which implies that

\[
\frac{1}{H(t, t_0)} \int_{t_0}^{t} H(t, s) \rho(s) \int_{a}^{b} p(s, \xi) \left\{ 1 - c[g(s, \xi)] \right\} d\sigma(\xi) - \frac{a[g(s, a)] \rho(s) h^2(t, s)}{4g'(s, a)} ds
\]

\[
= \frac{1}{H(t, t_0)} \left[ \int_{t_0}^{t_1} H(t, s) \rho(s) \int_{a}^{b} p(s, \xi) \left\{ 1 - c[g(s, \xi)] \right\} d\sigma(\xi) - \frac{a[g(s, a)] \rho(s) h^2(t, s)}{4g'(s, a)} \right] ds
\]

\[
\times \left[ H(t, s) \rho(s) \int_{a}^{b} p(s, \xi) \left\{ 1 - c[g(s, \xi)] \right\} d\sigma(\xi) - \frac{a[g(s, a)] \rho(s) h^2(t, s)}{4g'(s, a)} \right] ds (16)
\]

\[
\leq z(t_1) + \int_{t_0}^{t_1} H(t, s) \rho(s) \int_{a}^{b} p(s, \xi) \left\{ 1 - c[g(s, \xi)] \right\} d\sigma(\xi) ds
\]

\[
\leq z(t_1) + \int_{t_0}^{t_1} \rho(s) \int_{a}^{b} p(s, \xi) \left\{ 1 - c[g(s, \xi)] \right\} d\sigma(\xi) ds.
\]

Letting \( t \to \infty \), and taking upper limits, we have

\[
\lim_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} H(t, s) \rho(s) \int_{a}^{b} p(s, \xi) \left\{ 1 - c[g(s, \xi)] \right\} d\sigma(\xi) - \frac{a[g(s, a)] \rho(s) h^2(t, s)}{4g'(s, a)} ds
\]

\[
\leq z(t_1) + \int_{t_0}^{t_1} \rho(s) \int_{a}^{b} p(s, \xi) \left\{ 1 - c[g(s, \xi)] \right\} d\sigma(\xi) ds = M < \infty,
\]

where \( M \) is a constant, which contradicts (8). This completes the proof of Theorem 1.

From Theorem 1, we can obtain the following corollary.

**COROLLARY 1.** If condition (8) of Theorem 1 is replaced by

\[
\int_{t_0}^{t} H(t, s) \rho(s) \int_{a}^{b} p(s, \xi) \left\{ 1 - c[g(s, \xi)] \right\} d\sigma(\xi) - \frac{a[g(s, a)] \rho(s) h^2(t, s)}{4g'(s, a)} ds = e, (18)
\]

\[
\lim_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \rho(s) \int_{a}^{b} p(s, \xi) \left\{ 1 - c[g(s, \xi)] \right\} d\sigma(\xi) ds < \infty, (19)
\]

then every solution of equation (1) is oscillatory.

**REMARK 1.** By introducing various \( H(t, s) \) from Theorem 1 or Corollary 1, we can obtain some oscillatory criteria of solution of equation (1). For example, let \( H(t, s) = (t - s)^{m-1}, t \geq s \geq t_0, \) in which \( m > 2 \) is an integer. By choosing

\[
h(t, s) = (t - s)^{(m-3)/2} \left[ m - 1 - (t - s) \frac{\rho'(s)}{\rho(s)} \right],
\]

it is easily known that Conditions of (I) and (II) hold, then, from Theorem 1 and Corollary 1, we have the following.

**COROLLARY 2.** Assume that there exist \( \frac{d}{dt} g(t, a) \) and function \( \rho(t) \in C([t_0, \infty), (0, \infty)) \) such that

\[
\lim_{t \to \infty} \frac{1}{m-1} \int_{t_0}^{t} (t - s)^{m-1} \rho(s) \int_{a}^{b} p(s, \xi) \left\{ 1 - c[g(s, \xi)] \right\} d\sigma(\xi)
\]

\[
- \frac{a[g(s, a)] \rho(s)}{4g'(s, a)} \left[ m - 1 - (t - s) \frac{\rho'(s)}{\rho(s)} \right]^2 (t - s)^{m-3} ds = \infty,
\]

then every solution of equation (1) is oscillatory.
COROLLARY 3. Assume that there exist \( \frac{d}{dt} g(t, a) \) and function \( \rho(t) \in C^\prime([t_0, \infty), (0, \infty)) \) such that

\[
\limsup_{t \to \infty} \frac{1}{t^{m-1}} \int_{t_0}^{t} (t-s)^{m-1} \rho(s) \int_a^b p(s, \xi) \left\{ 1 - c[g(s, \xi)] \right\} d\sigma(\xi) \, ds = \infty, \tag{21}
\]

\[
\limsup_{t \to \infty} \frac{1}{t^{m-1}} \int_{t_0}^{t} a \frac{g(s, a)}{g'(s, a)} \frac{\rho(s)}{\rho'(s)} \left[ m - 1 - (t-s) \frac{\rho'(s)}{\rho(s)} \right]^2 (t-s)^{m-3} \, ds < \infty, \tag{22}
\]

then every solution of equation (1) is oscillatory.

EXAMPLE. Consider the following equation:

\[
\left[ \frac{1}{t} [x(t) - (1-e^{-t}) x(t-1)] \right]' + \int_a^b e^{t+2\xi} x[t+\xi] d\xi = 0, \quad t \geq 1, \tag{23}
\]

in which \( a(t) = 1/t, c(t) = 1-e^{-t}, p(t, \xi) = e^{t+2\xi}, g(t, \xi) = t+\xi, \tau = 1, a = 0, b = 1, \) and \( t_0 = 1. \)

Choosing \( \rho(s) = s \) and \( m = 3, \) then

\[
\limsup_{t \to \infty} \frac{1}{t^{m-1}} \int_{t_0}^{t} (t-s)^{m-1} \rho(s) \int_a^b p(s, \xi) \left\{ 1 - c[g(s, \xi)] \right\} d\sigma(\xi) \, ds = \infty
\]

and

\[
\limsup_{t \to \infty} \frac{1}{t^{m-1}} \int_{t_0}^{t} a \frac{g(s, a)}{g'(s, a)} \frac{\rho(s)}{\rho'(s)} \left[ m - 1 - (t-s) \frac{\rho'(s)}{\rho(s)} \right]^2 (t-s)^{m-3} \, ds < \infty
\]

then, from Corollary 3, every solution of equation (23) is oscillatory.

If taking \( \rho(t) \equiv 1, \) then we have the following.

COROLLARY 4. Assume that \( \frac{d}{dt} g(t, a) \) exists and

\[
\limsup_{t \to \infty} \frac{1}{t^{m-1}} \int_{t_0}^{t} (t-s)^{m-1} \int_a^b p(s, \xi) \left\{ 1 - c[g(s, \xi)] \right\} d\sigma(\xi) \, ds = \infty, \tag{24}
\]

\[
\limsup_{t \to \infty} \frac{1}{t^{m-1}} \int_{t_0}^{t} a \frac{g(s, a)}{g'(s, a)} \left[ m - 1 \right]^2 (t-s)^{m-3} \, ds < \infty, \tag{25}
\]

then every solution of equation (1) is oscillatory.

Let

\[
A(t) = \int_{t_0}^{t} \frac{1}{a(s)} \, ds, \quad t \geq t_0,
\]

and choosing \( H(t, s) = (A(t) - A(s))^\lambda(t), \) in which \( \lambda > 1 \) is a constant, then

\[
h(t, s) = (A(t) - A(s))^{\lambda/2 - 1} \left[ \frac{\lambda}{a(s)} - (A(t) - A(s)) \frac{\rho'(s)}{\rho(s)} \right],
\]

thus, from Corollary 1, we have the following.
COROLLARY 5. Assume that there exist $\frac{d}{dt} g(t, a)$ and a function $\rho(t) \in C'(\left[t_0, \infty\right), (0, \infty))$ such that

$$
\limsup_{t \to \infty} A^{-\lambda}(t) \int_{t_0}^{t} \left( A(t) - A(s) \right)^{\lambda} \rho(s) \int_{a}^{b} p(s, \xi) \left\{ 1 - c[g(s, \xi)] \right\} d\sigma(\xi) ds = \infty, \quad (26)
$$

$$
\limsup_{t \to \infty} A^{-\lambda}(t) \int_{t_0}^{t} \frac{a[g(s, a)] \rho(s)}{g'(s, a)} \left( A(t) - A(s) \right)^{\lambda - 2} \left[ \frac{\lambda}{a(s)} - \frac{A(t) - A(s)}{a(s)} \right]^{2} ds < \infty, \quad (27)
$$

then every solution of equation (1) is oscillatory.

If $H(t, s) = \varphi(t - s)$, taking $\rho(t) \equiv 1$, $t \geq s \geq 0$, in which $\varphi(t)$ is an any function satisfying $\varphi(t) \in C'(\left[t_0, \infty\right), (0, \infty))$, $\varphi(0) = 0$, $\varphi'(t) \geq 0$, $t \geq 0$. Choosing

$$
h(t, s) = \begin{cases} \frac{\varphi'(t - s)}{\sqrt{\varphi(t - s)}}, & t > s, \\ 0, & t = s, \end{cases}
$$

then the conditions of (I) and (II) are satisfied, thus we have the following.

COROLLARY 6. Assume that $\frac{d}{dt} g(t, a)$ exists and

$$
\limsup_{t \to \infty} \frac{1}{\varphi(t)} \int_{t_0}^{t} \left[ \varphi(t - s) \int_{a}^{b} p(s, \xi) \left( 1 - c[g(s, \xi)] \right) d\sigma(\xi) - \frac{a[g(s, a)] \varphi'(t - s)}{4\varphi(t - s) g'(s, a)} \right] ds = \infty, \quad (28)
$$

then every solution of equation (1) is oscillatory.

THEOREM 2. Assume that the conditions of Theorem 1 hold, and

$$
0 < \inf_{t \geq t_0} \left[ \liminf_{t \to \infty} H(t, s) \right] \leq \infty, \quad (29)
$$

$$
\limsup_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \frac{a[g(s, a)] \rho(s) h^2(t, s)}{g'(s, a)} ds < \infty. \quad (30)
$$

If there exists a function $\varphi(t) \in C([t_0, \infty), R)$ satisfying

$$
\limsup_{t \to \infty} \frac{1}{H(t, u)} \int_{u}^{t} \left[ H(t, s) \rho(s) \int_{a}^{b} p(s, \xi) \left( 1 - c[g(s, \xi)] \right) d\sigma(\xi) \right. \\
- \left. \frac{a[g(s, a)] \rho(s) h^2(t, s)}{4g'(s, a)} \right] ds \geq \varphi(u), \quad u \geq t_0,
$$

$$
\limsup_{t \to \infty} \frac{1}{H(t, t_0)} \int_{0}^{t} \frac{g'(u, a)}{a[g(u, a)] \rho(s)} \varphi^2(u) du = \infty, \quad \varphi_+(u) = \max_{u \geq t_0} \left\{ \varphi(u), 0 \right\}, \quad (31)
$$

then every solution of equation (1) is oscillatory.

PROOF. Assume that there exists a nonoscillatory solution $x(t)$ of equation (1) on $\left[t_0, \infty\right)$, such that $x(t) \neq 0$ on $\left[t_0, \infty\right)$. Without loss of generality, assume that $x(t) > 0$, $t \geq t_0$. Then, proceeding as in the proof of Theorem 1, for $t > u \geq t_1 \geq t_0$, we have

$$
\frac{1}{H(t, u)} \int_{u}^{t} \left[ H(t, s) \rho(s) \int_{a}^{b} p(s, \xi) \left( 1 - c[g(s, \xi)] \right) d\sigma(\xi) \right. \\
- \left. \frac{a[g(s, a)] \rho(s) h^2(t, s)}{4g'(s, a)} \right] ds \\
\leq z(u) - \frac{1}{H(t, u)} \int_{u}^{t} \left[ \sqrt{\frac{H(t, s) g'(s, a)}{a[g(s, a)] \rho(s)} z(s) + \frac{\sqrt{a[g(s, a)] \rho(s) h(t, s)}}{2\sqrt{g'(s, a)}}} \right]^2 ds.
$$
Let $t \to \infty$, and taking upper limits, we have

$$
\limsup_{t \to \infty} \frac{1}{H(t,u)} \int_{t_1}^{t} \left[ H(t,s) \rho(s) \int_{a}^{b} p(s,\xi) \left\{ 1 - c[g(s,\xi)] \right\} d\sigma(\xi) - \frac{a[g(s,a)] \rho(s) h^2(t,s)}{4g'(s,a)} \right] ds
$$

\begin{align*}
\leq z(u) & - \liminf_{t \to \infty} \frac{1}{H(t,u)} \int_{t_1}^{t} \left[ \frac{H(t,s) g'(s,a)}{a[g(s,a)] \rho(s)} z(s) + \frac{a[g(s,a)] \rho(s) h(t,s)}{2 \sqrt{g'(s,a)}} \right]^2 ds,
\end{align*}

thus, from (31), we have

$$
z(u) \geq \varphi(u) + \liminf_{t \to \infty} \frac{1}{H(t,u)} \int_{t_1}^{t} \left[ \frac{H(t,s) g'(s,a)}{a[g(s,a)] \rho(s)} z(s) + \frac{a[g(s,a)] \rho(s) h(t,s)}{2 \sqrt{g'(s,a)}} \right]^2 ds,
$$

then $z(u) \geq \varphi(u)$, and

$$
\liminf_{t \to \infty} \frac{1}{H(t,u)} \int_{t_1}^{t} \left[ \frac{H(t,s) g'(s,a)}{a[g(s,a)] \rho(s)} z(s) + \frac{a[g(s,a)] \rho(s) h(t,s)}{2 \sqrt{g'(s,a)}} \right]^2 ds \leq z(u) - \varphi(u) \quad (33)
$$

where $M$ is a constant. On the other hand, for $t > t_1$, we have

$$
\liminf_{t \to \infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} \left[ \frac{H(t,s) g'(s,a)}{a[g(s,a)] \rho(s)} z(s) + \frac{a[g(s,a)] \rho(s) h(t,s)}{2 \sqrt{g'(s,a)}} \right]^2 ds \geq \liminf_{t \to \infty} \left[ \frac{1}{H(t,t_1)} \int_{t_1}^{t} H(t,s) g'(s,a) z^2(s) ds + \frac{1}{H(t,t_1)} \int_{t_1}^{t} \sqrt{H(t,s) h(t,s) z(s)} ds \right].
$$

Let

$$
v(t) = \frac{1}{H(t,t_1)} \int_{t_1}^{t} H(t,s) g'(s,a) z^2(s) ds,
$$

$$
w(t) = \frac{1}{H(t,t_1)} \int_{t_1}^{t} \sqrt{H(t,s) h(t,s) z(s)} ds.
$$

From (33) and (34), we have

$$
\liminf_{t \to \infty} [v(t) + w(t)] < \infty.
$$

Now we can claim that

$$
\int_{t_1}^{\infty} \frac{g'(s,a)}{a[g(s,a)] \rho(s)} z^2(s) ds < \infty, \quad t > t_1.
$$

In fact, assume the contrary, that

$$
\int_{t_1}^{\infty} \frac{g'(s,a)}{a[g(s,a)] \rho(s)} z^2(s) ds = \infty, \quad t > t_1.
$$

From (29), there exists a constant $L > 0$ such that

$$
\inf_{s \geq t_0} \left[ \liminf_{t \to \infty} \frac{H(t,s)}{H(t,t_0)} \right] > L > 0,
$$

then it follows from (38), for any positive number $\mu > 0$, that there exists a $T > t_1$ such that

$$
\int_{t_1}^{t} \frac{g'(s,a)}{a[g(s,a)] \rho(s)} z^2(s) ds \geq \frac{\mu}{L}, \quad t > T,
$$

where $\mu$ is a constant.
then, for \( t \geq T > t_1 \), integrating by parts, we have

\[
v(t) = \frac{1}{H(t, t_1)} \int_{t_1}^{t} H(t, s) \frac{g'(s, a)}{[g(s, a)] \rho(s)} z^2(s) \, ds
\]

\[
= \frac{1}{H(t, t_1)} \int_{t_1}^{t} H(t, s) \left[ \int_{t_1}^{s} \frac{g'(u, a)}{[g(u, a)] \rho(u)} z^2(u) \, du \right] \left[ -\frac{\partial H(t, s)}{\partial s} \right] ds
\]

\[
\geq \frac{1}{H(t, t_1)} \int_{t_1}^{t} \left[ \int_{t_1}^{s} \frac{g'(u, a)}{[g(u, a)] \rho(u)} z^2(u) \, du \right] \left[ -\frac{\partial H(t, s)}{\partial s} \right] ds
\]

\[
\geq \frac{\mu}{L H(t, t_1)} \int_{t_1}^{t} \left[ -\frac{\partial H(t, s)}{\partial s} \right] ds
\]

\[
= \frac{\mu}{L H(t, t_1)} \left( T - H(t, t_0) \right) > 0.
\]

It follows from (39) that

\[
\liminf_{t \to \infty} \frac{H(t, s)}{H(t, t_0)} > L > 0, \quad s \geq t_0,
\]

therefore, there exists a \( t_2 > T \) such that

\[
\frac{H(t, T)}{H(t, t_0)} \geq L, \quad t > t_2.
\]

From (41) and (42), we have

\[
v(t) \geq \mu, \quad t > t_2.
\]

Since \( \mu \) is a arbitrary, thus, we have

\[
\lim_{t \to \infty} v(t) = \infty.
\]

From (36), that is, \( \liminf_{t \to \infty} [v(t) + w(t)] < \infty \), then there is a convergence subsequence, for convenience, we can take \( \{t_n\} \to \infty \) on \([t_1, \infty)\) such that \( \lim_{n \to \infty} t_n = \infty \) and satisfying

\[
\lim_{n \to \infty} [v(t_n) + w(t_n)] = \liminf_{t \to \infty} [v(t) + w(t)] < \infty.
\]

Thus, there exists a positive integer \( N_1 \) and constant \( M \) such that

\[
v(t_n) + w(t_n) < M, \quad n > N_1,
\]

and from (43), we have

\[
\lim_{n \to \infty} v(t_n) = \infty.
\]

Furthermore, from (44) and (45), we obtain

\[
\lim_{n \to \infty} w(t_n) = -\infty,
\]

and for any \( \varepsilon \in (0, 1) \), there exists a positive integer \( N_2 \) such that

\[
\frac{w(t_n)}{v(t_n)} + 1 < \varepsilon, \quad n > N_2,
\]

then

\[
\frac{w(t_n)}{v(t_n)} < \varepsilon - 1 < 0.
\]
From (46) and (47), we have
\[
\lim_{t \to \infty} \frac{w(t_n)}{v(t_n)} w(t_n) = \infty.
\] (48)

On the other hand, by using the Schwartz inequality, for \( t \geq t_1 \), we obtain
\[
0 \leq w^2(t_n) = \frac{1}{H^2(t_n, t_1)} \left\{ \int_{t_1}^{t_n} \sqrt{H(t_n, s)} h(t_n, s) z(s) \, ds \right\}^2 \leq \frac{1}{H(t_n, t_1)} \int_{t_1}^{t_n} \frac{a[g(s, a)] \rho(s)}{g'(s, a)} h(t_n, s) \, ds \right\} \times \frac{1}{H(t_n, t_1)} \int_{t_1}^{t_n} a[g(s, a)] \rho(s) h(t_n, s) \, ds \right\}.
\]

\[
\left(1 \leq H(t_n, t_1) \right) \int_{t_1}^{t_n} \frac{a[g(s, a)] \rho(s)}{g'(s, a)} h(t_n, s) \, ds,
\]

\[
\lim_{t \to \infty} H(t, t_0) \lim_{t \to \infty} (t - s) \rho(s) = 0,
\]

\[
\frac{w^2(t_n)}{v(t_n)} \leq \frac{1}{H(t_n, t_1)} \int_{t_1}^{t_n} \frac{a[g(s, a)] \rho(s)}{g'(s, a)} h(t_n, s) \, ds.
\] (49)

According to (42), we obtain
\[
\frac{1}{H(t_n, t_1)} \leq \frac{1}{H(t_n, t_0)} = \frac{1}{H(t_n, t_1) H(t_n, t_0)} \leq \frac{1}{LH(t_n, t_0)},
\] (50)

therefore, from (49) and (50), we have
\[
\frac{w^2(t_n)}{v(t_n)} \leq \frac{1}{LH(t_n, t_0)} \int_{t_1}^{t_n} \frac{a[g(s, a)] \rho(s)}{g'(s, a)} h(t_n, s) \, ds.
\] (51)

from (51) and (48), we have
\[
\lim_{n \to \infty} \frac{1}{LH(t_n, t_0)} \int_{t_1}^{t_n} \frac{a[g(s, a)] \rho(s)}{g'(s, a)} h(t_n, s) \, ds = \infty,
\] (52)

thus,
\[
\limsup_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \frac{a[g(s, a)] \rho(s)}{g'(s, a)} h(t, s) \, ds = \infty,
\] (53)

which contradicts (30). Therefore, from (37) and \( z(s) \geq \varphi(s) \), we have
\[
\int_{t_1}^{t} \frac{g'(s, a)}{a[g(s, a)] \rho(s)} \varphi^2(s) \, ds \leq \int_{t_1}^{t} \frac{g'(s, a)}{a[g(s, a)] \rho(s)} z^2(s) \, ds < \infty,
\] (54)

which contradicts (32). This completes the proof of Theorem 2.

**Remark 2.** By introducing various \( H(t, s) \) from Theorem 2, we can obtain some oscillatory criteria of solution of equation (1). For example, let \( H(t, s) = (t - s)^{m-1}, \ t \geq s \geq t_0, \) in which \( m > 2 \) is a integer. Then,
\[
h(t, s) = (t - s)^{m-3/2} \left[ m - 1 - (t - s) \frac{\rho'(s)}{\rho(s)} \right],
\]

\[
\lim_{t \to \infty} \frac{H(t, s)}{H(t, t_0)} = \lim_{t \to \infty} \frac{(t - s)^{m-1}}{(t - t_0)^{m-1}} = 1 > 0,
\]

thus, the conditions of Theorem 2 are satisfied, and we have the following.
COROLLARY 7. Assume that there exist functions \( \rho(t) \in C([t_0, \infty), (0, \infty)) \) and \( \varphi(t) \in C([t_0, \infty), R) \) satisfying

\[
\limsup_{t \to \infty} \frac{1}{t^{m-1}} \int_{t_0}^{t} \frac{a \rho(s)}{g'(s, a)} (t-s)^{m-3} \left[ m-1 - (t-s) \frac{\rho'(s)}{\rho(s)} \right]^2 ds < \infty, \tag{55}
\]

\[
\limsup_{t \to \infty} \frac{1}{t^{m-1}} \int_{t_0}^{t} \left\{ (t-s)^{m-1} \rho(s) \int_{a}^{b} p(s, \xi) \left( 1 - c |g(s, \xi)| \right) d\xi \right\} \frac{m-1 - (t-s) \frac{\rho'(s)}{\rho(s)} }{4g'(s, a)} ds \geq \varphi(u), \quad u \geq t_0, \tag{56}
\]

\[
\limsup_{t \to \infty} \frac{1}{t^{m-1}} \int_{t_0}^{t} \frac{g'(u, a)}{a \rho(u, a) \rho(s)} \varphi_+^2(u) du = \infty, \quad \varphi_+ (u) = \max \{ \varphi(u), 0 \}, \tag{57}
\]

then every solution of equation (1) is oscillatory.

THEOREM 3. Assume that the conditions of Theorem 1 and (29) hold, and

\[
\liminf_{t \to \infty} \frac{1}{H(t, u)} \int_{t_0}^{t} H(t, s) \rho(s) \int_{a}^{b} p(s, \xi) \left( 1 - c |g(s, \xi)| \right) d\sigma(\xi) ds < \infty. \tag{58}
\]

If there exists a function \( \varphi(t) \in C([t_0, \infty), R) \) satisfying

\[
\liminf_{t \to \infty} \frac{1}{H(t, u)} \int_{t_0}^{t} H(t, s) \rho(s) \int_{a}^{b} p(s, \xi) \left( 1 - c |g(s, \xi)| \right) d\sigma(\xi) \frac{m-1 - (t-s) \frac{\rho'(s)}{\rho(s)} }{4g'(s, a)} ds \geq \varphi(u), \quad u \geq t_0, \tag{59}
\]

\[
\limsup_{t \to \infty} \frac{1}{H(t, u)} \int_{t_0}^{t} \frac{g'(u, a)}{a \rho(u, a) \rho(s)} \varphi_+^2(u) du = \infty, \quad \varphi_+ (u) = \max \{ \varphi(u), 0 \}, \tag{60}
\]

then every solution of equation (1) is oscillatory.

PROOF. Assume that there exists a nonoscillatory solution \( x(t) \) of equation (1) on \([t_0, \infty)\), such that \( x(t) \neq 0 \) on \([t_0, \infty)\). Without loss of generality, assume that \( x(t) > 0, t \geq t_0 \). Then, proceeding as in the proof of Theorem 1, for \( t > u \geq t_1 \geq t_0 \), we have

\[
\frac{1}{H(t, u)} \int_{u}^{t} \left[ H(t, s) \rho(s) \int_{a}^{b} p(s, \xi) \left( 1 - c |g(s, \xi)| \right) d\sigma(\xi) - \frac{a |g(s, a)| \rho(s) h^2(t, s)}{4g'(s, a)} \right] ds \leq \varphi(u) - \frac{1}{H(t, u)} \int_{u}^{t} \left[ \frac{\sqrt{H(t, s) g'(s, a)}}{a |g(s, a)| \rho(s)} \varphi_+^2(s) + \frac{\sqrt{a |g(s, a)| \rho(s) h(t, s)}}{2 \sqrt{g'(s, a)}} \right]^2 ds.
\]

Let \( t \to \infty \), and taking lower limits, we have

\[
\liminf_{t \to \infty} \frac{1}{H(t, u)} \int_{u}^{t} \left[ H(t, s) \rho(s) \int_{a}^{b} p(s, \xi) \left( 1 - c |g(s, \xi)| \right) d\sigma(\xi) - \frac{a |g(s, a)| \rho(s) h^2(t, s)}{4g'(s, a)} \right] ds \leq \varphi(u) - \limsup_{t \to \infty} \frac{1}{H(t, u)} \int_{u}^{t} \left[ \frac{\sqrt{H(t, s) g'(s, a)}}{a |g(s, a)| \rho(s)} \varphi_+^2(s) + \frac{\sqrt{a |g(s, a)| \rho(s) h(t, s)}}{2 \sqrt{g'(s, a)}} \right]^2 ds,
\]

which implies that

\[
\varphi(u) \leq \varphi(u), \quad u \geq t_0,
\]
and
\[
\limsup_{t \to \infty} \frac{1}{H(t, u)} \int_u^t \left[ \frac{H(t, s) g'(s, a)}{a \left[ g(s, a) \right] \rho(s)} z(s) + \frac{\sqrt{a \left[ g(s, a) \right] \rho(s) h(t, s)}}{2 \sqrt{g'(s, a)}} \right]^2 \, ds \leq z(u) - \varphi(u) \quad (61)
\]

Let \( v(t) \) and \( w(t) \) as same as Theorem 2, then, from (61), we have
\[
\limsup_{t \to \infty} [v(t) + w(t)] \leq M_1 < \infty. \quad (62)
\]

From (59), we have
\[
\varphi(t_0) \leq \liminf_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \left[ H(t, s) \rho(s) \int_a^b p(s, \xi) \left\{ 1 - c \left[ g(s, \xi) \right] \right\} \, d\sigma(\xi) \right. \\
- \left. \frac{a \left[ g(s, a) \right] \rho(s) h^2(t, s)}{4g'(s, a)} \right] \, ds
\]
\leq \liminf_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t H(t, s) \rho(s) \int_a^b p(s, \xi) \left\{ 1 - c \left[ g(s, \xi) \right] \right\} \, d\sigma(\xi) \, ds \\
- \limsup_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \frac{a \left[ g(s, a) \right] \rho(s) h^2(t, s)}{4g'(s, a)} \, ds, \quad (63)
\]
from (58) and (63), we have
\[
\limsup_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \frac{a \left[ g(s, a) \right] \rho(s) h^2(t, s)}{4g'(s, a)} \, ds < \infty. \quad (64)
\]

Thus, there exists a sequence \( \{t_n\}^\infty_{n=1} \) on \([t_1, \infty)\) such that \( \lim_{n \to \infty} t_n = \infty \) and satisfying
\[
\lim_{n \to \infty} \frac{1}{H(t_n, t_0)} \int_{t_0}^{t_n} \frac{a \left[ g(s, a) \right] \rho(s) h^2(t, s)}{4g'(s, a)} \, ds = \limsup_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \frac{a \left[ g(s, a) \right] \rho(s) h^2(t, s)}{4g'(s, a)} \, ds < \infty, \quad (65)
\]
then (30) holds in Theorem 2. The following proof is similar to Theorem 2, we omit the details. This completes the proof of Theorem 3.

REFERENCES